3

Kinematics of a Continuum

The branch of mechanics in which materials are treated as continuous is known as **continuum mechanics**. Thus, in this theory, one speaks of an infinitesimal volume of material, the totality of which forms a **body**. One also speaks of a **particle** in a continuum, meaning, in fact an infinitesimal volume of material. This chapter is concerned with the kinematics of such particles.

3.1 Description of Motions of a Continuum

In particle kinematics, the path line of a particle is described by a vector function of time, i.e.,

$$\mathbf{r} = \mathbf{r}(t) \tag{ia}$$

where $\mathbf{r}(t) = x(t)\mathbf{e}_1 + y(t)\mathbf{e}_2 + z(t)\mathbf{e}_3$ is the position vector. In component form, the above equation reads:

$$x = x(t), \quad y = y(t) \quad \text{and} \quad z = z(t) \tag{ib}$$

If there are N particles, there are N pathlines, each is described by one of the equations:

$$\mathbf{r}_n = \mathbf{r}_n(t), \quad n = 1,2...N \tag{ii}$$

That is, for the particle number 1, the path line is given by $\mathbf{r}_1(t)$, for the particle number 2, it is given by $\mathbf{r}_2(t)$, etc.

For a continuum, there are not only infinitely many particles, but within each and every neighborhood of a particle there are infinitely many other particles. Therefore, it is not possible to identify particles by assigning each of them a number in the same way as in the kinematics of particles. However, it is possible to identify them by the positions they occupy at some reference time t_0. For example, if a particle of a continuum was at the position (1,2,3) at the reference time t_o, the set of coordinates(1,2,3) can be used to identify this particle. In general, therefore, if a particle of a continuum was at the position (X_1,X_2,X_3) at the reference time t_o, the set of coordinate (X_1,X_2,X_3) can be used to identity this particle. Thus, in general,

the path lines of every particle in a continuum can be described by a vector equation of the

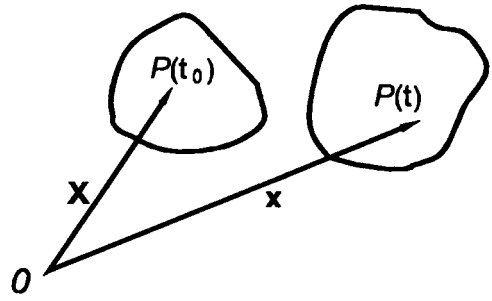

Fig. 3.1

form

$$\mathbf{x} = \mathbf{x}(\mathbf{X},t) \quad \text{with} \quad \mathbf{x}(\mathbf{X},t_o) = \mathbf{X} \tag{3.1.1}$$

where $\mathbf{x} = x_1\mathbf{e}_1+x_2\mathbf{e}_2+x_3\mathbf{e}_3$ is the position vector at time t for the particle P which was at $\mathbf{X} = X_1\mathbf{e}_1+X_2\mathbf{e}_2+X_3\mathbf{e}_3$ (see Fig. F3.1).

In component form, Eq. (3.1.1) takes the form:

$$\begin{aligned} x_1 &= x_1\,(X_1,X_2,X_3,t) \\ x_2 &= x_2\,(X_1,X_2,X_3,t) \\ x_3 &= x_3\,(X_1,X_2,X_3,t) \end{aligned} \tag{3.1.2a}$$

or

$$x_i = x_i\,(X_1,X_2,X_3,t) \quad \text{with} \quad x_i\,(X_1,X_2,X_3,t_o) = X_i \tag{3.1.2b}$$

In Eqs. (3.1.2), the triple (X_1,X_2,X_3) serves to identify the different particles of the body and is known as **material coordinates.** Equation (3.1.1) or Eqs. (3.1.2) is said to define a motion for a continuum; these equations describe the **pathline** for every particle in the continuum. They may also be called the **kinematic equations of motion.**

Example 3.1.1

Consider the motion

$$\mathbf{x} = \mathbf{X} + ktX_2\,\mathbf{e}_1 \tag{i}$$

where $\mathbf{x} = x_1\mathbf{e}_1+x_2\mathbf{e}_2+x_3\mathbf{e}_3$ is the position vector at time t for a particle which was at $\mathbf{X} = X_1\mathbf{e}_1+X_2\mathbf{e}_2+X_3\mathbf{e}_3$ at $t=0$. Sketch the configuration at time t for a body which at $t=0$ has the shape of a cube of unit sides as shown in Fig. 3.2.

Solution. In component form, Eq. (i) becomes

$$x_1 = X_1+ktX_2 \tag{iia}$$

$$x_2 = X_2 \tag{iib}$$

$$x_3 = X_3 \tag{iic}$$

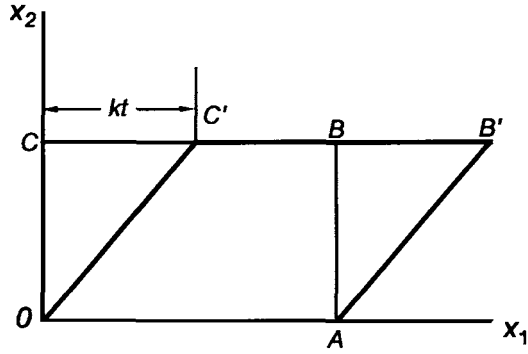

Fig. 3.2

At $t=0$, the particle O is located at (0,0,0). Thus, for this particle, the material coordinates are

$$X_1=0,\ X_2=0 \text{ and } X_3=0.$$

Substituting these values for X_i in Eq. (ii), we get, for all time t, $(x_1, x_2, x_3) = (0,0,0)$. In other words, this particle remains at (0,0,0) at all times.

Similarly, the material coordinates for the particle A are

$$(X_1,X_2,X_3) = (1,0,0)$$

and the position for A at time t is

$$(x_1,x_2,x_3) = (1,0,0)$$

Thus, the particle A also does not move with time. In fact, since the material coordinates for the points along the material line OA are

$$(X_1, X_2, X_3) = (X_1,0,0)$$

Therefore, for them, the positions at time t are

$$(x_1, x_2, x_3) = (X_1,0,0)$$

so that the whole material line OA is motionless.

On the other hand, the material coordinates for the material line CB are

$$(X_1, X_2, X_3) = (X_1, 1, 0)$$

so that according Eq. (ii)

$$(x_1, x_2, x_3) = (X_1 + kt, 1, 0)$$

In other words, the material line has moved horizontally through a distance of kt (see Fig. 3.2).

The material coordinates for the material line OC are $(X_1, X_2, X_3) = (0, X_2, 0)$, so that for the particles along this line $(x_1, x_2, x_3) = (ktX_2, X_2, 0)$. The fact that $x_1 = ktX_2$ means that the straight material line OC remains a straight line OC' at time t as shown in Fig. 3.2. The situation for the material line AB is similar. Thus, at time t, the side view of the cube changes from that of a square to a parallelogram as shown. Since $x_3 = X_3$ at all time for all particles, it is clear that all motions are parallel to the plane $x_3 = 0$. The motion given in this example is known as **simple shearing motion.**

Example 3.1.2

Let

$$Y_1 = -X_1, \; Y_2 = X_2, \text{ and } Y_3 = X_3. \tag{i}$$

Express the simple shearing motion given in Example 3.1.1 in terms of (Y_1, Y_2, Y_3)

Solution. Straight forward substitutions give

$$\begin{aligned} x_1 &= -Y_1 + ktY_2 \\ x_2 &= Y_2 \\ x_3 &= Y_3. \end{aligned} \tag{ii}$$

These equations, i.e.,

$$x_i = x_i(Y_1, Y_2, Y_3, t) \tag{iii}$$

obviously also describe the simple shearing motion just as the equations given in the previous example. The triples (Y_1, Y_2, Y_3) are also material coordinates in that they also identify the particles in the continuum although they are not the coordinates of the particles at any time. This example demonstrates the fact that while the positions of the particles at some reference time t_0 can be used as the material coordinates, the material coordinates need not be the positions of the particle at any time. However, within this book, all material coordinates will be coordinates of the particles at some reference time.

Example 3.1.3

The position at time t, of a particle initially at (X_1,X_2,X_3), is given by the equations:

$$x_1 = X_1 + (X_1 + X_2)t, \quad x_2 = X_2 + (X_1 + X_2)t, \quad x_3 = X_3 \tag{i}$$

(a) Find the velocity at $t=2$ for the particle which was at (1,1,0) at the reference time.

(b) Find the velocity at $t=2$ for the particle which is at the position (1,1,0) at t = 2.

Solution. (a)

$$v_1 = \left(\frac{\partial x_1}{\partial t}\right)_{X_i-\text{fixed}} = X_1 + X_2, \quad v_2 = \left(\frac{\partial x_2}{\partial t}\right)_{X_i-\text{fixed}} = X_1 + X_2, \quad v_3 = 0 \tag{ii}$$

For the particle $(X_1,X_2,X_3) = (1,1,0)$, the velocity at $t = 2$ (and any time t) is

$$v_1 = 1 + 1 = 2, \quad v_2 = 1 + 1 = 2, \quad v_3 = 0 \tag{iii}$$

i.e.,

$$\mathbf{v} = 2\mathbf{e}_1 + 2\mathbf{e}_2 \tag{iv}$$

(b)To calculate the reference position (X_1,X_2,X_3) which was occupied by the particle which is at $(x_1,x_2,x_3) = (1,1,0)$ at $t = 2$, we substitute the value of $(x_1,x_2,x_3) = (1,1,0)$ and $t = 2$ in Eq. (i) and solve for (X_1,X_2,X_3), i.e.,

$$1 = 3X_1 + 2X_2, \qquad 1 = 2X_1 + 3X_2 \tag{v}$$

Thus, $X_1 = \frac{1}{5}$, $X_2 = \frac{1}{5}$. Substituting these values in Eq. (ii), we obtain

$$v_1 = \frac{2}{5}, \quad v_2 = \frac{2}{5}, \quad v_3 = 0 \tag{vi}$$

3.2 Material Description and Spatial Description

When a continuum is in motion, its temperature Θ, its velocity $\mathbf{v}$, its stress tensor $\mathbf{T}$ (to be defined in the next chapter) may change with time. We can describe these changes by:

I. Following the particles, i.e., we express Θ, $\mathbf{v}$, $\mathbf{T}$ as functions of the particles (identified by the material coordinates, (X_1,X_2,X_3)) and time t. In other words, we express

$$\Theta = \hat{\Theta}(X_1,X_2,X_3,t) \tag{3.2.1a}$$

$$\mathbf{v} = \hat{\mathbf{v}}(X_1,X_2,X_3,t) \tag{3.2.1b}$$

$$\mathbf{T} = \hat{\mathbf{T}}(X_1,X_2,X_3,t) \tag{3.2.1c}$$

Such a description is known as the **material description.** Other names for it are: **Lagrangian description** and **reference description.**

II. Observing the changes at fixed locations, i.e., we express, Θ,**v**,**T** etc. as functions of fixed position and time. Thus,†

$$\Theta = \tilde{\Theta}(x_1,x_2,x_3,t) \tag{3.2.2a}$$

$$\mathbf{v} = \tilde{\mathbf{v}}(x_1,x_2,x_3,t) \tag{3.2.2b}$$

$$\mathbf{T} = \tilde{\mathbf{T}}(x_1,x_2,x_3,t) \tag{3.2.2c}$$

Such a description is known as a **spatial description** or **Eulerian description.** The triple (x_1,x_2,x_3) locates the fixed position of points in the physical space and is known as the **spatial coordinates.** The spatial coordinates x_i of a particle at any time t are related to the material coordinates X_i of the particle by Eq. (3.1.2). We note that in this description, what is described (or measured) is the change of quantities at a fixed location as a function of time. Spatial positions are occupied by different particles at different times. Therefore, the spatial description does not provide direct information regarding changes in particle properties as they move about. The material and spatial descriptions are, of course, related by the motion. That is, if the motion is known then, one description can be obtained from the other as illustrated by the following example.

Example 3.2.1

Given the motion of a continuum to be

$$x_1 = X_1 + kt\,X_2, \qquad x_2 = X_2, \qquad x_3 = X_3 \tag{i}$$

If the temperature field is given by the spatial description

$$\Theta = x_1 + x_2 \tag{ii}$$

(a)find the material description of temperature and (b)obtain the velocity and rate of change of temperature for particular material particles and express the answer in both a material and a spatial description.

Solution. (a)Substituting (i) into (ii), we obtain

$$\Theta = X_1 + (kt+1)X_2. \tag{iii}$$

(b) Since a particular material particle is designated by a specific **X**, its velocity will be given by

† Note: the superposed ^ and the superposed ~ are used to distinguish different functions for the same dependent variable.

$$v_i = \left(\frac{\partial x_i}{\partial t}\right)_{X_i-\text{fixed}} \tag{iv}$$

so that from Eq. (i)

$$v_1 = kX_2,\ v_2 = v_3 = 0 \tag{v}$$

This is the material description of the velocity field. To obtain the spatial description, we make use of Eq. (i) again, where we have $x_2 = X_2$, so that

$$v_1 = kx_2,\ \ v_2 = v_3 = 0 \tag{vi}$$

From Eq. (iii), the rate of change of temperature for particular material particles is given by

$$\left(\frac{\partial \Theta}{\partial t}\right)_{X_i-\text{fixed}} = kX_2 = kx_2 \tag{vii}$$

We note that even though the given temperature field is independent of time, each particle experiences changes of temperature, since it flows from one spatial position to another.

3.3 Material Derivative

The time rate of change of a quantity (such as temperature or velocity or stress tensor) of a material particle, is known as a material derivative. We shall denote the material derivative by D/Dt.

(i)When a material description of the quantity is used, we have

$$\Theta = \hat{\Theta}(X_1, X_2, X_3, t) \tag{3.3.1}$$

Thus,

$$\frac{D\Theta}{Dt} = \left(\frac{\partial \hat{\Theta}}{\partial t}\right)_{X_i-\text{fixed}} \tag{3.3.2}$$

(ii) When a spatial description of the quantity is used, we have

$$\Theta = \tilde{\Theta}(x_1, x_2, x_3, t) \tag{3.3.3}$$

where x_i, the positions of material particles at time t, are related to the material coordinates by the motion $x_i = \hat{x}_i(X_1, X_2, X_3, t)$. Then,

$$\left(\frac{D\Theta}{Dt}\right) = \left(\frac{\partial \hat{\Theta}}{\partial t}\right)_{X_i-\text{fixed}} = \frac{\partial \tilde{\Theta}}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial \tilde{\Theta}}{\partial x_2}\frac{\partial x_2}{\partial t} + \frac{\partial \tilde{\Theta}}{\partial x_3}\frac{\partial x_3}{\partial t} + \left(\frac{\partial \tilde{\Theta}}{\partial t}\right)_{x_i-\text{fixed}} \tag{i}$$

where $\frac{\partial x_1}{\partial t}, \frac{\partial x_2}{\partial t}, \frac{\partial x_3}{\partial t}$ are to be obtained with fixed values of the X_i 's. When rectangular Cartesian coordinates are used, these are the velocity components v_i of the particle X_i. Thus, the material derivative in rectangular coordinates is

$$\frac{D\Theta}{Dt} = \frac{\partial\Theta}{\partial t} + v_1\frac{\partial\Theta}{\partial x_1} + v_2\frac{\partial\Theta}{\partial x_2} + v_3\frac{\partial\Theta}{\partial x_3} \tag{3.3.4a}$$

or,

$$\frac{D\Theta}{Dt} = \frac{\partial\Theta}{\partial t} + \mathbf{v}\cdot\nabla\Theta \tag{3.3.4b}$$

where it should be emphasized that these equations are for Θ in a spatial description, i.e., $\Theta = \tilde{\Theta}(x_1,x_2,x_3,t)$. Note that if the temperature field is independent of time and if the velocity of a particle is perpendicular to $\nabla\Theta$ (i.e, the particle is moving along the path of constant Θ) then , as expected $\frac{D\Theta}{Dt}=0$.

Note again that Eq. (3.3.4a) is valid only for rectangular Cartesian coordinates, whereas Eq. (3.3.4b) has the advantage that it is valid for all coordinate systems. For a specific coordinate system, all that is needed is the appropriate expression for the gradient. For example, in cylindrical coordinate (r, θ, z),

$$\mathbf{v} = v_r\mathbf{e_r} + v_\theta\mathbf{e}_\theta + v_z\mathbf{e_z} \tag{3.3.5}$$

and from Eq. (2D2.3)

$$\nabla\Theta = \frac{\partial\Theta}{\partial r}\mathbf{e_r} + \frac{1}{r}\frac{\partial\Theta}{\partial\theta}\mathbf{e}_\theta + \frac{\partial\Theta}{\partial z}\mathbf{e_z} \tag{3.3.6}$$

Thus,

$$\frac{D\Theta}{Dt} = \frac{\partial\Theta}{\partial t} + v_r\frac{\partial\Theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial\Theta}{\partial\theta} + v_z\frac{\partial\Theta}{\partial z} \tag{3.3.7}$$

In spherical coordinates (r,θ,ϕ)

$$\mathbf{v} = v_r\,\mathbf{e}_r + v_\theta\,\mathbf{e}_\theta + v_\phi\,\mathbf{e}_\phi \tag{3.3.8}$$

and from Eq. (2D3.9)

$$(\nabla\Theta)_r = \frac{\partial\Theta}{\partial r} \qquad (\nabla\Theta)_\theta = \frac{1}{r}\frac{\partial\Theta}{\partial\theta} \qquad (\nabla\Theta)_\phi = \frac{1}{r\sin\theta}\frac{\partial\Theta}{\partial\phi} \tag{3.3.9}$$

Thus,

$$\frac{D\Theta}{Dt} = \frac{\partial\Theta}{\partial t} + v_r\frac{\partial\Theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial\Theta}{\partial\theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial\Theta}{\partial\phi} \tag{3.3.10}$$

Example 3.3.1

Use Eq. (3.3.4), obtain $\frac{D\Theta}{Dt}$ for the motion and temperature field given in the previous example.

Solution. From Example 3.2.1, we have

$$\mathbf{v} = (kx_2)\mathbf{e}_1 \tag{i}$$

and

$$\Theta = x_1 + x_2 \tag{ii}$$

The gradient of Θ is simply

$$\nabla\Theta = \mathbf{e}_1 + \mathbf{e}_2 \tag{iii}$$

Therefore,

$$\frac{D\Theta}{Dt} = 0 + (kx_2)\mathbf{e}_1 \cdot (\mathbf{e}_1 + \mathbf{e}_2) = kx_2 \tag{iv}$$

which agrees with the previous example.

3.4 Acceleration of a Particle in a Continuum

The acceleration of a particle is the rate of change of velocity of the particle. It is therefore the material derivative of velocity. If the motion of a continuum is given by Eq. (3.1.1), i.e.,

$$\mathbf{x} = \mathbf{x}(\mathbf{X},t) \quad \text{with } \mathbf{x}(\mathbf{X},t_o) = \mathbf{X}$$

then the velocity $\mathbf{v}$, at time t, of a particle $\mathbf{X}$ is given by

$$\mathbf{v} = \left(\frac{\partial \mathbf{x}}{\partial t}\right)_{X_i-\text{fixed}} \equiv \frac{D\mathbf{x}}{Dt} \tag{3.4.1}$$

and the acceleration $\mathbf{a}$, at time t, of a particle $\mathbf{X}$ is given by

$$\mathbf{a} = \left(\frac{\partial \mathbf{v}}{\partial t}\right)_{X_i-\text{fixed}} \equiv \frac{D\mathbf{v}}{Dt} \tag{3.4.2}$$

Thus, if the material description of velocity, $\mathbf{v}(\mathbf{X},t)$ is known (or is obtained from Eq. (3.4.1), then the acceleration is very easily computed, simply taking the partial derivative with respect to time of the function $\mathbf{v}(X,t)$. On the other hand, if only the spatial description of velocity [i.e., $\mathbf{v} = \mathbf{v}(x,t)$] is known, the computation of acceleration is not as simple.

(A)Rectangular Cartesian Coordinates (x_1,x_2,x_3). With

$$\mathbf{v} = v_1(x_1,x_2,x_3,t)\mathbf{e}_1 + v_2(x_1,x_2,x_3,t)\mathbf{e}_2 + v_3(x_1,x_2,x_3,t)\mathbf{e}_3 \tag{3.4.3}$$

we have, since the base vectors $\mathbf{e}_1$,$\mathbf{e}_2$, and $\mathbf{e}_3$ do not change with time

$$\frac{D\mathbf{v}}{Dt} = \frac{Dv_1}{Dt}\mathbf{e}_1 + \frac{Dv_2}{Dt}\mathbf{e}_2 + \frac{Dv_3}{Dt}\mathbf{e}_3 \tag{3.4.4}$$

where

$$\frac{Dv_i}{Dt} = \frac{\partial v_i}{\partial t} + v_1\frac{\partial v_i}{\partial x_1} + v_2\frac{\partial v_i}{\partial x_2} + v_3\frac{\partial v_i}{\partial x_3} \tag{3.4.5a}$$

i.e,

$$a_i = \frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j} \tag{3.4.5b}$$

Or, in a form valid for all coordinate systems:

$$\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} + (\nabla\mathbf{v})\,\mathbf{v} \tag{3.4.5c}$$

In dyadic notation, the above equation is written as

$$\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\tilde{\nabla}\,\mathbf{v} \tag{3.4.5d}$$

where $\tilde{\nabla} = \mathbf{e}_m\dfrac{\partial}{\partial x_m}$.

(B)Cylindrical Coordinates (r,θ,z). With

$$\mathbf{v} = v_r(r,\theta,z)\mathbf{e}_r + v_\theta(r,\theta,z)\mathbf{e}_\theta + v_z(r,\theta,z)\mathbf{e}_z \tag{3.4.6}$$

and,[see Eq. (2D2.4)]

$$[\nabla\mathbf{v}] = \begin{bmatrix} \dfrac{\partial v_r}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_r}{\partial\theta} - v_\theta\right) & \dfrac{\partial v_r}{\partial z} \\ \dfrac{\partial v_\theta}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_\theta}{\partial\theta} + v_r\right) & \dfrac{\partial v_\theta}{\partial z} \\ \dfrac{\partial v_z}{\partial r} & \dfrac{1}{r}\dfrac{\partial v_z}{\partial v_\theta} & \dfrac{\partial v_z}{\partial z} \end{bmatrix} \tag{3.4.7}$$

we have,

$$a_r = \frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\left(\frac{\partial v_r}{\partial\theta} - v_\theta\right) + v_z\frac{\partial v_r}{\partial z} \tag{3.4.8a}$$

$$a_\theta = \frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\left(\frac{\partial v_\theta}{\partial\theta} + v_r\right) + v_z\frac{\partial v_\theta}{\partial z} \tag{3.4.8b}$$

$$a_z = \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \tag{3.4.8c}$$

(C) Spherical Coordinates (r,θ,ϕ). With

$$\mathbf{v} = v_r(r,\theta,\phi)\mathbf{e}_r + v_\theta(r,\theta,\phi)\mathbf{e}_\theta + v_\phi(r,\theta,\phi)\mathbf{e}_\phi \tag{3.4.9}$$

and, [see Eq. (2D3.17)]

$$[\nabla \mathbf{v}] = \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right) & \frac{1}{r\sin\theta}\left(\frac{\partial v_r}{\partial \phi} - v_\phi \sin\theta\right) \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) & \frac{1}{r\sin\theta}\left(\frac{\partial v_\theta}{\partial \phi} - v_\phi \cos\theta\right) \\ \frac{\partial v_\phi}{\partial r} & \frac{1}{r}\frac{\partial v_\phi}{\partial \theta} & \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot\theta}{r} \end{bmatrix} \tag{3.4.10}$$

we have,

$$a_r = \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right) + \frac{v_\phi}{r\sin\theta}\left(\frac{\partial v_r}{\partial \phi} - v_\phi \sin\theta\right) \tag{3.4.11a}$$

$$a_\theta = \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) + \frac{v_\phi}{r\sin\theta}\left(\frac{\partial v_\theta}{\partial \phi} - v_\phi \cos\theta\right) \tag{3.4.11b}$$

$$a_\phi = \frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\left(\frac{\partial v_\phi}{\partial \phi} + v_r \sin\theta + v_\theta \cos\theta\right) \tag{3.4.11c}$$

Example 3.4.1

(a) Find the velocity field associated with the motion of a rigid body rotating with angular velocity $\boldsymbol{\omega} = \omega \mathbf{e}_3$ in Cartesian and in cylindrical coordinates.

(b) Using the velocity field of part (a), evaluate the acceleration field.

Solution. (a) For a rigid body rotation

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x} \tag{i}$$

In Cartesian coordinates

$$\mathbf{v} = \omega \mathbf{e}_3 \times (x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3) \tag{ii}$$

i.e.,

$$v_1 = -\omega x_2, \quad v_2 = \omega x_1, \quad v_3 = 0 \tag{iii}$$

In cylindrical coordinates

$$\mathbf{v} = \omega \mathbf{e}_z \times (r\mathbf{e}_r) = \omega r \mathbf{e}_\theta \tag{iv}$$

i.e.,

$$v_r = 0, \quad v_\theta = \omega\, r, \quad v_z = 0 \tag{v}$$

(b)We can use either Eq. (iii) or Eq. (v) to find the acceleration.

Using Eq. (iii) and Eq. (3.4.5b), we obtain

$$a_1 = 0+(-\omega\, x_2)(0)+(\omega\, x_1)(-\omega)+(0)(0)=-\omega^2 x_1$$

$$a_2 = 0+(-\omega\, x_2)(\omega)+(\omega\, x_1)(0)+(0)(0)=-\omega^2 x_2$$

$$a_3 = 0$$

i.e.,

$$\mathbf{a}=-\omega^2(x_1\mathbf{e}_1+x_2\mathbf{e}_2) \tag{vi}$$

Or, using Eq. (v) and Eqs. (3.4.8), we obtain

$$a_r = 0+(0)(0)+\frac{v_\theta}{r}\left(0-v_\theta\right)+(0)(0) = -\frac{v_\theta^{\,2}}{r} = -\omega^2 r$$

$$a_\theta = 0+(0)(\omega)+\frac{v_\theta}{r}\left(0+0\right)+(0)(0) = 0$$

$$a_z = 0$$

i.e.,

$$\mathbf{a}=-\omega^2 r\mathbf{e}_r \tag{vii}$$

We note that $(x_1\mathbf{e}_1+x_2\mathbf{e}_2) = r\mathbf{e}_r$ so that (vi) and (vii) are the same. We also note that in this example, even though at every spatial point there is no change of velocity with time, for every material point, there is a rate of change of velocity due to a change of direction at every point as it moves along a circular path giving rise to a centripetal acceleration.

Example 3.4.2

Given the velocity field

$$v_1 = \frac{x_1}{1+t} \qquad v_2 = \frac{x_2}{1+t} \qquad v_3 = \frac{x_3}{1+t} \tag{i}$$

(a) Find the acceleration field and (b) find the pathline $x_i = x_i\,(\mathbf{X},t)$

Solution. (a) With

$$v_i = \frac{x_i}{1+t}, \quad i = 1,2,3 \tag{ii}$$

we have

$$\frac{\partial v_i}{\partial t} = -\frac{x_i}{(1+t)^2} \tag{iii}$$

Also, since

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

so that

$$\frac{\partial v_i}{\partial x_j} = \frac{\delta_{ij}}{1+t} \tag{iv}$$

therefore,

$$v_j \frac{\partial v_i}{\partial x_j} = v_j \frac{\delta_{ij}}{1+t} = \frac{v_i}{1+t} = \frac{x_i}{(1+t)^2} \tag{v}$$

Thus,

$$a_i = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{x_i}{(1+t)^2} + \frac{x_i}{(1+t)^2} = 0 \tag{vi}$$

i.e.,

$$\mathbf{a} = \mathbf{0} \tag{vii}$$

We note that in this example, even though at any spatial position (except the origin), the velocity is observed to be changing with time, but the actual velocity of a particular particle is a constant, with zero acceleration.

(b)Since

$$v_i = \left(\frac{\partial x_i}{\partial t}\right)_{\mathbf{X}-\text{fixed}} = \frac{x_i}{1+t} \qquad i = 1, 2, 3 \tag{viii}$$

therefore,

$$\int_{X_1}^{x_1} \frac{dx_1}{x_1} = \int_0^t \frac{dt}{1+t} \tag{ix}$$

so that

$$\ln x_1 - \ln X_1 = \ln(1+t) \tag{x}$$

Thus,

$$x_1 = X_1(1+t) \tag{xi}$$

Similarly,

$$x_2=X_2(1+t) \quad \text{and} \quad x_3 = X_3(1+t) \tag{xii}$$

3.5 Displacement Field

The displacement of a particle from position P to position Q is the vector PQ. Thus, the displacement vector of a particle, from the reference position to the position at time t, is given by

$$\mathbf{u}=\mathbf{x}(\mathbf{X},t)-\mathbf{X} \tag{3.5.1}$$

From the above equation, it is clear that whenever the pathline $\mathbf{x}(\mathbf{X},t)$ of a particle is known, its displacement field is also known. Thus, the motion of a continuum can be described either by the pathlines equation Eq. (3.1.1) or by its displacement vector field as given by Eq. (3.5.1).

Example 3.5.1

The position at time t, of a particle initially at (X_1,X_2,X_3) is given by

$$x_1 = X_1+(X_1+X_2)t, \;\; x_2 = X_2+(X_1+X_2)t, \;\; x_3 = X_3 \tag{i}$$

Find the displacement field.

Solution.

$$u_1 = x_1-X_1=(X_1+X_2)t, \;\; u_2=x_2-X_2 = (X_1+X_2)t, \;\; u_3=x_3-X_3=0 \tag{ii}$$

Example 5.2

The deformed configuration of a continuum is given by

$$x_1 = \frac{1}{2}X_1, \;\; x_2 = X_2, \;\; x_3 = X_3 \tag{i}$$

Find the displacement field.

Solution. The displacement components are:

$$u_1 = \frac{1}{2}X_1-X_1 = -\frac{1}{2}X_1, \;\; u_2 = X_2-X_2=0, \;\; u_3 = X_3-X_3 = 0 \tag{ii}$$

This displacement field represents a *uniaxial contraction* (the state of confined compression).

3.6 Kinematic Equation For Rigid Body Motion

(a) *Rigid body translation*: For this motion, the kinematic equation of motion is given by

$$\mathbf{x} = \mathbf{X} + \mathbf{c}(t) \tag{3.6.1}$$

where $\mathbf{c}(0) = \mathbf{0}$. We note that the displacement vector, $\mathbf{u} = \mathbf{x} - \mathbf{X} = \mathbf{c}(t)$, is independent of $\mathbf{X}$. That is, every material point is displaced in an identical manner, with the same magnitude and the same direction at time t.

(b) *Rigid body rotation about a fixed point*: For this motion, the kinematic equation of motion is given by

$$\mathbf{x} - \mathbf{b} = \mathbf{R}(t)(\mathbf{X} - \mathbf{b}) \tag{3.6.2}$$

where $\mathbf{R}(t)$ is a proper orthogonal tensor (i.e., a rotation tensor, see Sect. 2B.10) with $\mathbf{R}(0) = \mathbf{I}$, and $\mathbf{b}$ is a constant vector. We note that the material point $\mathbf{X} = \mathbf{b}$ is always at the spatial point $\mathbf{x} = \mathbf{b}$ so that the rotation is about the fixed point $\mathbf{x} = \mathbf{b}$.

If the rotation is about the origin, then $\mathbf{b} = \mathbf{0}$ and $\mathbf{x} = \mathbf{R}(t)\mathbf{X}$.

Example 3.6.1

Show that for motions given by Eq. (3.6.2) there is no change in distance between any pair of material points.

Solution. Consider two material points $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$, we have, from Eq. (3.6.2)

$$\mathbf{x}^{(1)} - \mathbf{x}^{(2)} = \mathbf{R}(t)(\mathbf{X}^{(1)} - \mathbf{X}^{(2)}) \tag{i}$$

That is, the material vector $\Delta\mathbf{X} \equiv \mathbf{X}^{(1)} - \mathbf{X}^{(2)}$ changes to $\Delta\mathbf{x} \equiv \mathbf{x}^{(1)} - \mathbf{x}^{(2)}$ where

$$\Delta\mathbf{x} = \mathbf{R}(t)\Delta\mathbf{X}. \tag{ii}$$

Now, the square of the length of $\Delta\mathbf{x}$ is given by

$$\Delta\mathbf{x} \cdot \Delta\mathbf{x} = \mathbf{R}(t)\Delta\mathbf{X} \cdot \mathbf{R}(t)\Delta\mathbf{X} \tag{iii}$$

The right side of the above equation is, according to the definition of transpose of a tensor $\Delta\mathbf{X} \cdot \mathbf{R}(t)\mathbf{R}^{\mathbf{T}}(t)\Delta\mathbf{X}$. and for an orthogonal tensor, $\mathbf{R}\mathbf{R}^{\mathbf{T}} = \mathbf{I}$, so that

$$\Delta\mathbf{x} \cdot \Delta\mathbf{x} = \Delta\mathbf{X} \cdot \Delta\mathbf{X} \tag{iv}$$

In other words, the length of $\Delta\mathbf{X}$ does not change.

(c)*General rigid body motion:* The kinematic equation describing a general rigid body motion is given by

$$\mathbf{x} = \mathbf{R}(t)(\mathbf{X} - \mathbf{b}) + \mathbf{c}(t) \tag{3.6.3}$$

where $\mathbf{R}(t)$ is a rotation tensor with $\mathbf{R}(0) = \mathbf{I}$ and $\mathbf{c}(t)$ is a vector with $\mathbf{c}(0) = \mathbf{b}$.

Equation (3.6.3) states that the motion is described by a translation $\mathbf{c}(t)$, of an arbitrary chosen material base point $\mathbf{X}=\mathbf{b}$ plus a rotation $\mathbf{R}(t)$.

Example 3.6.2

From Eq. (3.6.3) derive the relation between the velocity of a general material point in the rigid body with the angular velocity of the body and the velocity of the arbitrary chosen material point.

Solution. Taking the material derivative of Eq. (3.6.3), we obtain

$$\mathbf{v} = \dot{\mathbf{R}}(\mathbf{X}-\mathbf{b})+\dot{\mathbf{c}}(t)$$

Now, from Eq. (3.6.3), we have

$$(\mathbf{X}-\mathbf{b}) = \mathbf{R}^T(\mathbf{x}-\mathbf{c}) \tag{ii}$$

Thus

$$\mathbf{v} = \dot{\mathbf{R}}\,\mathbf{R}^T(\mathbf{x}-\mathbf{c}) + \dot{\mathbf{c}}(t) \tag{iii}$$

Since $\mathbf{R}\mathbf{R}^T = \mathbf{I}$, $\dot{\mathbf{R}}\mathbf{R}^T+\mathbf{R}\dot{\mathbf{R}}^T = \mathbf{0}$, so that $\dot{\mathbf{R}}\mathbf{R}^T$ is antisymmetric which is equivalent to a dual (or axial) vector $\boldsymbol{\omega}$ [see Sect. 2B16], thus,

$$\mathbf{v} = \boldsymbol{\omega}\times(\mathbf{x}-\mathbf{c}) + \dot{\mathbf{c}}(t) \tag{iv}$$

If we measure the position vector $\mathbf{r}$ for the general material point from the position at time t of the chosen material base point, i.e., $\mathbf{r} = (\mathbf{x}-\mathbf{c})$, then

$$\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r} + \dot{\mathbf{c}}(t) \tag{v}$$

3.7 Infinitesimal Deformations

There are many important engineering problems which involves structural members or machine parts, for which the displacement of every material point is very small (mathematically infinitesimal) under design loadings. In this section, we derive the tensor which characterizes the deformation of such bodies.

Consider a body, having a particular configuration at some reference time t_o, changes to another configuration at time t. Referring to Fig. 3.3, a typical material point P undergoes a displacement $\mathbf{u}$, so that it arrives at the position

$$\mathbf{x} = \mathbf{X}+\mathbf{u}(\mathbf{X},t) \tag{i}$$

A neighboring point Q at $\mathbf{X}+d\mathbf{X}$ arrives at $\mathbf{x}+d\mathbf{x}$ which is related to $\mathbf{X}+d\mathbf{X}$ by:

$$\mathbf{x}+d\mathbf{x} = \mathbf{X}+d\mathbf{X}+\mathbf{u}(\mathbf{X}+d\mathbf{X},t) \tag{ii}$$

Subtracting Eq. (i) from Eq. (ii), we obtain

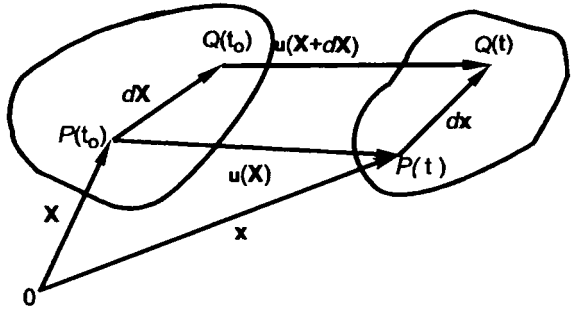

Fig. 3.3

$$d\mathbf{x} = d\mathbf{X} + \mathbf{u}(\mathbf{X}+d\mathbf{X},t) - \mathbf{u}(\mathbf{X},t) \tag{iii}$$

Using the definition of gradient of a vector function [see Eq. (2C3.1)], Eq. (iii) becomes

$$d\mathbf{x} = d\mathbf{X} + (\nabla\mathbf{u})d\mathbf{X} \tag{3.7.1a}$$

where $\nabla\mathbf{u}$ is a second-order tensor known as the **displacement gradient**. The matrix of $\nabla\mathbf{u}$ with respect to rectangular Cartesian coordinates (with $\mathbf{X} = X_i\mathbf{e_i}$ and $\mathbf{u} = u_i\mathbf{e}_i$) is

$$[\nabla\mathbf{u}] = \begin{bmatrix} \dfrac{\partial u_1}{\partial X_1} & \dfrac{\partial u_1}{\partial X_2} & \dfrac{\partial u_1}{\partial X_3} \\ \dfrac{\partial u_2}{\partial X_1} & \dfrac{\partial u_2}{\partial X_2} & \dfrac{\partial u_2}{\partial X_3} \\ \dfrac{\partial u_3}{\partial X_1} & \dfrac{\partial u_3}{\partial X_2} & \dfrac{\partial u_3}{\partial X_3} \end{bmatrix} \tag{3.7.1b}$$

Example 3.7.1

Given the following displacement components

$$u_1 = kX_2^2, \quad u_2 = u_3 = 0. \tag{i}$$

(a) Sketch the deformed shape of the unit square $OABC$ in Fig. 3.4

(b) Find the deformed vector (i.e., $d\mathbf{x}^{(1)}$ and $d\mathbf{x}^{(2)}$) of the material elements $d\mathbf{X}^{(1)} = dX_1\mathbf{e_1}$ and $d\mathbf{X}^{(2)} = dX_2\mathbf{e_2}$ which were at the point C.

(c) determine the ratio of the deformed to the undeformed lengths of the differential elements (known as **stretch**) of part (b) and the change in angle between these elements.

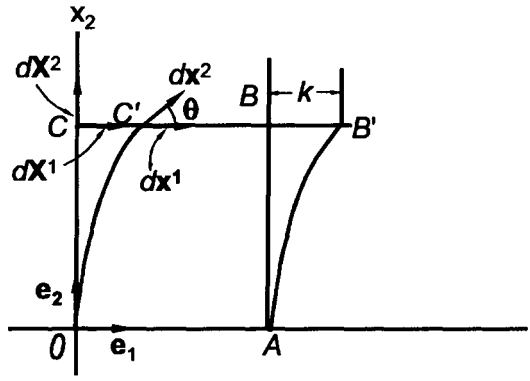

Fig. 3.4

Solution. (a) For the material line OA, $X_2 = 0$, therefore, $u_1 = u_2 = u_3 = 0$. That is, the line is not displaced. For the material $CB, X_2 = 1, u_1 = k$, the line is displaced by k units to the right. For the material line OC and $AB, u_1 = kX_2^2$, the lines become parabolic in shape. Thus, the deformed shape is given by $OAB'C'$ in Fig. 3.4.

(b) For the material point C, the matrix of the displacement gradient is

$$[\nabla \mathbf{u}] = \begin{bmatrix} 0 & 2kX_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{X_2=1} = \begin{bmatrix} 0 & 2k & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{ii}$$

Therefore, from Eq. (3.7.1a)

$$d\mathbf{x}^{(1)} = d\mathbf{X}^{(1)} + (\nabla \mathbf{u})d\mathbf{X}^{(1)} = dX_1\mathbf{e}_1 + 0 = dX_1\mathbf{e}_1 \tag{iii}$$

$$d\mathbf{x}^{(2)} = d\mathbf{X}^{(2)} + (\nabla \mathbf{u})d\mathbf{X}^{(2)} = dX_2\mathbf{e}_2 + 2kdX_2\mathbf{e}_1 = dX_2(\mathbf{e}_2 + 2k\mathbf{e}_1) \tag{iv}$$

(c) From Eqs. (iii) and (iv), we have $|d\mathbf{x}^{(1)}| = dX_1$, $|d\mathbf{x}^{(2)}| = dX_2(1+4k^2)^{1/2}$, thus,

$$\frac{|d\mathbf{x}^{(1)}|}{|d\mathbf{X}^{(1)}|} = 1 \quad \text{and} \quad \frac{|d\mathbf{x}^{(2)}|}{|d\mathbf{X}^{(2)}|} = (1+4k^2)^{(1/2)} \tag{v}$$

and

$$\cos\theta = \frac{d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)}}{|d\mathbf{x}^{(1)}|\,|d\mathbf{x}^{(2)}|} = \frac{2k}{(1+4k^2)^{1/2}} \tag{vi}$$

If k is very small, we have the case of small deformations and by the binomial theorem, we have , keeping only the first power of k,

$$\frac{|d\mathbf{x}^{(1)}|}{|d\mathbf{X}^{(1)}|}=1 \quad \text{and} \quad \frac{|d\mathbf{x}^{(2)}|}{|d\mathbf{X}^{(2)}|}=(1+2k^2)\approx 1 \tag{vii}$$

and

$$\cos\theta=2k \tag{viii}$$

if γ denote the decrease in angle, then

$$\cos\theta=\cos(\frac{\pi}{2}-\gamma)=\sin\gamma=2k$$

That is, for small k,

$$\gamma=2k \tag{ix}$$

We can write Eq. (3.7.1a), i.e., $d\mathbf{x} = d\mathbf{X}+(\nabla\mathbf{u})d\mathbf{X}$ as

$$d\mathbf{x} = \mathbf{F}d\mathbf{X} \tag{3.7.2}$$

where

$$\mathbf{F} = \mathbf{I}+\nabla\mathbf{u} \tag{3.7.3}$$

To find the relationship between ds, the length of $d\mathbf{x}$ and dS, the length of $d\mathbf{X}$, we take the dot product of Eq. (3.7.2) with itself:

$$d\mathbf{x}\cdot d\mathbf{x} = \mathbf{F}d\mathbf{X}\cdot\mathbf{F}d\mathbf{X} = d\mathbf{X}\cdot\mathbf{F}^{\mathbf{T}}\mathbf{F}d\mathbf{X} \tag{3.7.4a}$$

i.e.,

$$(ds)^2 = d\mathbf{X}\cdot\mathbf{F}^{\mathbf{T}}\mathbf{F}d\mathbf{X} \tag{3.7.4b}$$

If $\mathbf{F}$ is an orthogonal tensor, then $\mathbf{F}^{\mathbf{T}}\mathbf{F} = \mathbf{I}$, and

$$(ds)^2 = (dS)^2$$

Thus, an orthogonal $\mathbf{F}$ corresponds to a rigid body motion (translation and/or rotation).

Now, from Eq. (3.7.3),

$$\mathbf{F}^{\mathbf{T}}\mathbf{F} = (\mathbf{I}+\nabla\mathbf{u})^T(\mathbf{I}+\nabla\mathbf{u}) = \mathbf{I}+\nabla\mathbf{u}+(\nabla\mathbf{u})^T+(\nabla\mathbf{u})^T\nabla\mathbf{u} \tag{3.7.5}$$

In this section, we shall consider only cases where the components of the displacement vector as well as their partial derivatives are all very small (mathematically, infinitesimal) so that the absolute value of every component of $(\nabla\mathbf{u})^T\nabla\mathbf{u}$ is a small quantity of higher order than those of the components of $\nabla\mathbf{u}$. For such a case, the above equation becomes:

$$\mathbf{F}^{\mathbf{T}}\mathbf{F} \approx \mathbf{I}+\nabla\mathbf{u}+(\nabla\mathbf{u})^T \equiv \mathbf{I} + 2\mathbf{E} \tag{3.7.6}$$

where

$$\mathbf{E} = \frac{1}{2}[(\nabla\mathbf{u})^T+\nabla\mathbf{u}\ ] \ \equiv \ \text{symmetric part of } \nabla\mathbf{u} \tag{3.7.7}$$

From Eq. (3.7.4b) and (3.7.6), it is clear that the tensor $\mathbf{E}$ characterizes the changes of lengths in the continuum undergoing small deformations. This tensor $\mathbf{E}$ is known as the **infinitesimal strain tensor.**

Consider two material elements $d\mathbf{X}^{(1)}$ and $d\mathbf{X}^{(2)}$. Due to motion, they become $d\mathbf{x}^{(1)}$ and $d\mathbf{x}^{(2)}$ at time t with $d\mathbf{x}^{(1)} = \mathbf{F}d\mathbf{X}^{(1)}$ and $d\mathbf{x}^{(2)} = \mathbf{F}d\mathbf{X}^{(2)}$. Taking the dot product of $d\mathbf{x}^{(1)}$ and $d\mathbf{x}^{(2)}$, we obtain

$$d\mathbf{x}^{(1)}\cdot d\mathbf{x}^{(2)} = d\mathbf{X}^{(1)}\cdot\mathbf{F}^T\mathbf{F}d\mathbf{X}^{(2)} \tag{3.7.8}$$

Thus, using Eq. (3.7.6), we have the important equation

$$d\mathbf{x}^{(1)}\cdot d\mathbf{x}^{(2)} = d\mathbf{X}^{(1)}\cdot d\mathbf{X}^{(2)}+2d\mathbf{X}^{(1)}\cdot\mathbf{E}d\mathbf{X}^{(2)} \tag{3.7.9}$$

This equation will be used in the next section to establish the meanings of the components of the infinitesimal strain tensor $\mathbf{E}$.

The components of the infinitesimal strain tensor $\mathbf{E}$ can be obtained easily from the components of the gradient of $\mathbf{u}$ given in Chapter 2. We have

(a) *In rectangular coordinates*:

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j}+\frac{\partial u_j}{\partial X_i}\right) \tag{3.7.10a}$$

or ,

$$[\mathbf{E}] = \begin{bmatrix} \dfrac{\partial u_1}{\partial X_1} & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial X_2}+\dfrac{\partial u_2}{\partial X_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial X_3}+\dfrac{\partial u_3}{\partial X_1}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial X_2}+\dfrac{\partial u_2}{\partial X_1}\right) & \dfrac{\partial u_2}{\partial X_2} & \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial X_3}+\dfrac{\partial u_3}{\partial X_2}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial X_3}+\dfrac{\partial u_3}{\partial X_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial X_3}+\dfrac{\partial u_3}{\partial X_2}\right) & \dfrac{\partial u_3}{\partial X_3} \end{bmatrix} \tag{3.7.10b}$$

(B) *In cylindrical coordinates*:

$$[\mathbf{E}] = \begin{bmatrix} \dfrac{\partial u_r}{\partial r} & \dfrac{1}{2}\left[\dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta}+\dfrac{\partial u_\theta}{\partial r}-\dfrac{u_\theta}{r}\right] & \dfrac{1}{2}\left[\dfrac{\partial u_r}{\partial z}+\dfrac{\partial u_z}{\partial r}\right] \\ \dfrac{1}{2}\left[\dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta}+\dfrac{\partial u_\theta}{\partial r}-\dfrac{u_\theta}{r}\right] & \dfrac{1}{r}\left(\dfrac{\partial u_\theta}{\partial \theta}+u_r\right) & \dfrac{1}{2}\left[\dfrac{\partial u_\theta}{\partial z}+\dfrac{1}{r}\dfrac{\partial u_z}{\partial \theta}\right] \\ \dfrac{1}{2}\left[\dfrac{\partial u_r}{\partial z}+\dfrac{\partial u_z}{\partial r}\right] & \dfrac{1}{2}\left[\dfrac{\partial u_\theta}{\partial z}+\dfrac{1}{r}\dfrac{\partial u_z}{\partial u_\theta}\right] & \dfrac{\partial u_z}{\partial z} \end{bmatrix} \tag{3.7.11}$$

(c) *In spherical coordinates*:

$$[\mathbf{E}] = \begin{bmatrix} \dfrac{\partial u_r}{\partial r} & \dfrac{1}{2}\left[\dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta}+\dfrac{\partial u_\theta}{\partial r}-\dfrac{u_\theta}{r}\right] & \dfrac{1}{2}\left[\dfrac{1}{r\sin\theta}\dfrac{\partial u_r}{\partial \phi}+\dfrac{\partial u_\phi}{\partial r}-\dfrac{u_\phi}{r}\right] \\ \dfrac{1}{2}\left[\dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta}+\dfrac{\partial u_\theta}{\partial r}-\dfrac{u_\theta}{r}\right] & \dfrac{1}{r}\left(\dfrac{\partial u_\theta}{\partial \theta}+u_r\right) & \dfrac{1}{2}\left[\dfrac{1}{r\sin\theta}\dfrac{\partial u_\theta}{\partial \phi}-\dfrac{u_\phi\cot\theta}{r}+\dfrac{1}{r}\dfrac{\partial u_\phi}{\partial \theta}\right] \\ \dfrac{1}{2}\left[\dfrac{1}{r\sin\theta}\dfrac{\partial u_r}{\partial \phi}+\dfrac{\partial u_\phi}{\partial r}-\dfrac{u_\phi}{r}\right] & \dfrac{1}{2}\left[\dfrac{1}{r\sin\theta}\dfrac{\partial u_\theta}{\partial \phi}-\dfrac{u_\phi\cot\theta}{r}+\dfrac{1}{r}\dfrac{\partial u_\phi}{\partial \theta}\right] & \dfrac{1}{r\sin\theta}\dfrac{\partial u_\phi}{\partial \phi}+\dfrac{u_r}{r}+\dfrac{u_\theta\cot\theta}{r} \end{bmatrix}$$

(3.7.12)

3.8 Geometrical Meaning of the Components of the Infinitesimal Strain Tensor

(a)*Diagonal elements of* $\mathbf{E}$

Consider the single material element $d\mathbf{X}^{(1)} = d\mathbf{X}^{(2)} = d\mathbf{X} = (dS)\mathbf{n}$, where $\mathbf{n}$ is a unit vector and dS is the length of $d\mathbf{X}$. Let ds denote the deformed length of $d\mathbf{x}^{(1)}$, i.e., $ds = |d\mathbf{x}^{(1)}|$. Then, Eq. (3.7.9) gives

$$(ds)^2-(dS)^2 = 2(dS)^2\mathbf{n}\cdot\mathbf{E}\mathbf{n}$$

Now, for small deformation $(ds)^2-(dS)^2 = (ds+dS)(ds-dS) \approx 2dS(ds-dS)$. Thus

$$\frac{ds-dS}{dS} = \mathbf{n}\cdot\mathbf{E}\mathbf{n} = E_{nn} \qquad \text{(no sum on } n\text{)} \tag{3.8.1}$$

This equation states that the **unit elongation** (i.e., the increase in length per unit original length) for the element which was in the direction $\mathbf{n}$, is given by $\mathbf{n}\cdot\mathbf{E}\mathbf{n}$. In particular, if the element was in the $\mathbf{e}_1$ direction in the reference state , then $\mathbf{n} = \mathbf{e}_1$, and $E_{11} = \mathbf{e}_1\cdot\mathbf{E}\mathbf{e}_1$ so that

E_{11} is the unit elongation for an element originally in the x_1−direction. Similarly,

E_{22} is the unit elongation for an element originally in the x_2−direction and

E_{33} is the unit elongation for an element originally in the x_3−direction.

These components (the diagonal elements of the tensor **E**) are also known as the **normal strains.**

(b)*The off diagonal elements*:

Let $d\mathbf{X}^{(1)} = dS_1\mathbf{m}$ and $d\mathbf{X}^{(2)} = dS_2\mathbf{n}$, where **m** and **n** are unit vectors perpendicular to each other. Then Eq. (3.7.9) gives

$$(ds_1)(ds_2)\cos\theta = 2(dS_1)(dS_2)\mathbf{m}\cdot\mathbf{E}\mathbf{n}$$

where θ is the angle between $d\mathbf{x}^{(1)}$ and $d\mathbf{x}^{(2)}$. If we let $\theta = (\pi/2)-\gamma$, then γ will measure the small decrease in angle between $d\mathbf{X}^{(1)}$ and $d\mathbf{X}^{(2)}$, known as the **shear strain.** Since

$$\cos(\frac{\pi}{2}-\gamma) = \sin\gamma$$

and for small strain

$$\sin\gamma \approx \gamma,\ \frac{ds_1}{dS_1}\approx 1,\ \frac{ds_2}{dS_2}\approx 1$$

therefore,

$$\gamma = 2\mathbf{m}\cdot\mathbf{E}\mathbf{n} \tag{3.8.2}$$

If the elements were in the direction of $\mathbf{e}_1$ and $\mathbf{e}_2$, then $\mathbf{m}\cdot\mathbf{E}\mathbf{n} = E_{12}$ so that according to Eq. (3.8.2):

$2E_{12}$ gives the decrease in angle between two elements initially in the x_1 and x_2 directions. Similarly,

$2E_{13}$ gives the decrease in angle between two elements initially in the x_1 and x_3 directions, and

$2E_{23}$ gives the decrease in angle between two elements initially in the x_2 and x_3 directions.

Example 3.8.1

Given the displacement components

$$u_1 = kX_2^2 \text{ and } u_2 = u_3 = 0,\ k = 10^{-4} \tag{i}$$

(a) Obtain the infinitesimal strain tensor **E**.

(b)Using the strain tensor **E**, find the unit elongation for the material elements $d\mathbf{X}^{(1)} = dX_1\mathbf{e}_1$ and $d\mathbf{X}^{(2)} = dX_2\mathbf{e}_2$, which were at the point $C(0,1,0)$ of Fig. 3.4 (which is reproduced here for convenience). Also, find the decrease in angle between these two elements.

(c) Compare the results with those of Example 3.7.1.

Solution. (a) We have

$$[\nabla \mathbf{u}] = \begin{bmatrix} 0 & 2kX_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{ii}$$

Therefore

$$[\mathbf{E}] = [\nabla u]^s = \begin{bmatrix} 0 & kX_2 & 0 \\ kX_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{iii}$$

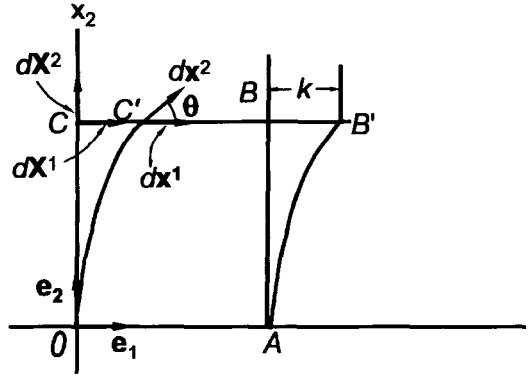

Fig. 3.4 (repeated)

(b) At the point C, $X_2=1$, therefore

$$[\mathbf{E}] = \begin{bmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{iv}$$

For the element $d\mathbf{X}^{(1)} = dX_1\mathbf{e}_1$, the unit elongation is E_{11}, which is zero. For the element $d\mathbf{X}^{(2)} = dX_2\mathbf{e}_2$, the unit elongation is E_{22} which is also zero. The decrease in angle between these elements is given by $2E_{12}$, which is equal to $2k$, i.e., 2×10^{-4} radians.

(c) In Example 3.7.1, we found that

$$\frac{|d\mathbf{x}^{(1)}|-|d\mathbf{X}^{(1)}|}{|d\mathbf{X}^{(1)}|} = 0, \quad \frac{|d\mathbf{x}^{(2)}|-|d\mathbf{X}^{(2)}|}{|d\mathbf{X}^{(2)}|} = (1+4k^2)^{1/2}-1 = 1+2k^2-1 = 2k^2(\approx 0) \tag{v}$$

and $\sin\gamma = 2k=2\times10^{-4}$ so that $\gamma \approx 2\times10^{-4}$.

We see that the results of this example is accurate up to the order of k.

Example 3.8.2

Given the displacement field

$$u_1 = k(2X_1+X_2^2),\ u_2 = k(X_1^2-X_2^2),\ u_3 = 0\ ; \qquad k=10^{-4} \tag{i}$$

(a)Find the unit elongation and the change of angle for the two material elements $d\mathbf{X}^{(1)} = dX_1\mathbf{e}_1$ and $d\mathbf{X}^{(2)} = dX_2\mathbf{e}_2$ that emanate from a particle designated by $\mathbf{X} = \mathbf{e}_1-\mathbf{e}_2$.

(b)Find the deformed position of these two elements $d\mathbf{X}^{(1)}$ and $d\mathbf{X}^{(2)}$.

Solution. (a) We evaluate $[\nabla\mathbf{u}]$ at $(X_1, X_2, X_3) = (1, -1, 0)$ as

$$[\nabla\mathbf{u}] = k\begin{bmatrix} 2 & -2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{ii}$$

and therefore the strain matrix is

$$[\mathbf{E}] = k\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{iii}$$

Since $E_{11} = E_{22} = 2k$, both elements have a unit elongation of 2×10^{-4}. Further, since $E_{12} = 0$, these line elements remain perpendicular to each other.

(b) From Eq. (3.7.1a)

$$[d\mathbf{x}^{(1)}]=[d\mathbf{X}^{(1)}]+[\nabla\mathbf{u}][d\mathbf{X}^{(1)}] = \begin{bmatrix} dX_1 \\ 0 \\ 0 \end{bmatrix} + k\begin{bmatrix} 2 & -2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} dX_1 \\ 0 \\ 0 \end{bmatrix} = dX_1\begin{bmatrix} 1+2k \\ 2k \\ 0 \end{bmatrix} \tag{iv}$$

and similarly

$$[d\mathbf{x}^{(2)}]=[d\mathbf{X}^{(2)}]+[\nabla\mathbf{u}][d\mathbf{X}^{(2)}] = dX_2\begin{bmatrix} -2k \\ 1+2k \\ 0 \end{bmatrix} \tag{v}$$

The deformed position of these elements is sketched in Fig. 3.5. Note from the diagram that

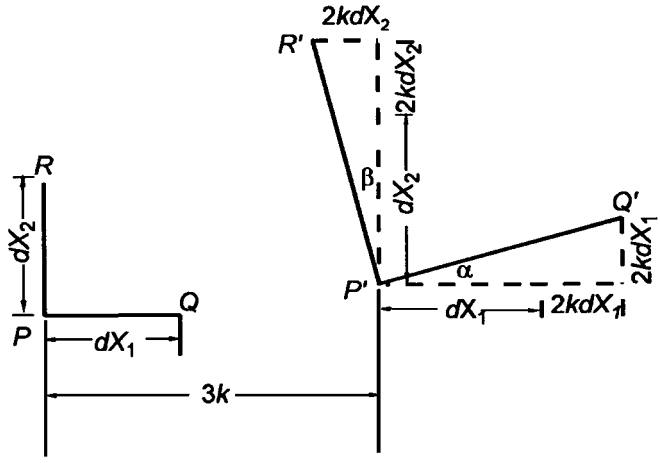

Fig. 3.5

$$\alpha \approx \tan\alpha = \frac{2kdX_1}{dX_1\,(1+k)} = \frac{2k}{1+k} \approx 2k \tag{vi}$$

and

$$\beta \approx \tan\beta = \frac{2kdX_2}{dX_2} = 2k \tag{vii}$$

Thus, as previously obtained, there is no change of angle between $d\mathbf{X}^{(1)}$ and $d\mathbf{X}^{(2)}$.

Example 3.8.3

A unit cube, with edges parallel to the coordinates axes, is given a displacement field

$$u_1 = kX_1,\ \ u_2 = u_3 = 0,\ \ k = 10^{-4} \tag{i}$$

Find the increase in length of the diagonal AB (see Fig. 3.6) (a) by using the infinitesimal strain tensor **E** and (b) by geometry

Solution. (a) The strain tensor is easily obtained to be

$$[\mathbf{E}] = \begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{ii}$$

Since the diagonal AB was originally in the direction $\mathbf{n} = \frac{\sqrt{2}}{2}(\mathbf{e}_1+\mathbf{e}_2)$, its unit elongation is given by

$$E_{nn} = \mathbf{n}\cdot\mathbf{E}\mathbf{n} = [\sqrt{2}/2,\sqrt{2}/2,0]\begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix} = \frac{k}{2} \qquad \text{(no sum on } n\text{)} \tag{iii}$$

Since $AB = \sqrt{2}$,

$$\Delta AB = \left(\frac{k}{2}\right)\sqrt{2} \tag{iv}$$

(b) Geometrically,

$$AB' - AB = [1+(1+k)^2]^{1/2} - \sqrt{2}$$

or,

$$\Delta AB = \sqrt{2}[(1+k+k^2/2)^{1/2} - 1] \tag{v}$$

To take advantage of the smallness of k, we expand the first term in the right hand side of Eq. (v) as

$$\left(1+k+\frac{k^2}{2}\right) = 1+\frac{1}{2}\left(k+\frac{k^2}{2}\right)+....\approx 1+\frac{k}{2} \tag{vi}$$

Therefore, in agreement with Part (a) , Eq. (iv),

$$\Delta AB = \sqrt{2}\left(\frac{k}{2}\right) \tag{vii}$$

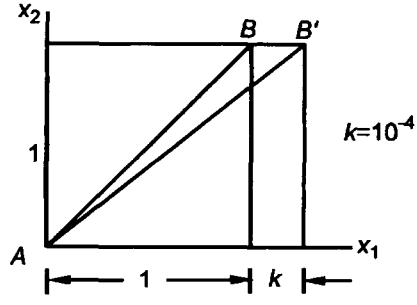

Fig.3.6

3.9 Principal Strain

Since the strain tensor **E** is symmetric, therefore, (see Section 2B.18) there exists at least three mutually perpendicular directions $\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3$ with respect to which the matrix of **E** is diagonal. That is

$$[\mathbf{E}]_{\mathbf{n}_i} = \begin{bmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{bmatrix} \tag{3.9.1}$$

Geometrically, this means that infinitesimal line elements in the directions of $\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3$ remain mutually perpendicular after deformation. These directions are known as the **principal directions** of strain. The unit elongation along the principal direction (i.e., E_1,E_2,E_3) are the eigenvalues of **E** , or **principal strains** , they include the maximum and the minimum normal strains among all directions emanating from the particle. For a given **E**, the principal strains are to be found from the characteristic equation of **E**, i.e.,

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0 \tag{3.9.2}$$

where

$$I_1 = E_{11}+E_{22}+E_{33} \tag{3.9.3a}$$

$$I_2 = \begin{vmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{vmatrix} + \begin{vmatrix} E_{11} & E_{13} \\ E_{31} & E_{33} \end{vmatrix} + \begin{vmatrix} E_{22} & E_{23} \\ E_{32} & E_{33} \end{vmatrix} \tag{3.9.3b}$$

$$I_3 = |E_{ij}| \tag{3.9.3c}$$

The coefficients $I_1,I_2,$ and I_3 are called the **principal scalar invariants** of the strain tensor.

3.10 Dilatation

The first scalar invariant of the infinitesimal strain tensor has a simple geometric meaning. For a specific deformation, consider the three material lines that emanate from a single point P and are in the principal directions. These lines define a rectangular parallelepiped whose sides have been elongated from the initial dimension

$$dS_1, dS_2 \text{ and } dS_3$$

to

$$dS_1(1+E_1),\ dS_2(1+E_2) \text{ and } dS_3(1+E_3)$$

where E_1,E_2 and E_3 are the principal strains. Hence the change $\Delta(dV)$ in this material volume dV is

$$\begin{aligned}\Delta(dV) &= (dS_1)(dS_2)(dS_3)(1+E_1)(1+E_2)(1+E_3)-(dS_1)(dS_2)(dS_3) \\ &= (dV)(E_1+E_2+E_3) + \text{ higher order terms in the } E_i\text{'s.}\end{aligned}$$

Thus, for small deformation

$$e \equiv \frac{\Delta(dV)}{dV} = E_1+E_2+E_3 = E_{11}+E_{22}+E_{33} \tag{3.10.1}$$

This unit volume change is known as **dilatation**. Note also that

$$e = E_{ii} = \frac{\partial u_i}{\partial X_i} = \text{div}\mathbf{u} \tag{3.10.2a}$$

In cylindrical coordinates,

$$e = \frac{\partial u_r}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}+\frac{\partial u_z}{\partial z} \tag{3.10.2b}$$

In spherical coordinates,

$$e = \frac{\partial u_r}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{2u_r}{r}+\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}+\frac{u_\theta\cot\theta}{r} \tag{3.10.2c}$$

3.11 The Infinitesimal Rotation Tensor

Equation (3.7.1), i.e., $d\mathbf{x} = d\mathbf{X} + (\nabla\mathbf{u})d\mathbf{X}$, can be written

$$d\mathbf{x} = d\mathbf{X}+(\mathbf{E}+\Omega)d\mathbf{X} \tag{3.11.1}$$

where $\mathbf{\Omega}$, the antisymmetric part of $\nabla\mathbf{u}$, is known as the **infinitesimal rotation tensor**. We see that the change of direction for $d\mathbf{X}$ in general comes from two sources, the infinitesimal deformation tensor $\mathbf{E}$ and the infinitesimal rotation tensor $\mathbf{\Omega}$. However, for any $d\mathbf{X}$ which is in the direction of an eigenvector of $\mathbf{E}$, there is no change of direction due to $\mathbf{E}$, only that due to $\mathbf{\Omega}$. Therefore, the tensor $\mathbf{\Omega}$ represents the infinitesimal rotation of the triad of the eigenvectors of $\mathbf{E}$. It can be described by a vector $\mathbf{t}^A$ in the sense that

$$\mathbf{t}^A \times d\mathbf{X} = \mathbf{\Omega}\, d\,\mathbf{X} \tag{3.11.2}$$

where (see Section 2B.16)

$$\mathbf{t}^A = \Omega_{32}\mathbf{e}_1+\Omega_{13}\mathbf{e}_2+\Omega_{21}\mathbf{e}_3 \tag{3.11.3}$$

Thus, $\Omega_{32},\Omega_{13},\Omega_{21}$ are the infinitesimal angles of rotation about $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$-axes, of the triad of material elements which are in the principal direction of $\mathbf{E}$.

3.12 Time Rate of Change of a Material Element

Let us consider a material element $d\mathbf{x}$ emanating from a material point $\mathbf{X}$ located at $\mathbf{x}$ at time t. We wish to compute $(D/Dt)d\mathbf{x}$, the rate of change of length and direction of the material element $d\mathbf{x}$. From $\mathbf{x} = \mathbf{x}(\mathbf{X},t)$, we have

$$d\mathbf{x} = \mathbf{x}(\mathbf{X}+d\mathbf{X},t)-\mathbf{x}(\mathbf{X},t) \tag{i}$$

Taking the material derivative of Eq. (i), we obtain

$$\left(\frac{D}{Dt}\right)d\mathbf{x} = \left(\frac{D}{Dt}\right)\mathbf{x}(\mathbf{X}+d\mathbf{X},t)-\left(\frac{D}{Dt}\right)\mathbf{x}(\mathbf{X},t) \tag{ii}$$

Now,

$$(D/Dt)\mathbf{x}(\mathbf{X},t) = \hat{\mathbf{v}}(\mathbf{X},t) = \tilde{\mathbf{v}}(\mathbf{x},t) \tag{3.12.1}$$

where $\hat{\mathbf{v}}(\mathbf{X},t)$ and $\tilde{\mathbf{v}}(\mathbf{x},t)$ are the material and the spatial description of the velocity of the particle $\mathbf{X}$, therefore Eq. (ii) becomes

$$\left(\frac{D}{Dt}\right)d\mathbf{x} = \hat{\mathbf{v}}(\mathbf{X}+d\mathbf{X},t)-\hat{\mathbf{v}}(\mathbf{X},t) = \tilde{\mathbf{v}}(\mathbf{x}+d\mathbf{x},t)-\tilde{\mathbf{v}}(\mathbf{x},t) \tag{iii}$$

Thus, from the definition (see Section 2C3.1) of the gradient of a vector function, we have

$$\left(\frac{D}{Dt}\right)d\mathbf{x} = (\nabla_{\mathbf{X}}\hat{\mathbf{v}})d\mathbf{X} \tag{3.12.2}$$

and

$$\left(\frac{D}{Dt}\right)d\mathbf{x} = (\nabla_{\mathbf{x}}\tilde{\mathbf{v}})d\mathbf{x} \tag{3.12.3}$$

In Eq. (3.12.2) the subscript $\mathbf{X}$ in $(\nabla_{\mathbf{X}}\hat{\mathbf{v}})$ emphasizes that $(\nabla_{\mathbf{X}}\hat{\mathbf{v}})$ is the gradient of the material description of the velocity field $\mathbf{v}$ and in Eq. (3.12.3) the subscript $\mathbf{x}$ in $(\nabla_{\mathbf{x}}\tilde{\mathbf{v}})$ emphasizes that $(\nabla_{\mathbf{x}}\tilde{\mathbf{v}})$ is the gradient of the spatial description of $\mathbf{v}$.

In the following, the spatial description of the velocity function will be used exclusively so that *the notation* $(\nabla\mathbf{v})$ *will be understood to mean* $(\nabla_{\mathbf{x}}\mathbf{v})$. Thus we write Eq. (3.12.3) simply as

$$\left(\frac{D}{Dt}\right)d\mathbf{x} = (\nabla\mathbf{v})d\mathbf{x} \tag{3.12.4}$$

With respect to rectangular Cartesian coordinates, the components of $(\nabla\mathbf{v})$ are given by

$$[\nabla\mathbf{v}] = \begin{bmatrix} \dfrac{\partial v_1}{\partial x_1} & \dfrac{\partial v_1}{\partial x_2} & \dfrac{\partial v_1}{\partial x_3} \\ \dfrac{\partial v_2}{\partial x_1} & \dfrac{\partial v_2}{\partial x_2} & \dfrac{\partial v_2}{\partial x_3} \\ \dfrac{\partial v_3}{\partial x_1} & \dfrac{\partial v_3}{\partial x_2} & \dfrac{\partial v_3}{\partial x_3} \end{bmatrix} \tag{3.12.5}$$

3.13 The Rate of Deformation Tensor

The velocity gradient $(\nabla \mathbf{v})$ can be decomposed into a symmetric part and an antisymmetric part as follows:

$$(\nabla \mathbf{v}) = \mathbf{D} + \mathbf{W} \tag{3.13.1}$$

where $\mathbf{D}$ is the symmetric part, i.e.,

$$\mathbf{D} = \frac{1}{2}[(\nabla \mathbf{v}) + (\nabla \mathbf{v})^T] \tag{3.13.2}$$

and $\mathbf{W}$ is the antisymmetric part, i.e.,

$$\mathbf{W} = \frac{1}{2}[(\nabla \mathbf{v}) - (\nabla \mathbf{v})^T] \tag{3.13.3}$$

The symmetric tensor $\mathbf{D}$ is known as the **rate of deformation tensor** and the antisymmetric tensor $\mathbf{W}$ is known as the **spin tensor**. The reason for these names will be apparent soon.

With respect to rectangular Cartesian coordinates, the components of $\mathbf{D}$ and $\mathbf{W}$ are given by:

$$[\mathbf{D}] = \begin{bmatrix} \dfrac{\partial v_1}{\partial x_1} & \dfrac{1}{2}\left(\dfrac{\partial v_1}{\partial x_2} + \dfrac{\partial v_2}{\partial x_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial v_1}{\partial x_3} + \dfrac{\partial v_3}{\partial x_1}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial v_1}{\partial x_2} + \dfrac{\partial v_2}{\partial x_1}\right) & \dfrac{\partial v_2}{\partial x_2} & \dfrac{1}{2}\left(\dfrac{\partial v_2}{\partial x_3} + \dfrac{\partial v_3}{\partial x_2}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial v_1}{\partial x_3} + \dfrac{\partial v_3}{\partial x_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial v_2}{\partial x_3} + \dfrac{\partial v_3}{\partial x_2}\right) & \dfrac{\partial v_3}{\partial x_3} \end{bmatrix} \tag{3.13.4}$$

$$[\mathbf{W}] = \begin{bmatrix} 0 & \dfrac{1}{2}\left(\dfrac{\partial v_1}{\partial x_2} - \dfrac{\partial v_2}{\partial x_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial v_1}{\partial x_3} - \dfrac{\partial v_3}{\partial x_1}\right) \\ -\dfrac{1}{2}\left(\dfrac{\partial v_1}{\partial x_2} - \dfrac{\partial v_2}{\partial x_1}\right) & 0 & \dfrac{1}{2}\left(\dfrac{\partial v_2}{\partial x_3} - \dfrac{\partial v_3}{\partial x_2}\right) \\ -\dfrac{1}{2}\left(\dfrac{\partial v_1}{\partial x_3} - \dfrac{\partial v_3}{\partial x_1}\right) & -\dfrac{1}{2}\left(\dfrac{\partial v_2}{\partial x_3} - \dfrac{\partial v_3}{\partial x_2}\right) & 0 \end{bmatrix} \tag{3.13.5}$$

With respect to cylindrical and spherical coordinates the matrices take the form given in Eq. (3.7.11) and Eq. (3.7.12).

We now show that the rate of change of length of $d\mathbf{x}$ is described by the tensor $\mathbf{D}$ whereas the rate of rotation of $d\mathbf{x}$ is described by the tensor $\mathbf{W}$.

Let $d\mathbf{x} = ds\mathbf{n}$, where $\mathbf{n}$ is a unit vector, then

$$d\mathbf{x} \cdot d\mathbf{x} = (ds)^2 \tag{i}$$

Taking the material derivatives of the above equation gives

$$2d\mathbf{x}\cdot\frac{D}{Dt}(d\mathbf{x}) = 2ds\frac{D}{Dt}(ds) \tag{ii}$$

Now, from Eq. (3.12.4) and (3.13.1)

$$d\mathbf{x}\cdot\frac{D}{Dt}(d\mathbf{x}) = d\mathbf{x}\cdot(\nabla\mathbf{v})d\mathbf{x} = d\mathbf{x}\cdot\mathbf{D}d\mathbf{x} + d\mathbf{x}\cdot\mathbf{W}d\mathbf{x} \tag{iii}$$

and by the definition of transpose of a tensor and the fact that $\mathbf{W}$ is an antisymmetric tensor (i.e., $\mathbf{W} = -\mathbf{W}^T$), we have

$$d\mathbf{x}\cdot\mathbf{W}d\mathbf{x} = d\mathbf{x}\cdot\mathbf{W}^T d\mathbf{x} \quad = -d\mathbf{x}\cdot\mathbf{W}d\mathbf{x} \tag{iv}$$

Thus,

$$d\mathbf{x}\cdot\mathbf{W}d\mathbf{x} = 0 \tag{v}$$

Therefore,

$$d\mathbf{x}\cdot\frac{D}{Dt}(d\mathbf{x}) = d\mathbf{x}\cdot\mathbf{D}d\mathbf{x} \tag{vi}$$

Equation (ii) then gives

$$ds\,\frac{D(ds)}{Dt} = d\mathbf{x}\cdot\mathbf{D}\,d\mathbf{x} \tag{3.13.6a}$$

With $d\mathbf{x} = ds\mathbf{n}$, Eq. (3.13.6a) can also be written:

$$\frac{1}{ds}\frac{D(ds)}{Dt} = \mathbf{n}\cdot\mathbf{D}\mathbf{n} = D_{nn} \qquad (\text{ no sum on } n\ ) \tag{3.13.6b}$$

Eq. (3.13.6b) states that for a material element in the direction of **n,** its **rate of extension** (i.e., rate of change of length per unit length) is given by D_{nn}(no sum on n). The rate of extension is also known as **stretching.** In particular

D_{11} = rate of extension for an element which is in the $\mathbf{e}_1$ direction

D_{22} = rate of extension for an element which is in the $\mathbf{e}_2$ direction and

D_{33} = rate of extension for an element which is in the $\mathbf{e}_3$ direction

We note that since $\mathbf{v}dt$ gives the infinitesimal displacement undergone by a particle during the time interval dt, the interpretation just given can be inferred from those for the infinitesimal strain components. Thus, we obviously will have the following results: [see also Prob. 3. 45(b)]:

$2\,D_{12}$ = rate of decrease of angle (from $\frac{\pi}{2}$) of two elements in $\mathbf{e}_1$ and $\mathbf{e}_2$ directions

$2\,D_{13}$ = rate of decrease of angle (from $\frac{\pi}{2}$) of two elements in $\mathbf{e}_1$ and $\mathbf{e}_3$ directions and

$2\,D_{23}$ = rate of decrease of angle (from $\frac{\pi}{2}$) of two elements in $\mathbf{e}_2$ and $\mathbf{e}_3$ directions.

These rates of decrease of angle are also known as the **rates of shear**, or **shearings**.

Also , the first scalar invariant of the rate of deformation tensor **D** gives the rate of change of volume per unit volume (see also Prob. 3.46). That is,

$$D_{11}+D_{22}+D_{33} = \frac{1}{dV}\frac{D(dV)}{Dt} \tag{3.13.7a}$$

Or, in terms of the velocity components, we have

$$\frac{1}{dV}\frac{D(dV)}{Dt} = \frac{\partial v_i}{\partial x_i} = \text{div}\mathbf{v} \tag{3.13.7b}$$

Since **D** is symmetric, we also have the result that there always exists three mutually perpendicular directions (eigenvectors of **D**) along which the stretchings (eigenvalues of **D**) include a maximum and a minimum value among all differential elements extending from a material point.

Example 3.13.1

Given the velocity field:

$$v_1 = kx_2 \qquad v_2 = v_3 = 0 \tag{i}$$

(a) Find the rate of deformation and spin tensor.

(b) Determine the rate of extension of the material elements:

$$d\mathbf{x}^{(1)} = (ds_1)\mathbf{e}_1,\ d\mathbf{x}^{(2)} = (ds_2)\mathbf{e}_2,\ \text{and}\ d\mathbf{x} = \frac{ds}{\sqrt{5}}(\mathbf{e}_1+2\mathbf{e}_2) \tag{ii}$$

(c) Find the maximum and minimum rates of extension.

Solution. (a) The matrix of the velocity gradient is

$$[\nabla\mathbf{v}] = \begin{bmatrix} 0 & k & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{iii}$$

so that

$$[\mathbf{D}] = [\nabla\mathbf{v}]^s = \begin{bmatrix} 0 & \frac{k}{2} & 0 \\ \frac{k}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{iv}$$

and

$$[\mathbf{W}] = [\nabla \mathbf{v}]^A = \begin{bmatrix} 0 & \frac{k}{2} & 0 \\ -\frac{k}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{v}$$

(b) The material element $d\mathbf{x}^{(1)}$ is currently in the $\mathbf{e}_1$-direction and therefore its rate of extension is equal to $D_{11} = 0$. Similarly, the rate of extension of $d\mathbf{x}^{(2)}$ is equal to $D_{22} = 0$. For the element $d\mathbf{x} = (ds)\mathbf{n}$, where $\mathbf{n} = \left(\frac{1}{\sqrt{5}}\right)(\mathbf{e}_1 + 2\mathbf{e}_2)$

$$\frac{1}{ds}\frac{D}{Dt}(ds) = \mathbf{n}\cdot\mathbf{D}\mathbf{n} = \frac{1}{5}[1, 2, 0]\begin{bmatrix} 0 & \frac{k}{2} & 0 \\ \frac{k}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \frac{2}{5}k \tag{vi}$$

(c) From the characteristic equation

$$|\mathbf{D} - \lambda\mathbf{I}| = -\lambda(\lambda^2 - k^2/4) = 0 \tag{vii}$$

we determine the eigenvalues of the tensor D as $\lambda = 0, \pm k/2$, therefore, $k/2$ is the maximum and $-k/2$ is the minimum rate of extension. The eigenvectors $\mathbf{n}_1 = \left(\frac{\sqrt{2}}{2}\right)(\mathbf{e}_1 + \mathbf{e}_2)$ and $\mathbf{n}_2 = \left(\frac{\sqrt{2}}{2}\right)(\mathbf{e}_1 - \mathbf{e}_2)$ give the directions of the elements having the maximum and the minimum stretching respectively.

3.14 The Spin Tensor and the Angular Velocity Vector

In section 2B.16 of Chapter 2, it was shown that an antisymmetric tensor **W** is equivalent to a vector $\boldsymbol{\omega}$ in the sense that for any vector **a**

$$\mathbf{W}\mathbf{a} = \boldsymbol{\omega}\times\mathbf{a} \tag{3.14.1}$$

The vector $\boldsymbol{\omega}$ is called the dual vector or axial vector of the tensor **W** and is related to the three nonzero components of **W** by the relation:

$$\boldsymbol{\omega} = -(W_{23}\mathbf{e}_1 + W_{31}\mathbf{e}_2 + W_{12}\mathbf{e}_3) \tag{3.14.2}$$

Now, since the spin tensor **W** is an antisymmetric tensor (by definition, the antisymmetric part of $\nabla\mathbf{v}$), therefore

$$\mathbf{W}d\mathbf{x} = \boldsymbol{\omega}\times d\mathbf{x} \tag{3.14.3}$$

and

$$\frac{D\mathbf{dx}}{Dt} = (\nabla\mathbf{v})d\mathbf{x} = (\mathbf{D}+\mathbf{W})d\mathbf{x} = \mathbf{D}d\mathbf{x}+\boldsymbol{\omega}\times d\mathbf{x} \tag{3.14.4}$$

We have already seen in the previous section that **W** does not contribute to the rate of change of length of the material vector $d\mathbf{x}$. Thus, Eq. (3.14.3) shows that its effect on $d\mathbf{x}$ is simply to rotate it (without changing its length) with an angular velocity $\boldsymbol{\omega}$.

It should be noted however, that the rate of deformation tensor **D** also contributes to the rate of change in direction of $d\mathbf{x}$ as well so that in general, most material vectors $d\mathbf{x}$ rotate with an angular velocity different from $\boldsymbol{\omega}$ (while changing their lengths). Indeed, it can be proven that in general, only the three material vectors which are in the principal direction of **D** do rotate with the angular velocity $\boldsymbol{\omega}$, (while changing their length). (see Prob. 3.47)

We also note that in fluid mechanics literature, 2**W** is called the **vorticity tensor**.

3.15 Equation of Conservation of Mass

If we follow an infinitesimal volume of material through its motion, its volume dV and density ρ may change, but its total mass ρdV will remain unchanged. That is,

$$\frac{D}{Dt}(\rho dV) = 0 \tag{3.15.1}$$

i.e.,

$$\rho\frac{D(dV)}{Dt}+dV\frac{D\rho}{Dt} = 0$$

Using Eq. (3.13.7), we obtain

$$\rho\frac{\partial v_i}{\partial x_i}+\frac{D\rho}{Dt} = 0 \tag{3.15.2a}$$

Or, in invariant form,

$$\rho\ \mathrm{div}\mathbf{v}+\frac{D\rho}{Dt} = 0 \tag{3.15.2b}$$

where in spatial description,

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t}+\mathbf{v}\cdot\nabla\rho \tag{3.15.3}$$

Equation (3.15.2) is the equation of conservation of mass, also known as the equation of continuity.

In Cartesian coordinates, Eq. (3.15.2b) reads:

$$\rho\left(\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}\right)+\frac{\partial\rho}{\partial t}+v_1\frac{\partial\rho}{\partial x_1}+v_2\frac{\partial\rho}{\partial x_2}+v_3\frac{\partial\rho}{\partial x_3} = 0 \tag{3.15.4}$$

In cylindrical coordinates, it reads:

$$\rho\left(\frac{\partial v_r}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r}{r}+\frac{\partial v_z}{\partial z}\right)+\frac{\partial \rho}{\partial t}+v_r\frac{\partial \rho}{\partial r}+\frac{v_\theta}{r}\frac{\partial \rho}{\partial \theta}+v_z\frac{\partial \rho}{\partial z}=0 \tag{3.15.5}$$

In spherical coordinates it reads:

$$\rho\left(\frac{\partial v_r}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{2v_r}{r}+\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}+\frac{v_\theta\cot\theta}{r}\right)+\frac{\partial \rho}{\partial t}+v_r\frac{\partial \rho}{\partial r}+\frac{v_\theta}{r}\frac{\partial \rho}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial \rho}{\partial \phi}=0 \tag{3.15.6}$$

For an **incompressible material**, the material derivative of the density is zero, and the mass conservation of equation reduces to simply:

$$\mathrm{div}\mathbf{v}=0 \tag{3.15.7}$$

or, in Cartesian coordinates

$$\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}=0. \tag{3.15.7a}$$

in cylindrical coordinates

$$\frac{\partial v_r}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r}{r}+\frac{\partial v_z}{\partial z}=0. \tag{3.15.7b}$$

and in spherical coordinates

$$\frac{\partial v_r}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{2v_r}{r}+\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}+\frac{v_\theta\cot\theta}{r}=0. \tag{3.15.7c}$$

Example 3.15.1

For the velocity field of Example 3.4.2,

$$v_i=\frac{x_i}{(1+t)}$$

find the density of a material particle as a function of time.

Solution. From the mass conservation equation

$$\frac{D\rho}{Dt}=-\rho\frac{\partial v_i}{\partial x_i}=-\rho\left[\frac{1}{1+t}+\frac{1}{1+t}+\frac{1}{1+t}\right]=-\frac{3\rho}{1+t} \tag{i}$$

Thus,

$$\int_{\rho_o}^{\rho} \frac{d\rho}{\rho} = -\int_{o}^{t} \frac{3dt}{1+t}$$

from which we obtain

$$\rho = \frac{\rho_o}{(1+t)^3} \tag{iii}$$

3.16 Compatibility Conditions for Infinitesimal Strain Components

When any three displacement functions u_1, u_2, and u_3 are given, one can always determine the six strain components in any region where the partial derivatives $\frac{\partial u_i}{\partial X_j}$ exist. On the other hand, when the six strain components $(E_{11}, E_{22}, E_{33}, E_{12}, E_{13}, E_{23})$ are arbitrarily prescribed in some region, in general, there may not exist three displacement functions (u_1, u_2, u_3), satisfying the six equations

$$\frac{\partial u_1}{\partial X_1} = E_{11} \tag{3.16.1}$$

$$\frac{\partial u_2}{\partial X_2} = E_{22} \tag{3.16.2}$$

$$\frac{\partial u_3}{\partial X_3} = E_{33} \tag{3.16.3}$$

$$\frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\right) = E_{12} \tag{3.16.4}$$

$$\frac{1}{2}\left(\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\right) = E_{23} \tag{3.16.5}$$

$$\frac{1}{2}\left(\frac{\partial u_3}{\partial X_1} + \frac{\partial u_1}{\partial X_3}\right) = E_{31} \tag{3.16.6}$$

For example, if we let

$$E_{11} = X_2^2,\ E_{22} = E_{33} = E_{12} = E_{13} = E_{23} = 0 \tag{i}$$

then, from Eq. (3.16.1) $\frac{\partial u_1}{\partial X_1} = X_2^2$ and from Eq. (3.16.2), $\frac{\partial u_2}{\partial X_2} = 0$, so that

$$u_1 = X_1 X_2^2 + f(X_2, X_3) \tag{ii}$$

and

$$u_2 = g(X_1, X_3) \tag{iii}$$

where f and g are arbitrary integration functions. Now, since $E_{12} = 0$, we must have, from Eq. (3.16.4)

$$\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} = 0 \tag{iv}$$

Using Eqs. (ii) and (iii), we get from Eq. (iv)

$$2X_1X_2 + \frac{\partial f(X_2, X_3)}{\partial X_2} + \frac{\partial g(X_1, X_3)}{\partial X_1} = 0 \tag{v}$$

Since the second or third term cannot have terms of the form X_1X_2, the above equation can never be satisfied. In other words, there is no displacement field corresponding to this given E_{ij}. That is, the given six strain components are not compatible with the three displacement-strain equations.

We now state the following theorem: If $E_{ij}(X_1, X_2, X_3)$ are continuous functions having continuous second partial derivatives in a simply connected region, then the necessary and sufficient conditions for the existence of single-valued continuous solutions u_1, u_2 and u_3 of the six equation Eq. (3.16.1) to Eq. (3.16.6) are

$$\frac{\partial^2 E_{11}}{\partial X_2^2} + \frac{\partial^2 E_{22}}{\partial X_1^2} = 2\frac{\partial^2 E_{12}}{\partial X_1 \partial X_2} \tag{3.16.7}$$

$$\frac{\partial^2 E_{22}}{\partial X_3^2} + \frac{\partial^2 E_{33}}{\partial X_2^2} = 2\frac{\partial^2 E_{23}}{\partial X_2 \partial X_3} \tag{3.16.8}$$

$$\frac{\partial^2 E_{33}}{\partial X_1^2} + \frac{\partial^2 E_{11}}{\partial X_3^2} = 2\frac{\partial^2 E_{31}}{\partial X_3 \partial X_1} \tag{3.16.9}$$

$$\frac{\partial^2 E_{11}}{\partial X_2 \partial X_3} = \frac{\partial}{\partial X_1}\left(\frac{-\partial E_{23}}{\partial X_1} + \frac{\partial E_{31}}{\partial X_2} + \frac{\partial E_{12}}{\partial X_3}\right) \tag{3.16.10}$$

$$\frac{\partial^2 E_{22}}{\partial X_3 \partial X_1} = \frac{\partial}{\partial X_2}\left(\frac{-\partial E_{31}}{\partial X_2} + \frac{\partial E_{12}}{\partial X_3} + \frac{\partial E_{23}}{\partial X_1}\right) \tag{3.16.11}$$

$$\frac{\partial^2 E_{33}}{\partial X_1 \partial X_2} = \frac{\partial}{\partial X_3}\left(\frac{-\partial E_{12}}{\partial X_3} + \frac{\partial E_{23}}{\partial X_1} + \frac{\partial E_{31}}{\partial X_2}\right) \tag{3.16.12}$$

These six equations are known as the equations of compatibility (or integrability conditions).

That these conditions are necessary can be easily proved as follows:

From

$$\frac{\partial u_1}{\partial X_1} = E_{11} \text{ and } \frac{\partial u_2}{\partial X_2} = E_{22} \tag{i}$$

we get

$$\frac{\partial^2 E_{11}}{\partial X_2^2} = \frac{\partial^3 u_1}{\partial X_2^2 \partial X_1} \text{ and } \frac{\partial^2 E_{22}}{\partial X_1^2} = \frac{\partial^3 u_2}{\partial X_1^2 \partial X_2} \tag{ii}$$

Now, since the left-hand sides of the above equations are, by postulate, continuous, therefore, the right-hand sides are continuous, and so the order of the differentiation is immaterial , so that

$$\frac{\partial^2 E_{11}}{\partial X_2^2} = \frac{\partial^2}{\partial X_1 \partial X_2}\left(\frac{\partial u_1}{\partial X_2}\right) \text{ and } \frac{\partial^2 E_{22}}{\partial X_1^2} = \frac{\partial^2}{\partial X_1 \partial X_2}\left(\frac{\partial u_2}{\partial X_1}\right) \tag{iii}$$

Thus, from Eqs. (iii) and Eq. (3.16.4)

$$\frac{\partial^2 E_{11}}{\partial X_2^2} + \frac{\partial^2 E_{22}}{\partial X_1^2} = \frac{\partial^2}{\partial X_1 \partial X_2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\right) = 2\frac{\partial^2 E_{12}}{\partial X_1 \partial X_2} \tag{iv}$$

The other five conditions can be similarly established. We omit the proof that the conditions are also sufficient (under the conditions stated in the theorem). In Example 3.16.3 below, we shall give an instance where the conditions are not sufficient for a region which is not simply-connected. (A region of space is said to be simply-connected if every closed curve drawn in the region can be shrunk to a point, by continuous deformation, without passing out of the boundaries of the region. For example, the solid prismatical bar represented in Fig. 3.7 is simply-connected whereas, the prismatical tube represented in Fig. 3.8 is not simply-connected).

It is worth noting the following two special cases of strain components where the compatibility conditions need not be considered because they are obviously satisfied:

(1)The strain components are obtained from given displacement components.

(2)The strain components are linear functions of coordinates.

Example 3.16.1

Will the strain components obtained from the displacements

$$u_1 = X_1^3,\ u_2 = e^{X_1},\ u_3 = \sin X_2 \tag{i}$$

be compatible?

Solution. Yes. There is no need to check, because the displacement **u** is given (and therefore exists!)

Example 3.16.2

Does the following strain field:

$$[\mathbf{E}] = \begin{bmatrix} 2X_1 & X_1+2X_2 & 0 \\ X_1+2X_2 & 2X_1 & 0 \\ 0 & 0 & 2X_3 \end{bmatrix} \tag{i}$$

represent a compatible strain field?

Solution. Since each term of the compatibility equations involves second derivatives of the strain components with respect to the coordinates, the above strain tensor with each component a linear function of X_1. X_2. X_3 will obviously satisfy them. The given strain components are obviously continuous functions having continuous second derivatives (in fact continuous derivatives of all orders) in any bounded region. Thus, the existence of single valued continuous displacement field in any bounded simply-connected region is ensured by the theorem stated above. In fact, it can be easily verified that

$$u_1 = X_1^2+X_2^2,\ u_2 = 2X_1X_2+X_1^2,\ u_3 = X_3^2 \tag{ii}$$

(to which of course, can be added any rigid body displacements) which is a single-valued continuous displacement field in any bounded region, including multiply-connected region.

Example 16.3

For the following strain field

$$E_{11} = \frac{-X_2}{X_1^2+X_2^2},\ E_{12} = \frac{X_1}{2(X_1^2+X_2^2)},\ E_{22} = E_{33} = E_{23} = E_{13} = 0 \tag{i}$$

does there exist single-valued continuous displacement fields for (a) the cylindrical body with the normal cross-section shown in Fig. 3.7 and (b) for the body with the normal cross-section shown in Fig. 3.8 and with the origin of the axis inside the hole of the cross-section.

Solution. Out of the six compatibility conditions, only the first one needs to be checked, the others are automatically satisfied. Now,

$$\frac{\partial E_{11}}{\partial X_2} = -\frac{(X_1^2+X_2^2)-X_2(2X_2)}{(X_1^2+X_2^2)^2} = \frac{X_2^2-X_1^2}{(X_1^2+X_2^2)^2} \tag{ii}$$

$$2\frac{\partial E_{12}}{\partial X_1} = \frac{(X_1^2+X_2^2)-X_1(2X_1)}{(X_1^2+X_2^2)^2} = \frac{X_2^2-X_1^2}{(X_1^2+X_2^2)^2} \tag{iii}$$

and

$$\frac{\partial^2 E_{22}}{\partial X_1^2} = 0 \tag{iv}$$

Thus, the equation

$$\frac{\partial^2 E_{11}}{\partial X_2^2}+\frac{\partial^2 E_{22}}{\partial X_1^2} = 2\frac{\partial^2 E_{12}}{\partial X_1 \partial X_2} \tag{v}$$

is satisfied, and the existence of solution is assured. In fact it can be easily verified that for the given E_{ij},

$$u_1 = \arctan\frac{X_2}{X_1},\ u_2 = 0,\ u_3 = 0 \tag{vi}$$

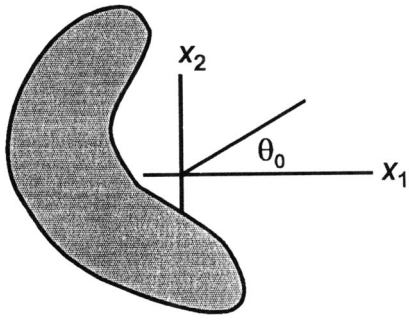

Fig. 3.7

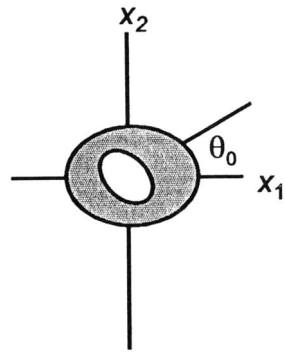

Fig. 3.8

(to which, of course, any rigid body displacement field can be added). Now arctan X_2/X_1 is a multiple-valued function, having infinitely many values corresponding to a point (X_1,X_2,X_3).

For example, for the point $(X_1,X_2,X_3) = (1,0,0)$, $\arctan X_2/X_1 = 0, 2\pi, 4\pi$, etc. It can be made a single-valued function by the restriction $\theta_o \leq \arctan X_2/X_1 < \theta_o + 2\pi$ for any θ_o. For a simply-connected region as that shown in Fig. 3.7, a θ_o can be chosen so that such a restriction makes Eq. (vi) a single-valued continuous displacement for the region. But for the body shown in Fig. 3.8, the function $u_1 = \arctan X_2/X_1$, under the same restriction is discontinuous along the line $\theta = \theta_o$ in the body (in fact, u_1 jumps by the value of 2π in crossing the line). Thus, for this so-called doubly-connected region, there does not exist single-valued continuous u_1 corresponding to the given E_{ij}, even though the compatibility equations are satisfied.

3.17 Compatibility Conditions For Rate Of Deformation

When any three velocity functions v_1, v_2, and v_3 are given, one can always determine the six rate of deformation components in any region where the partial derivatives $\partial v_i/\partial x_j$ exist. On the other hand, when the six components D_{ij} are arbitrarily prescribed in some region, in general, there does not exist any velocity field v_i, satisfying the six equations

$$\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right) = D_{ij} \tag{3.17.1}$$

The compatibility conditions for the rate of deformation components are similar to those of the infinitesimal strain components [Eqs. (3.16.7-12)], i.e.,

$$\frac{\partial^2 D_{11}}{\partial x_2^2}+\frac{\partial^2 D_{22}}{\partial x_1^2} = 2\frac{\partial^2 D_{12}}{\partial x_1 \partial x_2} \tag{3.17.2a}$$

$$\frac{\partial^2 D_{22}}{\partial x_3^2}+\frac{\partial^2 D_{33}}{\partial x_2^2} = 2\frac{\partial^2 D_{23}}{\partial x_2 \partial x_3} \tag{3.17.2b}$$

$$\frac{\partial^2 D_{33}}{\partial x_1^2}+\frac{\partial^2 D_{11}}{\partial x_3^3} = 2\frac{\partial^2 D_{13}}{\partial x_1 \partial x_3} \tag{3.17.2c}$$

etc. It should be emphasized that if one deals directly with differentiable velocity functions $v_i(x_1,x_2,x_3,t)$, (as is often the case in fluid mechanics), the question of compatibility does not arise.

3.18 Deformation Gradient

We recall that the general motion of a continuum is described by

$$\mathbf{x}=\mathbf{x}(\mathbf{X}, t) \tag{3.18.1}$$

where $\mathbf{x}$ is the spatial position at time t, of a material particle with a material coordinate $\mathbf{X}$. A material element $d\mathbf{X}$ at the reference configuration is transformed, through motion, into a material element $d\mathbf{x}$ at time t. The relation between $d\mathbf{X}$ and $d\mathbf{x}$ is given by

$$d\mathbf{x} = \mathbf{x}(\mathbf{X}+d\mathbf{X}, t)-\mathbf{x}(\mathbf{X}, t) = (\nabla \mathbf{x})d\mathbf{X} \tag{3.18.2}$$

i.e.,

$$d\mathbf{x} = \mathbf{F}d\mathbf{X} \tag{3.18.3}$$

where the tensor

$$\mathbf{F} = \nabla\mathbf{x} \tag{3.18.4}$$

is called the **deformation gradient** at $\mathbf{X}$. The notation $\nabla\mathbf{x}$ is an abbreviation for the notation $\nabla_{\mathbf{X}}\mathbf{x}$ where the subscript $\mathbf{X}$ indicates that the gradient is with respect to $\mathbf{X}$ for the function $\mathbf{x}(\mathbf{X}, t)$. We note that with $\mathbf{x} = \mathbf{X} + \mathbf{u}$, where $\mathbf{u}$ is the displacement vector,

$$\mathbf{F} = \mathbf{I} + \nabla\mathbf{u} \tag{3.18.5}$$

Example 3.18.1

Given the following motion:

$$x_1 = X_1+X_1^2 t,\ \ x_2 = X_2-X_2t-X_3 t,\ \ x_3 = X_3+X_2t-X_3 t \tag{i}$$

where both x_i and X_i are rectangular Cartesian coordinates. Find the deformation gradient at $t = 0$ and at $t = 1$.

Solution. For rectangular Cartesian coordinates,

$$[\mathbf{F}] = \begin{bmatrix} \dfrac{\partial x_1}{\partial X_1} & \dfrac{\partial x_1}{\partial X_2} & \dfrac{\partial x_1}{\partial X_3} \\ \dfrac{\partial x_2}{\partial X_1} & \dfrac{\partial x_2}{\partial X_2} & \dfrac{\partial x_2}{\partial X_3} \\ \dfrac{\partial x_3}{\partial X_1} & \dfrac{\partial x_3}{\partial X_2} & \dfrac{\partial x_3}{\partial X_3} \end{bmatrix} \tag{ii}$$

Thus, from Eq. (i) and (ii),

$$[\mathbf{F}] = \begin{bmatrix} 1+2X_1t & 0 & 0 \\ 0 & 1-t & -t \\ 0 & t & 1-t \end{bmatrix} \tag{iii}$$

From Eq. (iii) we have at $t = 0$, $\mathbf{F} = \mathbf{I}$, and $d\mathbf{x} = d\mathbf{X}$.

At $t = 1$, for all elements

$$[\mathbf{F}] = \begin{bmatrix} 1+2X_1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{iv}$$

3.19 Local Rigid Body Displacements

In Section 3.6, we discussed the case where the entire body undergoes rigid body displacements from the configuration at a reference time t_o to that at a particular time t. For a body in a general motion, however, it is possible that the body as a whole undergoes deformations while some (infinitesimally) small volumes of material inside the body undergo rigid body displacements. For example, for the motion given in the last example, at $t = 1$ and $X_1 = 0$,

$$[\mathbf{F}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{i}$$

It is easily to verify that the above $\mathbf{F}$ is a rotation tensor $\mathbf{R}$ (i.e., $\mathbf{F}\mathbf{F}^{\mathbf{T}} = \mathbf{I}$ and $\det \mathbf{F} = +1$). Thus, all infinitesimal material volumes with material coordinates $(0, X_2, X_3)$ undergo a rigid body displacement from the reference position to the position at $t = 1$.

3.20 Finite Deformation

Deformations at a material point $\mathbf{X}$ of a body are characterized by changes of distances between any pair of material points within the small neighborhood of $\mathbf{X}$. Since, through motion, a material element $d\mathbf{X}$ becomes $d\mathbf{x} = \mathbf{F}d\mathbf{X}$, whatever deformation there may be at $\mathbf{X}$, is embodied in the deformation gradient $\mathbf{F}$. We have already seen that if $\mathbf{F}$ is a proper orthogonal tensor, then there is no deformation at $\mathbf{X}$. In the following, we first consider the case where the deformation gradient $\mathbf{F}$ is a symmetric tensor before going to more general cases.

We shall use the notation $\mathbf{U}$ for a deformation gradient $\mathbf{F}$ that is symmetric. Thus, for a symmetric deformation gradient, we write

$$d\mathbf{x} = \mathbf{U}d\mathbf{X} \tag{3.20.1}$$

In this case, the material within a small neighborhood of $\mathbf{X}$ is said to be in a state of **pure stretch** deformation (from the reference configuration). Of course, Eq. (3.20.1) includes the special case where the motion is homogeneous, i.e., $\mathbf{x} = \mathbf{U}\mathbf{X}$, ($\mathbf{U}$ = constant tensor) in which case the entire body is in a state of pure stretch.

Since **U** is real and symmetric, there exists three mutually perpendicular directions, with respect to which, the matrix of **U** is diagonal. Thus, if $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are these principal directions, with eigenvalues $\lambda_1, \lambda_2, \lambda_3$, then, for $d\mathbf{X}^{(1)} = dX_1\mathbf{e}_1$, Eq. (3.20.1) gives $d\mathbf{x}^{(1)} = \lambda_1 dX_1\mathbf{e}_1$, i.e.,

$$d\mathbf{x}^{(1)} = \lambda_1 d\mathbf{X}^{(1)} \tag{3.20.2a}$$

Similarly, for $d\mathbf{X}^{(2)} = dX_2\mathbf{e}_2$ and $d\mathbf{X}^{(3)} = dX_3\mathbf{e}_3$, we have

$$d\mathbf{x}^{(2)} = \lambda_2 d\mathbf{X}^{(2)} \tag{3.20.2b}$$

$$d\mathbf{x}^{(3)} = \lambda_3 d\mathbf{X}^{(3)} \tag{3.20.2c}$$

We see that along each of these three directions, the deformed element is in the same direction as the undeformed element. If the eigenvalues are distinct, these will be the only elements which do not change their directions. The ratio of the deformed length to the original length is called the **stretch**, i.e.,

$$\text{Stretch} \equiv \frac{|d\mathbf{x}|}{|d\mathbf{X}|} \tag{3.20.3}$$

Thus, the eigenvalues of **U** are the principal stretches; they include the maximum and the minimum stretches.

Example 3.20.1

Given that at time t,

$$\begin{aligned} x_1 &= 3X_1 \\ x_2 &= 4X_2 \\ x_3 &= X_3 \end{aligned} \tag{i}$$

Referring to Fig. 3.9, find the stretches for the following material line (a)OP (b)OQ and (c)OB.

Solution. The matrix of the deformation gradient for this given motion is

$$[\mathbf{F}] = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is a symmetric matrix and is independent of X_i (i.e., the same for all material points). Thus, the given deformation is a homogeneous pure stretch deformation. The eigenvectors are obviously (see Sect. 2B.17, Example 2B17.2) $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ with corresponding eigenvalues, 3, 4 and 1. Thus:

(a)At the deformed state, the line OP triples its original length and remains parallel to the x_1-axis, i.e., stretch $\equiv \lambda_1 = 3$.

(b)At the deformed state, the line OQ quadruple its original length and remains parallel to the x_2- axis; stretch $\equiv\lambda_2 = 4$.

(c)The line OB has an original length of 1.414. In the deformed state, it has a length of 5, thus, the stretch is 5/1.414. Originally, the line OB makes an angle of 45° with the x_1 -axis; in the deformed state, it makes an angle of $\tan^{-1}(4/3)$. In other words, the material line OB changes its direction from OB to OB' (see Fig. 3.9).

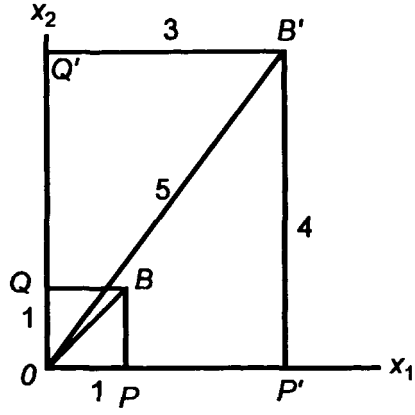

Fig. 3.9

Example 3.20.2

For a material sphere with center at **X** and described by $|d\mathbf{X}| = \varepsilon$, under a symmetric deformation gradient **U**, what does the sphere become after the deformation?

Solution. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be the principal directions for **U**, then with respect $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ a material element $d\mathbf{X}$ can be written

$$d\mathbf{X} = dX_1\mathbf{e}_1+dX_2\mathbf{e}_2+dX_3\mathbf{e}_3 \tag{i}$$

In the deformed state, this material vector becomes

$$d\mathbf{x} = dx_1\mathbf{e}_1+dx_2\mathbf{e}_2+dx_3\mathbf{e}_3 \tag{ii}$$

Since **F** is diagonal, with diagonal element λ_1, λ_2, λ_3, therefore $d\mathbf{x}=\mathbf{F}d\mathbf{X}$ gives

$$dx_1 = \lambda_1 dX_1,\ dx_2 = \lambda_2 dX_2,\ dx_3 = \lambda_3 dX_3, \tag{iii}$$

thus, the sphere :

$$(dX_1)^2+(dX_2)^2+(dX_3)^2 = \varepsilon^2 \tag{iv}$$

becomes

$$\left(\frac{dx_1}{\lambda_1}\right)^2 + \left(\frac{dx_2}{\lambda_2}\right)^2 + \left(\frac{dx_3}{\lambda_3}\right)^2 = \varepsilon^2 \tag{v}$$

This is the equation of an ellipsoid with its axis parallel to the eigenvectors of **U**. (see Fig. 3.10).

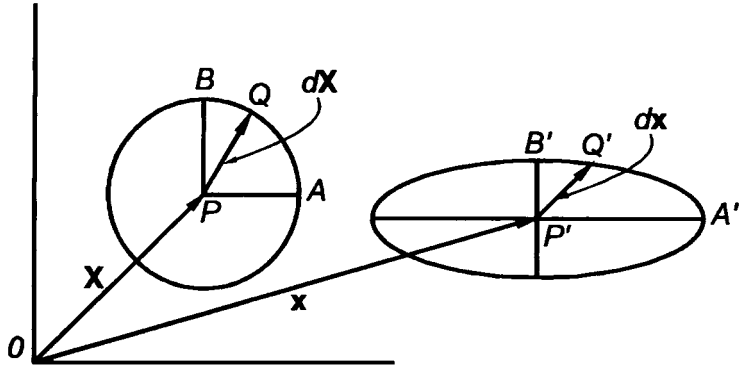

Fig. 3.10

3.21 Polar Decomposition Theorem

In the previous two sections, we considered two special deformation gradients **F**: a proper orthogonal **F** (denoted by **R**) describing rigid body displacements and a symmetric **F** (denoted by **U**) describing pure stretch deformation tensor. It can be shown that for any real tensor **F** with a nonzero determinant (i.e., $\mathbf{F}^{-1}$exists), one can always decompose it into the product of a proper orthogonal tensor and a symmetric tensor. That is

$$\mathbf{F} = \mathbf{RU} \tag{3.21.1}$$

or,

$$\mathbf{F} = \mathbf{VR} \tag{3.21.2}$$

In the above two equations, **U** and **V** are positive definite symmetric tensors and **R** (the same in both equations) is a proper orthogonal tensor. Eqs. (3.21.1) and (3.21.2) are known as the **polar decomposition theorem.** The decomposition is unique in that there is only one **R**, one **U** and one **V** for the above equations. The proof of this theorem consists of two steps : (1) Establishing a procedure which always enables one to obtain a symmetric tensor **U** and a proper orthogonal tensor **R** (or a symmetric tensor **V** and a proper orthogonal tensor **R**) which satisfies Eq. (3.21.1) (or, Eq. (3.21.2)) and (2) proving that the **U**, **V** and **R** so obtained are unique. The procedures for obtaining the tensors **U**, **V**, and **R** for a given **F** will be

demonstrated in Example 3.22.1 and 3.23.1. The proof of the uniqueness of the decompositions will be given in Example 3.22.2.

For any material element $d\mathbf{X}$ at $\mathbf{X}$, the deformation gradient transforms it (i.e., $d\mathbf{X}$) into a vector $d\mathbf{x}$:

$$d\mathbf{x} = \mathbf{F}\, d\mathbf{X} = \mathbf{RU}\, d\mathbf{X} \tag{3.21.3}$$

Now, $\mathbf{U}dX$ describes a pure stretch deformation (Section 3.20) in which there are three mutually perpendicular directions (the eigenvectors of $\mathbf{U}$) along which the material element $d\mathbf{X}$ stretches (i.e., becomes longer or shorter) but does not rotate. Figure 3.10 depicts the effect of $\mathbf{U}$ on a spherical volume $|d\mathbf{X}|$ = constant ; the spherical volume at $\mathbf{X}$ becomes an ellipsoid at $\mathbf{x}$. (See Example 3.20.2) The effect of $\mathbf{R}$ in $\mathbf{R}(\mathbf{U}\, d\mathbf{X})$ is then simply to rotate this ellipsoid through a rigid body rotation.(See Fig. 3.11)

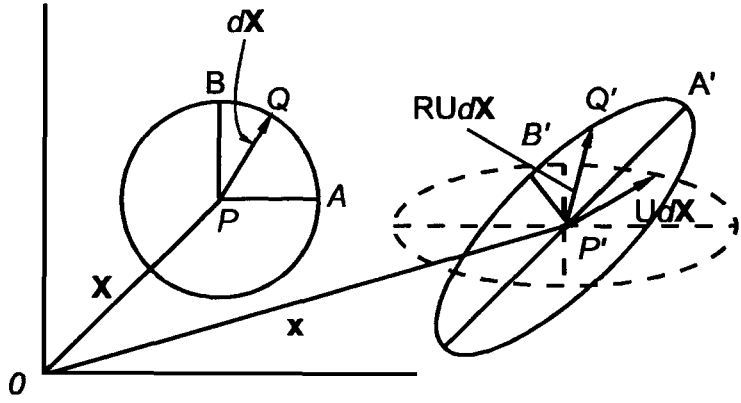

Fig. 3.11

Similarly, the effect of the same deformation gradient can be viewed as a rigid body rotation (described $\mathbf{R}$) of the sphere followed by a pure stretch of the sphere resulting in the same ellipsoid as described in the last paragraph.

From the polar decomposition theorem, $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$, it follows immediately that

$$\mathbf{U} = \mathbf{R}^T\mathbf{V}\,\mathbf{R} \tag{3.21.4}$$

Example 3.21.1

Show that if the eigenvector of $\mathbf{U}$ is $\mathbf{n}$, then the eigenvector for $\mathbf{V}$ is $\mathbf{Rn}$; the eigenvalues for both $\mathbf{U}$ and $\mathbf{V}$ are the same

Solution. Let $\mathbf{n}$ be an eigenvector for $\mathbf{U}$ with eigenvalue λ, then

$$\mathbf{Un} = \lambda \mathbf{n}. \tag{i}$$

so that

$$\mathbf{RUn} = \lambda \mathbf{Rn} \tag{ii}$$

Since $\mathbf{RU} = \mathbf{VR} = \mathbf{F}$, therefore, from Eq. (ii), we have

$$\mathbf{V}(\mathbf{Rn}) = \lambda(\mathbf{Rn}) \tag{iii}$$

Thus, $\mathbf{Rn}$ is an eigenvector of $\mathbf{V}$ with eigenvalue λ.

3.22 Calculation of the Stretch Tensors From the Deformation Gradient

From a given $\mathbf{F}$, we have $\mathbf{F} = \mathbf{R}\,\mathbf{U}$, thus,

$$\mathbf{F}^T\mathbf{F} = (\mathbf{RU})^T(\mathbf{RU}) = \mathbf{U}^T\mathbf{R}^T\mathbf{RU} = \mathbf{U}^T\mathbf{U} \tag{i}$$

That is,

$$\mathbf{U}^2 = \mathbf{F}^T\mathbf{F} \tag{3.22.1}$$

From which the positive definite symmetric tensor $\mathbf{U}$ can be calculated as (See Examples below).

$$\mathbf{U} = (\mathbf{F}^T\mathbf{F})^{1/2} \tag{3.22.2}$$

Once $\mathbf{U}$ is obtained, $\mathbf{R}$ can be obtained from the equation

$$\mathbf{R} = \mathbf{FU}^{-1} \tag{3.22.3}$$

Since

$$\mathbf{U}^{-1}\mathbf{F}^T\mathbf{FU}^{-1} = \mathbf{U}^{-1}\mathbf{U}^2\mathbf{U}^{-1} = \mathbf{I} \tag{i}$$

therefore, [note that $\mathbf{U}$ is symmetric],

$$(\mathbf{FU}^{-1})^T\mathbf{FU}^{-1} = \mathbf{I} \tag{ii}$$

Thus, from Eq. (3.22.3),

$$\mathbf{R}^T\mathbf{R} = \mathbf{I} \tag{iii}$$

Eq. (iii) states that the tensor $\mathbf{R}$ obtained from Eq. (3.22.3) is indeed an orthogonal tensor.

The left stretch tensor $\mathbf{V}$ can be obtained from

$$\mathbf{V} = \mathbf{FR}^T = \mathbf{RUR}^{\mathbf{T}} \tag{3.22.4}$$

Example 3.22.1

Given

$$x_1 = X_1,\ x_2 = -3X_3,\ x_3 = 2X_2 \tag{i}$$

Find (a) the deformation gradient **F**, (b) the right stretch tensor **U**, and (c) the rotation tensor **R** and (d) the left stretch tensor **V**.

Solution. (a)

$$[\mathbf{F}] = \begin{bmatrix} \dfrac{\partial x_1}{\partial X_1} & \dfrac{\partial x_1}{\partial X_2} & \dfrac{\partial x_1}{\partial X_3} \\ \dfrac{\partial x_2}{\partial X_1} & \dfrac{\partial x_2}{\partial X_2} & \dfrac{\partial x_2}{\partial X_3} \\ \dfrac{\partial x_3}{\partial X_1} & \dfrac{\partial x_3}{\partial X_2} & \dfrac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 2 & 0 \end{bmatrix} \tag{ii}$$

(b)

$$[\mathbf{U}^2] = [\mathbf{F}]^T[\mathbf{F}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix} \tag{iii}$$

Thus, the positive definite tensor **U** is given by

$$[\mathbf{U}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \tag{iv}$$

(c)

$$[\mathbf{R}] = [\mathbf{F}][\mathbf{U}^{-1}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{v}$$

(d)

$$[\mathbf{V}] = [\mathbf{F}][\mathbf{R}]^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{vi}$$

We can also obtain **V** from

$$[\mathbf{V}] = [\mathbf{R}][\mathbf{U}][\mathbf{R}]^T \tag{3.22.2}$$

In this example, the calculation of $[\mathbf{U}]$ and $[\mathbf{R}]$ are simple because $\mathbf{F}^T\mathbf{F}$ happens to be diagonal. If not, one can first diagonalize it to obtain $[\mathbf{U}]$ and $[\mathbf{U}]^{-1}$ as diagonal matrices

with respect to the principal axes of $\mathbf{F}^T\mathbf{F}$. After that, one then uses the transformation law discussed in Chapter 2 to obtain the matrices with respect to the $\mathbf{e}_i$ basis. (See Example 3.23.1 below).

Example 3.22.2

(a) Show that if $\mathbf{F} = \mathbf{R}_1\mathbf{U}_1 = \mathbf{R}_2\mathbf{U}_2$, then $\mathbf{R}_1 = \mathbf{R}_2$ and $\mathbf{U}_1 = \mathbf{U}_2$

(b) Show that if $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}'$, then $\mathbf{R} = \mathbf{R}'$

Solution. (a) From $\mathbf{R}_1\mathbf{U}_1 = \mathbf{R}_2\mathbf{U}_2$, we have $(\mathbf{R}_1\mathbf{U}_1^T) = (\mathbf{R}_2\mathbf{U}_2)^T$

Thus, $\mathbf{U}_1\mathbf{R}_1^T = \mathbf{U}_2\mathbf{R}_2^T$, so that $\mathbf{U}_1\mathbf{R}_1^T(\mathbf{R}_1\mathbf{U}_1) = \mathbf{U}_2\mathbf{R}_2^T\mathbf{R}_2\mathbf{U}_2$

In other words, $\mathbf{U}_1^2 = \mathbf{U}_2^2$. Since both $\mathbf{U}_1$ and $\mathbf{U}_2$ are positive definite, therefore

$$\mathbf{U}_1 = \mathbf{U}_2 = \mathbf{U}$$

and from $\mathbf{R}_1\mathbf{U} = \mathbf{R}_2\mathbf{U}$, it follows,

$$\mathbf{R}_1 = \mathbf{R}_2 = \mathbf{R}$$

(b) Since

$$\mathbf{V}\mathbf{R}' = (\mathbf{R}'\mathbf{R}'^{-1})\mathbf{V}\mathbf{R}' = \mathbf{R}'(\mathbf{R}'^{-1}\mathbf{V}\mathbf{R}')$$

thus,

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{R}'(\mathbf{R}'^{-1}\mathbf{V}\mathbf{R}')$$

Noting that $(\mathbf{R}'^{-1}\mathbf{V}\mathbf{R}')$ is symmetric, from the result of part (a), we have

$$\mathbf{R} = \mathbf{R}'$$

From the decomposition theorem we see that what is responsible for the deformation of a volume of material in a continuum in general motion is the stretch tensor, either $\mathbf{U}$ (the right stretch tensor) or $\mathbf{V}$ (the left stretch tensor). Obviously, $\mathbf{U}^2$ and $\mathbf{V}^2$ also characterize the deformation, as are many other tensors related to them. In the following sections, we discuss those tensors which have been commonly used to describe finite deformations for a continuum.

3.23 Right Cauchy-Green Deformation Tensor

Let

$$\mathbf{C} = \mathbf{U}^2 \tag{3.23.1}$$

where $\mathbf{U}$ is the right stretch tensor. The tensor $\mathbf{C}$ is known as the **right Cauchy-Green deformation tensor** (also known as the **Green's deformation tensor**). We note that if there is no deformation, $\mathbf{U} = \mathbf{C} = \mathbf{I}$.

Using Eq. (3.22.1), we have

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} \tag{3.23.2}$$

The components of $\mathbf{C}$ have very simple geometric meanings which are described below.

Consider two material elements $d\mathbf{x}^{(1)} = \mathbf{F}d\mathbf{X}^{(1)}$ and $d\mathbf{x}^{(2)} = \mathbf{F}d\mathbf{X}^{(2)}$, we have

$$d\mathbf{x}^{(1)}\cdot d\mathbf{x}^{(2)} = \mathbf{F}d\mathbf{X}^{(1)}\cdot\mathbf{F}d\mathbf{X}^{(2)} = d\mathbf{X}^{(1)}\cdot\mathbf{F}^T\mathbf{F}d\mathbf{X}^{(2)} \tag{3.23.3}$$

i.e.,

$$d\mathbf{x}^{(1)}\cdot d\mathbf{x}^{(2)} = d\mathbf{X}^{(1)}\cdot\mathbf{C}d\mathbf{X}^{(2)} \tag{3.23.4}$$

Thus, if $d\mathbf{x} = ds\mathbf{n}$, is the deformed vector of the material element $d\mathbf{X} = dS\mathbf{e}_1$ then Eq. (3.23.4) gives

$$(ds)^2 = (dS)^2\mathbf{e}_1\cdot\mathbf{C}\mathbf{e}_1 \quad \text{for } d\mathbf{X}^{(1)} = d\mathbf{X}^{(2)} = dS\mathbf{e}_1$$

That is

$$C_{11} = \left(\frac{ds}{dS}\right)^2 \text{ for a material element } d\mathbf{X} = dS\mathbf{e}_1 \tag{3.23.5a}$$

similarly,

$$C_{22} = \left(\frac{ds}{dS}\right)^2 \text{ for a material element } d\mathbf{X} = dS\mathbf{e}_2 \tag{3.23.5b}$$

$$C_{33} = \left(\frac{ds}{dS}\right)^2 \text{ for a material element } d\mathbf{X} = dS\mathbf{e}_3 \tag{3.23.5c}$$

By considering two material elements $d\mathbf{X}^{(1)} = dS_1\mathbf{e}_1$ and $d\mathbf{X}^{(2)} = dS_2\mathbf{e}_2$ which deform into $d\mathbf{x}^{(1)} = ds_1\mathbf{m}$ and $d\mathbf{x}^{(1)} = ds_1\mathbf{n}$ where $\mathbf{m}$ and $\mathbf{n}$ are unit vectors having an angle of β between them, then Eq. (3.23.4) gives

$$ds_1 ds_2\cos\beta = dS_1 dS_2\mathbf{e}_1\cdot\mathbf{C}\mathbf{e}_2 \tag{3.23.6}$$

That is

$$C_{12} = \frac{ds_1\, ds_2}{dS_1 dS_2}\cos(d\mathbf{x}^{(1)}, d\mathbf{x}^{(2)}) \quad \text{for } d\mathbf{X}^{(1)} = dS_1\mathbf{e}_1 \text{ and } d\mathbf{X}^{(2)} = dS_2\mathbf{e}_2 \tag{3.23.6a}$$

Similarly

$$C_{13} = \frac{ds_1\, ds_3}{dS_1 dS_3}\cos(d\mathbf{x}^{(1)}, d\mathbf{x}^{(3)}) \quad \text{for } d\mathbf{X}^{(1)} = dS_1\mathbf{e}_1 \text{ and } d\mathbf{X}^{(3)} = dS_3\mathbf{e}_3 \tag{3.23.6b}$$

and

$$C_{23} = \frac{ds_2\, ds_3}{dS_2 dS_3}\cos(d\mathbf{x}^{(2)}, d\mathbf{x}^{(3)}) \quad \text{for } d\mathbf{X}^{(2)} = dS_2\mathbf{e}_2 \text{ and } d\mathbf{X}^{(3)} = dS_3\mathbf{e}_3 \tag{3.23.6c}$$

Example 3.23.1

Given

$$x_1 = X_1 + 2X_2, \quad x_2 = X_2, \quad x_3 = X_3 \tag{i}$$

(a) Obtain **C**

(b) Obtain the principal values of **C** and the corresponding principal directions

(c) Obtain the matrix of **U** and $\mathbf{U}^{-1}$ with respect to the principal directions

(d) Obtain the matrix of **U** and $\mathbf{U}^{-1}$ with respect to the $\mathbf{e}_i$ basis

(e) Obtain the matrix of **R** with respect to the $\mathbf{e}_i$ basis

Solution. (a) From Eq. (i), we obtain,

$$[\mathbf{F}] = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{ii}$$

Thus,

$$[\mathbf{C}] = [\mathbf{F}]^T[\mathbf{F}] = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{iii}$$

The eigenvalues of **C** and their corresponding eigenvectors are easily found to be

$$\begin{aligned} C_1 &= 5.828, \quad \mathbf{n}_1 = \left(\frac{1}{2.613}\right)[\mathbf{e}_1 + 2.414\mathbf{e}_2] = [0.3827\mathbf{e}_1 + 0.9238\mathbf{e}_2] \\ C_2 &= 0.1716, \quad \mathbf{n}_2 = \left(\frac{1}{1.0824}\right)[\mathbf{e}_1 - 0.4142\mathbf{e}_2] = [0.9238\mathbf{e}_1 - 0.3827\mathbf{e}_2] \\ C_3 &= 1, \quad \mathbf{n}_3 = \mathbf{e}_3 \end{aligned} \tag{iv}$$

(b) The matrix of **C** with respect to the principal axis of **C** is

$$[\mathbf{C}] = \begin{bmatrix} 5.828 & 0 & 0 \\ 0 & 0.1716 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{v}$$

(c) The matrix of $\mathbf{U}$ and $\mathbf{U}^{-1}$ with respect to the principal axis of $\mathbf{C}$ are given by

$$[\mathbf{U}]_{\mathbf{n}_i} = \begin{bmatrix} \sqrt{5.828} & 0 & 0 \\ 0 & \sqrt{0.1716} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2.414 & 0 & 0 \\ 0 & 0.4142 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{vi}$$

$$[\mathbf{U}^{-1}]_{\mathbf{n}_i} = \begin{bmatrix} 0.4142 & 0 & 0 \\ 0 & 2.4143 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{vii}$$

(d) The matrix of $\mathbf{U}$ and $\mathbf{U}^{-1}$ with respect to $\mathbf{e}_i$ axes is given by

$$\begin{aligned} [\mathbf{U}]_{\mathbf{e}_i} &= \begin{bmatrix} 0.3827 & 0.9238 & 0 \\ 0.9238 & -0.3827 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2.414 & 0 & 0 \\ 0 & 0.4142 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.3827 & 0.9238 & 0 \\ 0.9238 & -0.3827 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0.7070 & 0.7070 & 0 \\ 0.7070 & 2.121 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{viii}$$

$$\begin{aligned} [\mathbf{U}^{-1}]_{\mathbf{e}_i} &= \begin{bmatrix} 0.3827 & 0.9238 & 0 \\ 0.9238 & -0.3827 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.4142 & 0 & 0 \\ 0 & 2.414 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.3827 & 0.9238 & 0 \\ 0.9238 & -0.3827 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2.121 & -0.7070 & 0 \\ -0.7070 & 0.7070 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{ix}$$

(e)

$$[\mathbf{R}]_{\mathbf{e}_i} = [\mathbf{F}][\mathbf{U}]^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2.121 & -0.707 & 0 \\ -0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.707 & 0.707 & 0 \\ -0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{x}$$

Using the same procedure as that used in the above example, one can obtain that in general, for

$$[\mathbf{F}] = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.23.7}$$

$$[\mathbf{R}] = (1+\frac{k^2}{4})^{-1/2}\begin{bmatrix} 1 & \frac{k}{2} & 0 \\ -\frac{k}{2} & 1 & 0 \\ 0 & 0 & (1+\frac{k^2}{4})^{1/2} \end{bmatrix} \tag{3.23.8}$$

Example 3.23.2

Consider the simple shear deformation given by

$$x_1 = X_1 + kX_2, \quad x_2 = X_2, \quad x_3 = X_3 \tag{i}$$

(a) What is the stretch for an element which was in the direction of $\mathbf{e}_1$

(b) What is the stretch for an element which was in the direction of $\mathbf{e}_2$

(c) What is the stretch for an element which was in the direction of $\mathbf{e}_1 + \mathbf{e}_2$

(d) In the deformed configuration, what is the angle between the two elements which were in the directions of $\mathbf{e}_1$ and $\mathbf{e}_2$

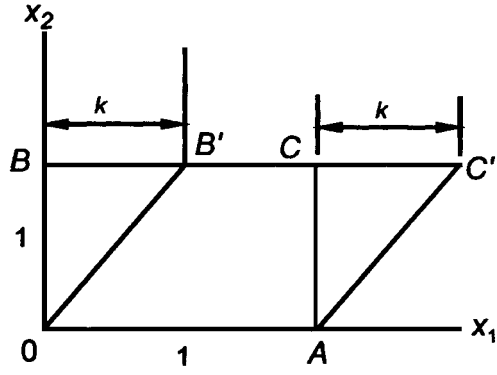

Fig. 3.12

Solution.

$$[\mathbf{F}] = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{ii}$$

$$[\mathbf{C}] = \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & k & 0 \\ k & 1+k^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{iii}$$

(a) for $d\mathbf{X} = dS_1\mathbf{e}_1$, $ds/dS = 1$

(b) for $d\mathbf{X} = dS_2\mathbf{e}_2$, $\quad ds/dS = \sqrt{1+k^2}$ [e.g. $OB' = \sqrt{1+k^2}\,OB$]

(c) for $d\mathbf{X} = (dS/\sqrt{2})(\mathbf{e}_1 + \mathbf{e}_2) = dS\mathbf{e}_1{}'$,

$$C'_{11} = \frac{1}{2}[1,1,0] \begin{bmatrix} 1 & k & 0 \\ k & 1+k^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 + k + \frac{k^2}{2}$$

thus, for this material element

$$ds/dS = 1 + k + \frac{k^2}{2}$$

(d) For $d\mathbf{X} = dS_1\mathbf{e}_1$ and $d\mathbf{X} = dS_2\mathbf{e}_2$

$$\cos(d\mathbf{x}^{(1)}, d\mathbf{x}^{(2)}) = \frac{dS_1}{ds_1}\frac{dS_2}{ds_2}C_{12} = \frac{k}{\sqrt{1+k^2}} \quad \left[\text{e.g. } \cos AOB' = \frac{k}{\sqrt{(1+k^2)}}\right]$$

Example 3.23.3

Show that (a) the eigenvectors of $\mathbf{U}$ and $\mathbf{C}$ are the same and (b) an element which was in the principal direction $\mathbf{n}$ of $\mathbf{C}$ becomes, in the deformed state, an element in the direction of $\mathbf{Rn}$.

Solution. (a) Since $\mathbf{Un} = \lambda\mathbf{n}$, therefore $\mathbf{U}^2\mathbf{n} = \lambda\mathbf{Un} = \lambda^2\mathbf{n}$

i.e.,

$$\mathbf{Cn} = \lambda^2\mathbf{n}$$

Thus, $\mathbf{n}$ is also an eigenvector of $\mathbf{C}$ with λ^2 as its eigenvalue.

(b) If $d\mathbf{X} = dS\mathbf{n}$ where $\mathbf{n}$ is a principal direction of $\mathbf{U}$ and $\mathbf{C}$, then $\mathbf{U}d\mathbf{X} = dS\mathbf{Un} = dS\,\lambda\mathbf{n}$ so that

$$d\mathbf{x} = \mathbf{F}d\mathbf{X} = \mathbf{RU}d\mathbf{X} = \lambda dS(\mathbf{Rn})$$

That is, the deformed vector is in the direction of $\mathbf{Rn}$.

3.24 Lagrangian Strain Tensor

Let

$$\mathbf{E}^* = \frac{1}{2}(\mathbf{C}-\mathbf{I}) \tag{3.24.1}$$

where $\mathbf{C}$ is the right Cauchy-Green deformation tensor and $\mathbf{I}$ is the identity tensor. The tensor $\mathbf{E}^*$ is known as the **Lagrangian Finite Strain tensor**. We note that if there is no deformation, $\mathbf{C} = \mathbf{I}$ and $\mathbf{E}^* = \mathbf{0}$.

From Eq. (3.23.4) , we have

$$d\mathbf{x}^{(1)}\cdot d\mathbf{x}^{(2)} - d\mathbf{X}^{(1)}\cdot d\mathbf{X}^{(2)} = d\mathbf{X}^{(1)}\cdot(\mathbf{C}-\mathbf{I})d\mathbf{X}^{(2)}$$

i.e.,

$$d\mathbf{x}^{(1)}\cdot d\mathbf{x}^{(2)} - d\mathbf{X}^{(1)}\cdot d\mathbf{X}^{(2)} = 2d\mathbf{X}^{(1)}\cdot\mathbf{E}^* d\mathbf{X}^{(2)} \tag{3.24.2}$$

For a material element $d\mathbf{X} = dS\mathbf{e}_1$, deforming into $d\mathbf{x} = ds\mathbf{n}$, where $\mathbf{n}$ is a unit vector, Eq. (3.24.2) gives

$$\mathbf{e}_1\cdot\mathbf{E}^*\mathbf{e}_1 = \frac{ds^2 - dS^2}{2dS^2} \tag{3.24.3}$$

Thus,

$$E^*_{11} = \frac{ds^2 - dS^2}{2dS^2} \quad \text{for } d\mathbf{X} = dS\mathbf{e}_1 \tag{3.24.3a}$$

Similarly,

$$E^*_{22} = \frac{ds^2 - dS^2}{2dS^2} \quad \text{for } d\mathbf{X} = dS\mathbf{e}_2 \tag{3.24.3b}$$

$$E^*_{33} = \frac{ds^2 - dS^2}{2dS^2} \quad \text{for } d\mathbf{X} = dS\mathbf{e}_3 \tag{3.24.3c}$$

We note that for infinitesimal deformations, Eqs.(3.24.3) reduces to Eq. (3.8.1)

By considering two material elements $d\mathbf{X}^{(1)} = dS_1\mathbf{e}_1$ and $d\mathbf{X}^{(2)} = dS_2\mathbf{e}_2$, deforming into $d\mathbf{x}^{(1)} = ds_1\mathbf{m}$ and $d\mathbf{x}^{(2)} = ds_2\mathbf{n}$, where $\mathbf{m}$ and $\mathbf{n}$ are unit vectors, then Eq(3.25.2) gives

$$2E^*_{12} = \frac{ds_1 ds_2}{dS_1 dS_2}\cos(\mathbf{n},\mathbf{m}) \tag{3.24.4}$$

We note that for infinitesimal deformations, Eq. (3.24.4) reduces to Eq. (3.8.2).

The meanings for $2E_{13}^*$ and $2E_{23}^*$ can be established in a similar fashion.

We can also express the components of $\mathbf{E}^*$ in terms of the displacement components. From Eq. (3.24.1), Eq. (3.23.2) and Eq. (3.18.5), we obtain immediately

$$\mathbf{E}^* = \frac{1}{2}[\nabla\mathbf{u}+(\nabla\mathbf{u})^T]+\frac{1}{2}(\nabla\mathbf{u})^T(\nabla\mathbf{u}) \tag{3.24.5a}$$

in component form,

$$E_{ij}^* = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j}+\frac{\partial u_j}{\partial X_i}\right)+\frac{1}{2}\frac{\partial u_m}{\partial X_i}\frac{\partial u_m}{\partial X_j} \tag{3.24.5b}$$

and in long form,

$$E*_{11} = \frac{\partial u_1}{\partial X_1} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial X_1}\right)^2 + \left(\frac{\partial u_2}{\partial X_1}\right)^2 + \left(\frac{\partial u_3}{\partial X_1}\right)^2\right] \tag{3.24.6a}$$

$$E*_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2}+\frac{\partial u_2}{\partial X_1}\right) + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial X_1}\right)\left(\frac{\partial u_1}{\partial X_2}\right) + \left(\frac{\partial u_2}{\partial X_1}\right)\left(\frac{\partial u_2}{\partial X_2}\right) + \left(\frac{\partial u_3}{\partial X_1}\right)\left(\frac{\partial u_3}{\partial X_2}\right)\right] \tag{3.24.6b}$$

Other components can be similarly written down.

We note that for small values of displacement gradients these equations reduce to those of the infinitesimal deformation tensor.

Example 3.24.1

For the simple shear deformation

$$x_1 = X_1 + kX_2,\ \ x_2 = X_2,\ \ x_3 = X_3$$

(a) Compute the Lagrangian Strain tensor $[\mathbf{E}^*]$

(b) Referring to Fig. 3.12, by a simple geometrical consideration, find the deformed length for the element *OB* in Fig. 3.12.

(c) Compare the results of (b) with E_{22}^*

Solution. (a) Using the [C] obtained in Example 3.23.2, we easily obtain from the equation $2\mathbf{E}^* = \mathbf{C}-\mathbf{I}$ that

$$[\mathbf{E}^*] = \begin{bmatrix} 0 & \frac{k}{2} & 0 \\ \frac{k}{2} & \frac{k^2}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) From Fig. 3.12, we see from geometry, that $OB' = OB\sqrt{1+k^2}$

(c) $E^*_{22} = \frac{k^2}{2}$. Thus

$$\frac{(\Delta s)^2 - (\Delta S)^2}{2(\Delta S)^2} = \frac{k^2}{2}$$

Thus, $\Delta s = \Delta S\sqrt{1+k^2}$, this result is the same as that of (b). We note that if k is small then $\Delta s = \Delta S$ to the first order of k.

Example 3.24.2

Consider the displacement components corresponding to a uniaxial strain field:

$$u_1 = kX_1, \quad u_2 = u_3 = 0 \tag{i}$$

(a) Calculate both the finite Lagrangian strain tensor $\mathbf{E}^*$ and the infinitesimal strain tensor $\mathbf{E}$.

(b) Use the finite strain tensor E^*_{11} and the infinitesimal strain tensor E_{11} to calculate $\frac{\Delta s}{\Delta S}$ for the element $\Delta\mathbf{X} = \Delta S\mathbf{e}_1$.

(c) For an element $\Delta\mathbf{X} = \frac{\Delta S}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2)$, calculate $\frac{\Delta s}{\Delta S}$ from both the finite strain tensor $\mathbf{E}^*$ and the infinitesimal strain tensor $\mathbf{E}$.

Solution. (a)

$$[\nabla\mathbf{u}] = \begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = [\nabla\mathbf{u}]^s \tag{ii}$$

Thus, the infinitesimal strain tensor gives

$$[\mathbf{E}] = [\nabla\mathbf{u}]^s = \begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = [\nabla\mathbf{u}] \tag{iii}$$

and

$$[\mathbf{E}^*] = [\nabla\mathbf{u}]^s + \frac{1}{2}[\nabla\mathbf{u}]^T[\nabla\mathbf{u}] = \begin{bmatrix} k(1+\frac{k}{2}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{iv}$$

(b) Based on $E_{11}^* = k + \frac{k^2}{2}$, $\quad \frac{(\Delta s)^2 - (\Delta S)^2}{2(\Delta S)^2} = k + \frac{k^2}{2}$, therefore

$$(\Delta s)^2 = [1 + 2k + k^2](\Delta S)^2, \ \Delta s = (1 + k)\,\Delta S. \tag{v}$$

Based on $E_{11} = k$, $\quad \frac{\Delta s - \Delta S}{\Delta S} = k$, therefore

$$\Delta s = (1 + k)\Delta S \tag{vi}$$

We see both the finite and the infinitesimal strain tensor components lead to the same answer whether k is large or small.

(c) Let $\mathbf{e}_1' = \frac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2)$ then,

$$E_{11}^{*\prime} = \frac{1}{2}[1,1,0]\begin{bmatrix} k + \frac{k^2}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{k}{2} + \frac{k^2}{4} \tag{vii}$$

Thus, $\frac{(\Delta s)^2 - (\Delta S)^2}{2(\Delta S)^2} = \frac{k}{2} + \frac{k^2}{4}$, from which we find $\Delta s = \sqrt{1 + k + k^2/2}\ \Delta S$. This result is easily confirmed by the geometry in Fig. 3.12 for any value of k. On the other hand, the infinitesimal strain component

$$E_{11}' = \frac{1}{2}[1,1,0]\begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{k}{2}$$

Thus, $\frac{\Delta s - \Delta S}{\Delta S} = \frac{k}{2}$, from which we find $\Delta s = (1 + \frac{k}{2})\,\Delta S$. From Fig 3.12, we can easily conclude that this result is acceptable only if k is very small.

This example demonstrates clearly that in the case of finite deformations, the concept of unit elongation (i.e., change of length per unit length) is inadequate.

3.25 Left Cauchy-Green Deformation Tensor

Let

$$\mathbf{B} = \mathbf{V}^2 \tag{3.25.1}$$

where $\mathbf{V}$ is the left stretch tensor. The tensor $\mathbf{B}$ is known as the **left Cauchy-Green deformation tensor** (also known as the **Finger deformation tensor**). We note that if there is no deformation, $\mathbf{V} = \mathbf{B} = \mathbf{I}$.

Since $\mathbf{F} = \mathbf{VR}$, and $\mathbf{R}^T\mathbf{R} = \mathbf{I}$, it is easily verified that

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T \tag{3.25.2}$$

Thus, one can calculate $\mathbf{B}$ directly from the deformation gradient $\mathbf{F}$.

Substituting $\mathbf{F} = \mathbf{RU}$ in Eq. (3.25.2), we obtain the relation between $\mathbf{B}$ and $\mathbf{C}$ as follows:

$$\mathbf{B} = \mathbf{R}\mathbf{C}\mathbf{R}^T \text{ and } \mathbf{C} = \mathbf{R}^T\mathbf{B}\mathbf{R} \tag{3.25.3}$$

We also note that if $\mathbf{n}$ is an eigenvector of $\mathbf{C}$ with eigenvalue λ, then $\mathbf{Rn}$ is an eigenvector of $\mathbf{B}$ with the same eigenvalue λ.

The components of $\mathbf{B}$ have simple geometric meanings which are described below:

Consider a material element $d\mathbf{X} = dS\mathbf{n}$, where $\mathbf{n} = \mathbf{R}^T\mathbf{e}_1$, $\mathbf{R}$ being the rotation tensor associated with the deformation gradient $\mathbf{F}$. Then from Eq. (3.23.4), we have

$$ds^2 = dS^2\mathbf{n}\cdot\mathbf{C}\mathbf{n} = dS^2\mathbf{R}^T\mathbf{e}_1\cdot\mathbf{C}\mathbf{R}^T\mathbf{e}_1 = dS^2\mathbf{e}_1\cdot(\mathbf{C}\mathbf{R}^T)^T\mathbf{R}^T\mathbf{e}_1 = dS^2\mathbf{e}_1\cdot\mathbf{R}\mathbf{C}\mathbf{R}^T\mathbf{e}_1 \tag{3.25.4}$$

That is

$$ds^2 = dS^2\mathbf{e}_1\cdot\mathbf{B}\mathbf{e}_1 \quad \text{for } d\mathbf{X} = dS(\mathbf{R}^T\mathbf{e}_1), \tag{3.25.5}$$

That is

$$B_{11} = \left(\frac{ds}{dS}\right)^2 \text{ for a material element } d\mathbf{X} = dS(\mathbf{R}^T\mathbf{e}_1) \tag{3.25.5a}$$

similarly,

$$B_{22} = \left(\frac{ds}{dS}\right)^2 \text{ for a material element } d\mathbf{X} = dS(\mathbf{R}^T\mathbf{e}_2) \tag{3.25.5b}$$

$$B_{33} = \left(\frac{ds}{dS}\right)^2 \text{ for a material element } d\mathbf{X} = dS(\mathbf{R}^T\mathbf{e}_3) \tag{3.25.5c}$$

By considering two material elements $d\mathbf{X}^{(1)} = dS_1(\mathbf{R}^T\mathbf{e}_1)$ and $d\mathbf{X}^{(2)} = dS_2(\mathbf{R}^T\mathbf{e}_2)$ which deform into $d\mathbf{x}^{(1)} = ds_1\mathbf{m}$ and $d\mathbf{x}^{(2)} = ds_2\mathbf{n}$ where $\mathbf{m}$ and $\mathbf{n}$ are unit vectors having an angle of β between them, then Eq. (3.23.4) gives

$$ds_1 ds_2 \cos\beta = dS_1 dS_2(\mathbf{R}^T\mathbf{e}_1)\cdot\mathbf{C}(\mathbf{R}^T\mathbf{e}_2) = dS_1\, dS_2\mathbf{e}_1\cdot\mathbf{B}\mathbf{e}_2 \tag{3.25.6}$$

That is

$$B_{12} = \frac{ds_1\, ds_2}{dS_1 dS_2}\cos(d\mathbf{x}_1, d\mathbf{x}_2) \quad \text{for } d\mathbf{X}^{(1)} = dS_1(\mathbf{R}^T\mathbf{e}_1) \text{ and } d\mathbf{X}^{(2)} = dS_2(\mathbf{R}^T\mathbf{e}_2) \tag{3.25.6a}$$

Similarly

$$B_{13} = \frac{ds_1\, ds_3}{dS_1 dS_3}\cos(d\mathbf{x}_1, d\mathbf{x}_3) \quad \text{for } d\mathbf{X}^{(1)} = dS_1(\mathbf{R}^T\mathbf{e}_1) \text{ and } d\mathbf{X}^{(3)} = dS_3(\mathbf{R}^T\mathbf{e}_3) \tag{3.25.6b}$$

and

$$B_{23} = \frac{ds_2\, ds_3}{dS_2 dS_3}\cos(d\mathbf{x}_2, d\mathbf{x}_3) \quad \text{for } d\mathbf{X}^{(2)} = dS_2(\mathbf{R}^T\mathbf{e}_2) \text{ and } d\mathbf{X}^{(3)} = dS_3(\mathbf{R}^T\mathbf{e}_3) \tag{3.25.6c}$$

We can also express the components of $\mathbf{B}$ in terms of the displacement components. Using Eq. (3.18.5) , we have

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = (\mathbf{I}+\nabla\mathbf{u})(\mathbf{I}+\nabla\mathbf{u})^T = \mathbf{I}+[\nabla\mathbf{u}+(\nabla\mathbf{u})^T]+(\nabla\mathbf{u})(\nabla\mathbf{u})^T \tag{3.25.7a}$$

and in component form,

$$B_{ij} = \delta ij + \left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) + \frac{\partial u_i}{\partial X_m}\frac{\partial u_j}{\partial X_m} \tag{3.25.7b}$$

We note that for small displacement gradients, $\frac{1}{2}(B_{ij} - \delta_{ij})$ reduces to $2E_{ij}$ of Eq. (3.7.10a).

Example 3.25.1

For the simple shear deformation

$$x_1 = X_1 + kX_2, \quad x_2 = X_2, \quad x_3 = X_3 \tag{i}$$

(a) Obtain the left Cauchy-Green deformation tensor.

(b) Calculate $\mathbf{R}^T\mathbf{e}_1$ and $\mathbf{R}^T\mathbf{e}_2$

(c) Sketch both the undeformed and the deformed position for an element which was in the direction of $\mathbf{R}^T\mathbf{e}_2$. Calculate the stretch from the geometry in the figure and compare it with B_{22}.

Solution. (a)

$$[\mathbf{B}] = [\mathbf{F}\mathbf{F}^{\mathbf{T}}] = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+k^2 & k & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{i}$$

(b) From Eq. (3.23.8), we have

$$[\mathbf{R}] = \begin{bmatrix} (1+\frac{k^2}{4})^{-1/2} & \frac{k}{2}(1+\frac{k^2}{4})^{-1/2} & 0 \\ -\frac{k}{2}(1+\frac{k^2}{4})^{-1/2} & (1+\frac{k^2}{4})^{-1/2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{ii}$$

Thus,

$$\mathbf{R}^T\mathbf{e}_2 = (1+k^2/4)^{-1/2}[-\frac{k}{2}\mathbf{e}_1 + \mathbf{e}_2] \tag{iii}$$

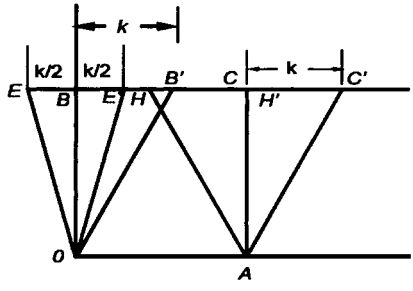

Fig. 3.13

(c) Referring to Fig. 3.13, OE was an element in the direction of $\mathbf{R}^T\mathbf{e}_2$. After deformation, it becomes OE', which obviously has the same length as OE. Thus, from geometry, the stretch for this element is unity. This checks with the value of B_{22} which is also unity.

3.26 Eulerian Strain Tensor

Let

$$\mathbf{e}^* \equiv \frac{1}{2}(\mathbf{I}-\mathbf{B}^{-1}) \tag{3.26.1}$$

where $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathbf{T}}$, then the tensor $\mathbf{e}^*$ is known as the **Eulerian Strain Tensor.** We note that if there if no deformation, then $\mathbf{B}^{-1} = \mathbf{I}$ and $\mathbf{e}^* = 0$.

The geometric meaning of the components of $\mathbf{e}^*$ and $\mathbf{B}^{-1}$ are described below:

From

$$d\mathbf{x} = \mathbf{F}d\mathbf{X} \tag{3.26.2}$$

we have

$$d\mathbf{X} = \mathbf{F}^{-1}d\mathbf{x} \tag{3.26.3}$$

where $\mathbf{F}^{-1}$is the inverse of $\mathbf{F}$. In rectangular Cartesian coordinates, Eq. (3.26.3) reads

$$dX_i = F_{ij}^{-1}dx_j \tag{3.26.4}$$

Thus,

$$F_{ij}^{-1} = \frac{\partial X_i}{\partial x_j} \tag{3.26.5}$$

where $X_i = X_i(x_1,x_2,x_3,t)$ is the inverse function of $x_i = x_i(X_1,X_2,X_3,t)$.

In other words, when rectangular Cartesian coordinates are used for both the reference and the current configuration,

$$[\mathbf{F}^{-1}] = \begin{bmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} & \frac{\partial X_1}{\partial x_3} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_2}{\partial x_3} \\ \frac{\partial X_3}{\partial x_1} & \frac{\partial X_3}{\partial x_2} & \frac{\partial X_3}{\partial x_3} \end{bmatrix} \tag{3.26.6}$$

Now,

$$d\mathbf{X}^{(1)}\cdot d\mathbf{X}^{(2)} = \mathbf{F}^{-1}d\mathbf{x}^{(1)}\cdot\mathbf{F}^{-1}d\mathbf{x}^{(2)} = d\mathbf{x}^{(1)}\cdot(\mathbf{F}^{-1})^T\mathbf{F}^{-1}d\mathbf{x}^{(2)} = d\mathbf{x}^{(1)}\cdot(\mathbf{F}\mathbf{F}^T)^{-1}d\mathbf{x}^{(2)}$$

i.e.,

$$d\mathbf{X}^{(1)}\cdot d\mathbf{X}^{(2)} = d\mathbf{x}^{(1)}\cdot\mathbf{B}^{-1}d\mathbf{x}^{(2)} \tag{3.26.7}$$

and

$$d\mathbf{x}^{(1)}\cdot d\mathbf{x}^{(2)}-d\mathbf{X}^{(1)}\cdot d\mathbf{X}^{(2)} = d\mathbf{x}^{(1)}\cdot(\mathbf{I}-\mathbf{B}^{-1})d\mathbf{x}^{(2)}$$

Or,

$$d\mathbf{x}^{(1)}\cdot d\mathbf{x}^{(2)}-d\mathbf{X}^{(1)}\cdot d\mathbf{X}^{(2)} = 2d\mathbf{x}^{(1)}\cdot\mathbf{e}^{*}\, d\mathbf{x}^{(2)} \tag{3.26.8}$$

Thus, if we consider a material element, which at time t is in the direction of $\mathbf{e}_1$,i.e., $d\mathbf{x} = ds\mathbf{e}_1$ and which at the reference time is $d\mathbf{X} = dS\mathbf{n}$, where $\mathbf{n}$ is a unit vector, then Eq. (3.26.7) and Eq. (3.26.8) give:

For $d\mathbf{x} = ds\mathbf{e}_1$

$$B_{11}^{-1} = \frac{dS^2}{ds^2}$$

and

$$e_{11}^{*} = \frac{(ds^2-dS^2)}{dS^2}$$

Similar meanings hold for the other diagonal elements of $\mathbf{B}^{-1}$ and $\mathbf{e}^{*}$.

By considering two material elements $d\mathbf{x}^{(1)} = ds_1\mathbf{e}_1$ and $d\mathbf{x}^{(2)} = ds_2\mathbf{e}_2$ at time t corresponding to $d\mathbf{X}^{(1)} = dS_1\mathbf{n}$ and $d\mathbf{X}^{(2)} = dS_2\mathbf{m}$ at the reference time, $\mathbf{n}$ and $\mathbf{m}$ are unit vectors, Eq. (3.26.7) and Eq. (3.26.8) give

$$B_{12}^{-1} = \frac{dS_1dS_2}{ds_1ds_2}\cos(\mathbf{n},\mathbf{m})$$

$$2e_{12}^{*} = 1-\frac{dS_1dS_2}{ds_1ds_2}\cos(\mathbf{n},\mathbf{m})$$

Similar meanings hold for the other off-diagonal elements of $\mathbf{B}^{-1}$ and $\mathbf{e}^{*}$.

We can also express $\mathbf{B}^{-1}$ and $\mathbf{e}^{*}$in terms of the displacement components:

From $\mathbf{u} = \mathbf{x}-\mathbf{X}$, we can write

$$\mathbf{X} = \mathbf{x}-\mathbf{u}(x_1,x_2,x_3,t) \tag{3.26.9a}$$

or,

$$X_i = x_i-u_i(x_1,x_2,x_3,t) \tag{3.26.9b}$$

where we have used the spatial description of the displacement field because we intend to differentiate this equation with respect to the spatial coordinates x_i. Thus,

$$\frac{\partial X_i}{\partial x_j} = \delta_{ij} - \frac{\partial u_i}{\partial x_j} \tag{3.26.10a}$$

or,

$$\mathbf{F}^{-1} = \mathbf{I} - \nabla_{\mathbf{x}}\mathbf{u} \tag{3.26.10b}$$

Thus , (dropping the subscript $\mathbf{x}$ from $\nabla_{\mathbf{x}}\,\mathbf{u}$)

$$\mathbf{B}^{-1} = (\mathbf{F}^{-1})^T\mathbf{F}^{-1} = (\mathbf{I}-\nabla_{\mathbf{x}}\mathbf{u})^T(\mathbf{I}-\nabla_{\mathbf{x}}\mathbf{u}) = \mathbf{I}-[\nabla_{\mathbf{x}}\mathbf{u}+(\nabla_{\mathbf{x}}\mathbf{u})^T]+(\nabla_{\mathbf{x}}\mathbf{u})^T(\nabla_{\mathbf{x}}\mathbf{u}) \tag{3.26.11}$$

and

$$\mathbf{e}^* = \frac{[\nabla_{\mathbf{x}}\mathbf{u}+(\nabla_{\mathbf{x}}\mathbf{u})^T]}{2} - \frac{(\nabla_{\mathbf{x}}\mathbf{u})^T(\nabla_{\mathbf{x}}\mathbf{u})}{2} \tag{3.26.12a}$$

In component form, Eq. (3.26.12a) is

$$e^*_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{1}{2}\frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j} \tag{3.26.12b}$$

and in long form,

$$e^*_{11} = \frac{\partial u_1}{\partial x_1} - \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\right)^2\right] \tag{3.26.13a}$$

$$e^*_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) - \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_1}\right)\left(\frac{\partial u_1}{\partial x_2}\right) + \left(\frac{\partial u_2}{\partial x_1}\right)\left(\frac{\partial u_2}{\partial x_2}\right) + \left(\frac{\partial u_3}{\partial x_1}\right)\left(\frac{\partial u_3}{\partial x_2}\right)\right] \tag{3.26.13b}$$

The other components can be similarly written down. We note that for infinitesimal deformation, $\frac{\partial u_i}{\partial X_j} \approx \frac{\partial u_i}{\partial x_j}$, Eq. (3.26.12) reduces to Eq. (3.7.10a).

Example 3.26.1

For the simple shear deformation

$$x_1 = X_1 + kX_2, \quad x_2 = X_2, \quad x_3 = X_3 \tag{i}$$

(a) Find $\mathbf{B}^{-1}$ and $\mathbf{e}^*$.

(b) Use the geometry in Fig. 3.13 to discuss the meaning of e_{11} and e_{22}.

Solution. (a)

$$[\mathbf{F}] = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad [\mathbf{F}^{-1}] = \begin{bmatrix} 1 & -k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{ii}$$

$$[\mathbf{B}^{-1}] = [\mathbf{F}^{-1}]^T[\mathbf{F}^{-1}] = \begin{bmatrix} 1 & 0 & 0 \\ -k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -k & 0 \\ -k & 1+k^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{iii}$$

$$[\mathbf{e}^*] = \frac{1}{2}[\mathbf{I}-\mathbf{B}^{-1}] = \begin{bmatrix} 0 & \frac{k}{2} & 0 \\ \frac{k}{2} & \frac{-k^2}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{iv}$$

(b) Since $e_{11}^* = 0$, an element which is in $\mathbf{e}_1$ direction in the deformed state (such as $B'C'$) had the same length in the undeformed state (BC in Fig. 3.13). Also since $e_{22}^* = -\frac{k^2}{2}$, an element which is in the $\mathbf{e}_2$ direction in the deformed state (such as AH') had a length AH given by the equation

$$(AH')^2-(AH)^2 = 2(AH')^2 e_{22}^* \tag{v}$$

from which one obtains

$$AH = (AH')\sqrt{1+k^2} \tag{vi}$$

This result checks with the geometry in Fig. 3.13.

3.27 Compatibility Conditions for Components of Finite Deformation Tensor

Whenever the three pathline equations (or equivalently, the three displacement functions) are given, one can always obtain the six components of $\mathbf{e}^*$ or $\mathbf{C}$ or $\mathbf{B}$ or $\mathbf{E}^*$ etc. by differentiation. On the other hand, if the six components of $\mathbf{e}^*$ etc. are given, there exist three displacement functions corresponding to the given strain components only when compatibility conditions for the components are satisfied. This is because in general, it is not possible to solve for three unknown functions from six differential equations. The compatibility conditions can in principle be obtained by the elimination of the three displacement components u_i from the six equations relating strain components with the displacement components such as Eqs. (3.26.12b) by partial differentiation and elimination as was done for the infinitesimal components (Section 3.16) The procedure is very lengthy and will be omitted. Only the conditions for e_{ij}^* are given below with the super * dropped for convenience:

$$\frac{\partial^2 e_{kn}}{\partial x_l\,\partial x_m}+\frac{\partial^2 e_{lm}}{\partial x_k\,\partial x_n}-\frac{\partial^2 e_{km}}{\partial x_l\,\partial x_n}-\frac{\partial^2 e_{ln}}{\partial x_k\,\partial x_m}$$
$$-\left(\frac{\partial e_{ks}}{\partial x_n}+\frac{\partial e_{ns}}{\partial x_k}-\frac{\partial e_{kn}}{\partial x_s}\right)\left(\frac{\partial e_{ls}}{\partial x_m}+\frac{\partial e_{ms}}{\partial x_l}-\frac{\partial e_{lm}}{\partial x_s}\right)+\left(\frac{\partial e_{ks}}{\partial x_m}+\frac{\partial e_{ms}}{\partial x_k}-\frac{\partial e_{km}}{\partial x_s}\right)\left(\frac{\partial e_{ls}}{\partial x_n}+\frac{\partial e_{ns}}{\partial x_l}-\frac{\partial e_{ln}}{\partial x_s}\right)$$
$$+2e_{rs}\left[-2\left(\frac{\partial e_{kr}}{\partial x_n}+\frac{\partial e_{nr}}{\partial x_k}-\frac{\partial e_{kn}}{\partial x_r}\right)\left(\frac{\partial e_{ls}}{\partial x_m}+\frac{\partial e_{ms}}{\partial x_l}-\frac{\partial e_{lm}}{\partial x_s}\right)-4\left(\frac{\partial e_{kr}}{\partial x_m}+\frac{\partial e_{mr}}{\partial x_k}-\frac{\partial e_{km}}{\partial x_r}\right)\left(\frac{\partial e_{ls}}{\partial x_n}+\frac{\partial e_{ns}}{\partial x_l}-\frac{\partial e_{ln}}{\partial x_s}\right)\right.$$
$$\left.+4\left(\frac{\partial e_{kr}}{\partial x_n}+\frac{\partial e_{nr}}{\partial x_k}-\frac{\partial e_{kn}}{\partial x_r}\right)\left(\frac{\partial e_{ls}}{\partial x_m}+\frac{\partial e_{ms}}{\partial x_l}-\frac{\partial e_{lm}}{\partial x_s}\right)\right]=0 \tag{3.27.1}$$

We note that for infinitesimal deformation, Eq. (3.27.1) reduces to

$$\frac{\partial^2 e_{kn}}{\partial x_l\,\partial x_m}+\frac{\partial^2 e_{lm}}{\partial x_k\,\partial x_n}-\frac{\partial^2 e_{km}}{\partial x_l\,\partial x_n}-\frac{\partial^2 e_{ln}}{\partial x_k\,\partial x_m}=0 \tag{3.27.2}$$

which are the same as those given in Sect. 3.16.

3. 28 Change of Area due to Deformation

Consider two material elements $d\mathbf{X}^{(1)} = dS_1\mathbf{e}_1$ and $d\mathbf{X}^{(2)} = dS_2\mathbf{e}_2$ emanating from $\mathbf{X}$.. The rectangular area formed by $d\mathbf{X}^{(1)}$ and $d\mathbf{X}^{(2)}$ at the reference time t_o is given by

$$d\mathbf{A}_o = d\mathbf{X}^{(1)} \times d\mathbf{X}^{(2)} = dS_1\, dS_2\, \mathbf{e}_3 = dA_o\mathbf{e}_3 \tag{3.28.1}$$

where dA_o is the magnitude of the undeformed area and $\mathbf{e}_3$ is normal to the area. At time t, $d\mathbf{X}^{(1)}$ deforms into $d\mathbf{x}^{(1)} = \mathbf{F}d\mathbf{X}^{(1)}$ and $d\mathbf{X}^{(2)}$ deforms into $d\mathbf{x}^{(2)} = \mathbf{F}d\mathbf{X}^{(2)}$ and the area is

$$d\mathbf{A} = \mathbf{F}d\mathbf{X}^{(1)} \times \mathbf{F}d\mathbf{X}^{(2)} = dS_1\, dS_2\, \mathbf{F}\mathbf{e}_1 \times \mathbf{F}\mathbf{e}_2 = dA_o\, \mathbf{F}\mathbf{e}_1 \times \mathbf{F}\mathbf{e}_2 \tag{3.28.2}$$

Thus, the orientation of the deformed area is normal to $\mathbf{F}\mathbf{e}_1$ and $\mathbf{F}\mathbf{e}_2$. Let this direction be denoted by the unit vector $\mathbf{n}$, i.e.,

$$\mathbf{dA} = dA\mathbf{n} \tag{3.28.3}$$

then,

$$dA\,\mathbf{n} = dA_o\,(\mathbf{F}\mathbf{e}_1 \times \mathbf{F}\mathbf{e}_2) \tag{i}$$

From the above equation, it is clear that

$$\mathbf{F}\mathbf{e}_1 \cdot dA\mathbf{n} = \mathbf{F}\mathbf{e}_2 \cdot dA\mathbf{n} = 0 \tag{ii}$$

and

$$\mathbf{F}\mathbf{e}_3 \cdot \mathbf{dA} = dA_o\,(\mathbf{F}\mathbf{e}_3 \cdot \mathbf{F}\mathbf{e}_1 \times \mathbf{F}\mathbf{e}_2) \tag{iii}$$

Recall that for any vectors **a**, **b**, and **c**,

$\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}$ = determinant whose rows are components of **a**, **b**, and **c**. Therefore

$$\mathbf{F}\mathbf{e}_3 \cdot \mathbf{F}\mathbf{e}_1 \times \mathbf{F}\mathbf{e}_2 = \det \mathbf{F} \tag{iv}$$

Eq. (iii) becomes

$$\mathbf{F}\mathbf{e}_3 \cdot dA\mathbf{n} = dA_o \det \mathbf{F} \tag{v}$$

Using the definition of transpose of a tensor, Eqs. (ii) become

$$\mathbf{e}_1 \cdot \mathbf{F}^T\mathbf{n} = \mathbf{e}_2 \cdot \mathbf{F}^T\mathbf{n} = 0 \tag{vi}$$

and Eq. (v) becomes

$$\mathbf{e}_3 \cdot \mathbf{F}^T\mathbf{n} = \left(\frac{dA_o}{dA}\right) \det \mathbf{F} \tag{vii}$$

Thus, $\mathbf{F}^T\mathbf{n}$ is in the direction of $\mathbf{e}_3$, so that

$$\mathbf{F}^T\mathbf{n} = \frac{dA_o}{dA}(\det \mathbf{F})\,\mathbf{e}_3 \tag{vii}$$

Therefore,

$$dA\,\mathbf{n} = dA_o\,(\det \mathbf{F})\,(\mathbf{F}^{-1})^T\mathbf{e}_3 \tag{3.28.4}$$

Equation (3.28.4) states that the deformed area has a normal in the direction of $(\mathbf{F}^{-1})^T\mathbf{e}_3$ and with a magnitude given by

$$dA = dA_o\,(\det \mathbf{F})\,|(\mathbf{F}^{-1})^T\mathbf{e}_3| \tag{3.28.5}$$

In deriving Eq. (3.28.4), we have chosen the initial area to be the rectangular area formed by the Cartesian base vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, it can be shown that the formula remains valid for any material area except that $\mathbf{e}_3$ be replaced by the normal vector of the undeformed area $\mathbf{n}_o$. That is in general,

$$dA\,\mathbf{n} = dA_o\,(\det \mathbf{F})\,(\mathbf{F}^{-1})^T\mathbf{n}_o \tag{3.28.6}$$

3.29 Change of Volume due to Deformation

Consider three material elements

$$d\mathbf{X}^{(1)} = dS_1\mathbf{e}_1,\quad d\mathbf{X}^{(2)} = dS_2\mathbf{e}_2 \text{ and } d\mathbf{X}^{(3)} = dS_3\mathbf{e}_3,$$

emanating from **X**.. The rectangular volume formed by $d\mathbf{X}^{(1)}$, $d\mathbf{X}^{(2)}$ and $d\mathbf{X}^{(3)}$ at the reference time t_o is given by

$$dV_o = dS_1\, dS_2\, dS_3 \tag{3.29.1}$$

At time t, $d\mathbf{X}^{(1)}$ deforms into $d\mathbf{x}^{(1)} = \mathbf{F}d\mathbf{X}^{(1)}$, $d\mathbf{X}^{(2)}$ deforms into $d\mathbf{x}^{(2)} = \mathbf{F}d\mathbf{X}^{(2)}$ and $d\mathbf{X}^{(3)}$deforms into $d\mathbf{x}^{(3)} = \mathbf{F}d\mathbf{X}^{(3)}$ and the volume is

$$\begin{aligned} dV &= \mathbf{F}d\mathbf{X}^{(1)} \cdot \mathbf{F}d\mathbf{X}^{(2)} \times \mathbf{F}d\mathbf{X}^{(3)} = dS_1\, dS_2\, dS_3\, (\mathbf{Fe}_1 \cdot \mathbf{Fe}_2 \times \mathbf{Fe}_3) \\ &= dV_o\, (\mathbf{Fe}_1 \cdot \mathbf{Fe}_2 \times \mathbf{Fe}_3) \end{aligned} \tag{3.29.2}$$

That is,

$$dV = (\det \mathbf{F})dV_o \tag{3.29.3}$$

Since $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ and $\mathbf{B} = \mathbf{F}\mathbf{F}^T$, therefore

$$\det\mathbf{C} = \det\mathbf{B} = (\det\mathbf{F})^2 \tag{3.29.4}$$

Thus, Eq. (3.29.3) can also be written as

$$dV = \sqrt{\det\mathbf{C}}\, dV_o = \sqrt{\det\mathbf{B}}\, dV_o \tag{3.29.5}$$

We note that for an incompressible material, $dV = dV_o$, so that

$$\det\mathbf{F} = \det\mathbf{C} = \det\mathbf{B} = 1 \tag{3.29.6}$$

We note also that due to Eq. (3.29.3), the conservation of mass equation can be written as:

$$\rho = \frac{\rho_o}{\det\mathbf{F}} \tag{3.29.7}$$

Example 3.29.1

Consider the deformation given by

$$x_1 = \lambda_1 X_1\ , \qquad x_2 = -\lambda_3 X_3\ , \qquad x_3 = \lambda_2 X_2 \tag{i}$$

(a)Find the deformed volume of the unit cube shown in Fig. 3.14.

(b)Find the deformed area of *OABC*.

(c) Find the rotation tensor and the axial vector of the antisymmetric part of the rotation tensor.

Solution. (a) From Eq. (i),

$$[\mathbf{F}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & -\lambda_3 \\ 0 & \lambda_2 & 0 \end{bmatrix}$$

Thus,

$$\det\mathbf{F} = \lambda_1 \lambda_2 \lambda_3 \tag{ii}$$

Since det **F** is a constant, from the equation

$$dV = (\det\mathbf{F})\, dV_o$$

we have, with $\Delta V_o = 1$,

$$\Delta V = \lambda_1 \lambda_2 \lambda_3 \tag{iii}$$

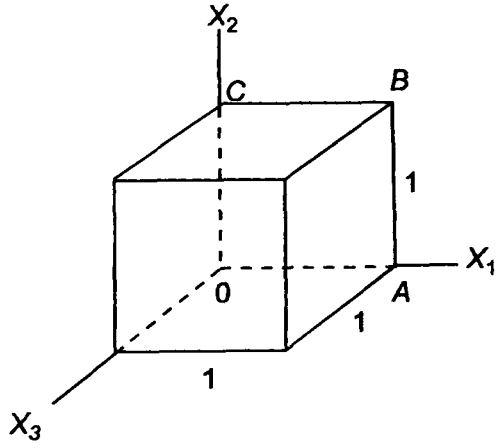

Fig. 3.14

(b) Using Eq. (3.28.6), with $\Delta A_o = 1$, and $\mathbf{n}_o = -\mathbf{e}_3$, we have

$$\Delta A\, \mathbf{n} = (1)\,(\det \mathbf{F})\,(\mathbf{F}^{-1})^T\, \mathbf{n}_o =$$

$$(\lambda_1 \lambda_2 \lambda_3) \begin{bmatrix} \frac{1}{\lambda_1} & 0 & 0 \\ 0 & 0 & -\frac{1}{\lambda_3} \\ 0 & \frac{1}{\lambda_2} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda_1 \lambda_2 \\ 0 \end{bmatrix} \tag{iv}$$

i.e.,

$$\Delta A\,\mathbf{n} = \lambda_1 \lambda_2\, \mathbf{e}_2$$

Thus, the area *OABC*, which was of unit area and having a normal in the direction of $-\mathbf{e}_3$ becomes an area whose normal is in the direction of $\mathbf{e}_2$ and with a magnitude of $\lambda_1 \lambda_2$.

(c)

$$[\mathbf{U}]^2 = [\mathbf{F}]^T[\mathbf{F}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 \\ 0 & -\lambda_3 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & -\lambda_3 \\ 0 & \lambda_2 & 0 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^3 \end{bmatrix}$$

$$[\mathbf{U}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{v}$$

$$[\mathbf{R}] = [\mathbf{F}][\mathbf{U}]^{-1} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & -\lambda_3 \\ 0 & \lambda_2 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\lambda_1} & 0 & 0 \\ 0 & \frac{1}{\lambda_2} & 0 \\ 0 & 0 & \frac{1}{\lambda_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{vi}$$

It is easily verified that $\mathbf{R}$ corresponds to a 90° rotation about the $\mathbf{e}_1$, which is the axial vector of the antisymmetric part of $\mathbf{R}$

3.30 Components of Deformation Tensors in other Coordinates

The deformation gradient $\mathbf{F}$ transforms a differential material element $d\mathbf{X}$ in the reference configuration into a material element $d\mathbf{x}$ in the current configuration in accordance with the equation

$$d\mathbf{x} = \mathbf{F}d\mathbf{X} \tag{3.30.1}$$

where

$$\mathbf{x} = \mathbf{x}(X_1, X_2, X_3, t)$$

describes the motion. If the same rectangular coordinate system is used for both the reference and the current configurations, then since the set of base vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is the same at every point, we have

$$\mathbf{e}_i \cdot d\mathbf{x} = \mathbf{e}_i \cdot \mathbf{F}(dX_m \mathbf{e}_m) = dX_m(\mathbf{e}_i \cdot \mathbf{F}\mathbf{e}_m) = F_{im} dX_m \tag{i}$$

That is

$$dx_i = F_{im} dX_m \tag{ii}$$

Thus

$$F_{im} = \frac{\partial x_i}{\partial X_m} \tag{3.30.2}$$

i.e.,

$$[\mathbf{F}] = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \tag{3.30.3}$$

We have already used this matrix for computing the components of **F** in a few examples above. The situation is more complicated if the base vectors at the reference configuration are different from those at the current configuration. Such situations arise not only in the case where different coordinate systems are used for the two configuration (for example, a rectangular coordinate system for the reference and a cylindrical coordinates for the current configuration , see (B) below) , but also in cases where the same curvilinear coordinates are used for the two configurations. The following are examples.

(A) Cylindrical coordinate system for both the reference and the current configuration

Let

$$r = r(r_o, \theta_o, z_o, t), \quad \theta = \theta(r_o, \theta_o, z_o, t), \qquad z = z(r_o, \theta_o, z_o, t) \tag{iii}$$

be the pathline equations. We shall show in the following that

$$\mathbf{F}\mathbf{e}_{or} = \left(\frac{\partial r}{\partial r_o}\right)\mathbf{e}_r + \left(\frac{r\,\partial\theta}{\partial r_o}\right)\mathbf{e}_\theta + \left(\frac{\partial z}{\partial r_o}\right)\mathbf{e}_z \tag{3.30.4a}$$

$$\mathbf{F}\mathbf{e}_{o\theta} = \left(\frac{\partial r}{r_o\partial\theta_o}\right)\mathbf{e}_r + \left(\frac{r\,\partial\theta}{r_o\partial\theta_o}\right)\mathbf{e}_\theta + \left(\frac{\partial z}{r_o\partial\theta_o}\right)\mathbf{e}_z \tag{3.30.4b}$$

$$\mathbf{F}\mathbf{e}_{oz} = \left(\frac{\partial r}{\partial z_o}\right)\mathbf{e}_r + \left(\frac{r\,\partial\theta}{\partial z_o}\right)\mathbf{e}_\theta + \left(\frac{\partial z}{\partial z_o}\right)\mathbf{e}_z \tag{3.30.4c}$$

and

$$\mathbf{F}^T\mathbf{e}_r = \left(\frac{\partial r}{\partial r_o}\right)\mathbf{e}_{or} + \left(\frac{\partial r}{r_o\partial\theta_o}\right)\mathbf{e}_{o\theta} + \left(\frac{\partial r}{\partial z_o}\right)\mathbf{e}_{oz} \tag{3.30.5a}$$

$$\mathbf{F}^T\mathbf{e}_\theta = \left(\frac{r\partial\theta}{\partial r_o}\right)\mathbf{e}_{or} + \left(\frac{r\partial\theta}{r_o\partial\theta_o}\right)\mathbf{e}_{o\theta} + \left(\frac{r\partial\theta}{\partial z_o}\right)\mathbf{e}_{oz} \tag{3.30.5b}$$

$$\mathbf{F}^T\mathbf{e}_z = \left(\frac{\partial z}{\partial r_o}\right)\mathbf{e}_{or} + \left(\frac{\partial z}{r_o\partial\theta_o}\right)\mathbf{e}_{o\theta} + \left(\frac{\partial z}{\partial z_o}\right)\mathbf{e}_{oz} \tag{3.30.5c}$$

where $\mathbf{e}_{oi}$ denotes base vectors at the reference position and $\mathbf{e}_i$ those at the current position.

Substituting

$$d\mathbf{x} = dr\mathbf{e}_r + rd\,\theta\mathbf{e}_\theta + dz\mathbf{e}_z \quad \text{and} \quad d\mathbf{X} = dr_o\mathbf{e}_{or} + r_o d\,\theta_o\mathbf{e}_{o\theta} + dz_o\mathbf{e}_{oz} \tag{iv}$$

in the equation $d\mathbf{x} = \mathbf{F}d\mathbf{X}$, we obtain

$$dr = dr_o(\mathbf{e}_r\cdot\mathbf{F}\mathbf{e}_{or}) + r_o d\,\theta_o\,(\mathbf{e}_r\cdot\mathbf{F}\mathbf{e}_{o\,\theta}) + dz_o(\mathbf{e}_r\cdot\mathbf{F}\mathbf{e}_{oz}) \tag{v}$$

$$rd\,\theta = dr_o(\mathbf{e}_\theta\cdot\mathbf{F}\mathbf{e}_{or}) + r_o d\,\theta_o\,(\mathbf{e}_\theta\cdot\mathbf{F}\mathbf{e}_{o\theta}) + dz_o(\mathbf{e}_\theta\cdot\mathbf{F}\mathbf{e}_{oz}) \tag{vi}$$

etc. Thus,

$$\mathbf{e}_r\cdot\mathbf{F}\mathbf{e}_{or} = \frac{\partial r}{\partial r_o}, \qquad \mathbf{e}_r\cdot\mathbf{F}\mathbf{e}_{o\theta} = \frac{\partial r}{r_o\partial\theta_o}, \qquad \mathbf{e}_r\cdot\mathbf{F}\mathbf{e}_{oz} = \frac{\partial r}{\partial z_o} \tag{3.30.6a}$$

$$\mathbf{e}_\theta\cdot\mathbf{F}\mathbf{e}_{or} = \frac{r\,\partial\theta}{\partial r_o}, \qquad \mathbf{e}_\theta\cdot\mathbf{F}\mathbf{e}_{o\theta} = \frac{r\partial\theta}{r_o\partial\theta_o}, \qquad \mathbf{e}_\theta\cdot\mathbf{F}\mathbf{e}_{oz} = \frac{r\partial\theta}{\partial z_o} \tag{3.30.6b}$$

$$\mathbf{e}_z\cdot\mathbf{F}\mathbf{e}_{or} = \frac{\partial z}{\partial r_o}, \qquad \mathbf{e}_z\cdot\mathbf{F}\mathbf{e}_{o\theta} = \frac{\partial z}{r_o\partial\theta_o}, \qquad \mathbf{e}_z\cdot\mathbf{F}\mathbf{e}_{oz} = \frac{\partial z}{\partial z_o} \tag{3.30.6c}$$

These equations are equivalent to Eqs. (3.30.4).

The matrix

$$[\mathbf{F}] = \begin{bmatrix} \dfrac{\partial r}{\partial r_o} & \dfrac{\partial r}{r_o\partial\theta_o} & \dfrac{\partial r}{\partial z_o} \\ \dfrac{r\partial\theta}{\partial r_o} & \dfrac{r\,\partial\theta}{r_o\partial\theta_o} & \dfrac{r\,\partial\theta}{\partial z_o} \\ \dfrac{\partial z}{\partial r_o} & \dfrac{\partial z}{r_o\partial\theta_o} & \dfrac{\partial z}{\partial z_o} \end{bmatrix}_{\{\mathbf{e}_i\},\ \{\mathbf{e}_{o_j}\}} \tag{3.30.7}$$

is based on two sets of bases, one at the reference configuration ($\mathbf{e}_{or}$, $\mathbf{e}_{o\theta}$, $\mathbf{e}_{oz}$) and the other the current configuration ($\mathbf{e}_r$,$\mathbf{e}_\theta$,$\mathbf{e}_z$). The components in this matrix is called **the two point components of the tensor F** with respect to ($\mathbf{e}_r$,$\mathbf{e}_\theta$,$\mathbf{e}_z$) and ($\mathbf{e}_{or}$, $\mathbf{e}_{o\theta}$, $\mathbf{e}_{oz}$).

By using the definition of transpose of a tensor, one can easily establish Eqs. (3.30.5) from Eq. (3.30.4). [see Prob. 3.73]

The components of the left Cauchy-Green tensor, with respect to the basis at the spatial position **x** can be obtained as follows. From the definition $\mathbf{B} = \mathbf{F}\mathbf{F}^T$, and by using Eqs. (3.30.4) and (3.30.5) we obtain

$$B_{rr} = \mathbf{e}_r \cdot \mathbf{B}\mathbf{e}_r = \mathbf{e}_r \cdot \mathbf{F}\mathbf{F}^T\mathbf{e}_r$$
$$= \left(\frac{\partial r}{\partial r_o}\right)\mathbf{e}_r \cdot \mathbf{F}\mathbf{e}_{or} + \left(\frac{\partial r}{r_o \partial \theta_o}\right)\mathbf{e}_r \cdot \mathbf{F}\mathbf{e}_{o\theta} + \left(\frac{\partial r}{\partial z_o}\right)\mathbf{e}_r \cdot \mathbf{F}\mathbf{e}_{oz}$$
$$= \left(\frac{\partial r}{\partial r_o}\right)^2 + \left(\frac{\partial r}{r_o \partial \theta_o}\right)^2 + \left(\frac{\partial r}{\partial z_o}\right)^2 \tag{vii}$$

Similarly,

$$B_{r\theta} = \mathbf{e}_r \cdot \mathbf{B}\mathbf{e}_\theta = \mathbf{e}_r \cdot \mathbf{F}\mathbf{F}^T\mathbf{e}_\theta$$
$$= \left(\frac{r\partial\theta}{\partial r_o}\right)\left(\frac{\partial r}{\partial r_o}\right) + \left(\frac{r\partial\theta}{r_o\partial\theta_o}\right)\left(\frac{\partial r}{r_o\partial\theta}\right) + \left(\frac{r\partial\theta}{\partial z_o}\right)\left(\frac{\partial r}{\partial z_o}\right) \tag{viii}$$

Other components can be obtained in the same way [see Prob. 3.74]. We list all the components below:

$$B_{rr} = \left(\frac{\partial r}{\partial r_o}\right)^2 + \left(\frac{\partial r}{r_o \partial \theta_o}\right)^2 + \left(\frac{\partial r}{\partial z_o}\right)^2 \tag{3.30.8a}$$

$$B_{\theta\theta} = \left(\frac{r\partial\theta}{\partial r_o}\right)^2 + \left(\frac{r\partial\theta}{r_o\partial\theta_o}\right)^2 + \left(\frac{r\partial\theta}{\partial z_o}\right)^2 \tag{3.30.8b}$$

$$B_{zz} = \left(\frac{\partial z}{\partial r_o}\right)^2 + \left(\frac{\partial z}{r_o \partial \theta_o}\right)^2 + \left(\frac{\partial z}{\partial z_o}\right)^2 \tag{3.30.8c}$$

$$B_{r\theta} = \left(\frac{\partial r}{\partial r_o}\right)\left(\frac{r\partial\theta}{\partial r_o}\right) + \left(\frac{\partial r}{r_o\partial\theta_o}\right)\left(\frac{r\partial\theta}{r_o\partial\theta_o}\right) + \left(\frac{\partial r}{\partial z_o}\right)\left(\frac{r\partial\theta}{\partial z_o}\right) \tag{3.30.8d}$$

$$B_{rz} = \left(\frac{\partial r}{\partial r_o}\right)\left(\frac{\partial z}{\partial r_o}\right) + \left(\frac{\partial r}{r_o\partial\theta_o}\right)\left(\frac{\partial z}{r_o\partial\theta_o}\right) + \left(\frac{\partial r}{\partial z_o}\right)\left(\frac{\partial z}{\partial z_o}\right) \tag{3.30.8e}$$

$$B_{z\theta} = \left(\frac{\partial z}{\partial r_o}\right)\left(\frac{r\partial\theta}{\partial r_o}\right) + \left(\frac{\partial z}{r_o\partial\theta_o}\right)\left(\frac{r\partial\theta}{r_o\partial\theta_o}\right) + \left(\frac{\partial z}{\partial z_o}\right)\left(\frac{r\partial\theta}{\partial z_o}\right) \tag{3.30.8f}$$

The components of $\mathbf{B}^{-1}$ can be obtained either by inverting the tensor $\mathbf{B}$ or by inverting the pathline equations. Let

$$r_o = r_o(r, \theta, z, t), \quad \theta_o = \theta_o(r, \theta, z, t), \quad z_o = z_o(r, \theta, z, t) \tag{ix}$$

be the inverse of Eq. (iii). Then from the equation $d\mathbf{X} = \mathbf{F}^{-1}d\mathbf{x}$, one can obtain

$$B_{rr}^{-1} = \left(\frac{\partial r_o}{\partial r}\right)^2 + \left(\frac{r_o \partial \theta_o}{\partial r}\right)^2 + \left(\frac{\partial z_o}{\partial r}\right)^2 \tag{3.30.8g}$$

$$B_{\theta\theta}^{-1} = \left(\frac{\partial r_o}{r\,\partial \theta}\right)^2 + \left(\frac{r_o \partial \theta_o}{r\,\partial \theta}\right)^2 + \left(\frac{\partial z_o}{r\,\partial \theta}\right)^2 \tag{3.30.8h}$$

$$B_{zz}^{-1} = \left(\frac{\partial r_o}{\partial z}\right)^2 + \left(\frac{r_o \partial \theta_o}{\partial z}\right)^2 + \left(\frac{\partial z_o}{\partial z}\right)^2 \tag{3.30.8i}$$

$$B_{r\theta}^{-1} = \left(\frac{\partial r_o}{\partial r}\right)\left(\frac{\partial r_o}{r\,\partial \theta}\right) + \left(\frac{r_o \partial \theta_o}{\partial r}\right)\left(\frac{r_o \partial \theta_o}{r\,\partial \theta}\right) + \left(\frac{\partial z_o}{\partial r}\right)\left(\frac{\partial z_o}{r\,\partial \theta}\right) \tag{3.30.8j}$$

$$B_{rz}^{-1} = \left(\frac{\partial r_o}{\partial r}\right)\left(\frac{\partial r_o}{\partial z}\right) + \left(\frac{r_o \partial \theta_o}{\partial r}\right)\left(\frac{r_o \partial \theta_o}{\partial z}\right) + \left(\frac{\partial z_o}{\partial r}\right)\left(\frac{\partial z_o}{\partial z}\right) \tag{3.30.8k}$$

$$B_{z\theta}^{-1} = \left(\frac{\partial r_o}{\partial z}\right)\left(\frac{\partial r_o}{r\,\partial \theta}\right) + \left(\frac{r_o \partial \theta_o}{\partial z}\right)\left(\frac{r_o \partial \theta_o}{r\,\partial \theta}\right) + \left(\frac{\partial z_o}{\partial z}\right)\left(\frac{\partial z_o}{r\,\partial \theta}\right) \tag{3.30.8l}$$

The components of the right Cauchy-Green tensor **C**, with respect to the basis at the reference position **X** can be obtained as follows. From the definition $\mathbf{C} = \mathbf{F}^T\mathbf{F}$, and by using Eqs. (3.30.4) and (3.30.5) we obtain

$$\begin{aligned} C_{r_o r_o} &= \mathbf{e}_{or} \cdot \mathbf{C}\, e_{or} = \mathbf{e}_{or} \cdot \mathbf{F}^T \mathbf{F} \mathbf{e}_{or} \\ &= \left(\frac{\partial r}{\partial r_o}\right) \mathbf{e}_{or} \cdot \mathbf{F}^T \mathbf{e}_r + \left(\frac{r \partial \theta}{\partial r_o}\right) \mathbf{e}_{or} \cdot \mathbf{F}^T \mathbf{e}_\theta + \left(\frac{\partial z}{\partial r_o}\right) \mathbf{e}_{or} \cdot \mathbf{F}^T \mathbf{e}_z \\ &= \left(\frac{\partial r}{\partial r_o}\right)^2 + \left(\frac{r\,\partial \theta}{\partial r_o}\right)^2 + \left(\frac{\partial z}{\partial r_o}\right)^2 \end{aligned} \tag{x}$$

Similarly,

$$\begin{aligned} C_{r_o \theta_o} &= \mathbf{e}_{or} \cdot \mathbf{C}\, e_{o\theta} = \mathbf{e}_{or} \cdot \mathbf{F}^T \mathbf{F} \mathbf{e}_{o\theta} \\ &= \left(\frac{\partial r}{r_o \partial \theta_o}\right)\left(\frac{\partial r}{\partial r_o}\right) + \left(\frac{r \partial \theta}{r_o \partial \theta_o}\right)\left(\frac{r \partial \theta}{\partial r_o}\right) + \left(\frac{\partial z}{r_o \partial \theta_o}\right)\left(\frac{\partial z}{\partial r_o}\right) \end{aligned} \tag{xi}$$

Other components can be obtained in the same way [see Prob. 3.75]. We list all the components below:

$$C_{r_o r_o} = \left(\frac{\partial r}{\partial r_o}\right)^2 + \left(\frac{r\,\partial \theta}{\partial r_o}\right)^2 + \left(\frac{\partial z}{\partial r_o}\right)^2 \tag{3.30.9a}$$

$$C_{\theta_o \theta_o} = \left(\frac{\partial r}{r_o \partial \theta_o}\right)^2 + \left(\frac{r \partial \theta}{r_o \partial \theta_o}\right)^2 + \left(\frac{\partial z}{r_o \partial \theta_o}\right)^2 \tag{3.30.9b}$$

$$C_{z_o z_o} = \left(\frac{\partial r}{\partial z_o}\right)^2 + \left(\frac{r\,\partial\theta}{\partial z_o}\right)^2 + \left(\frac{\partial z}{\partial z_o}\right)^2 \tag{3.30.9c}$$

$$C_{r_o\theta_o} = \left(\frac{\partial r}{\partial r_o}\right)\left(\frac{\partial r}{r_o\partial\theta_o}\right) + \left(\frac{r\partial\theta}{\partial r_o}\right)\left(\frac{r\partial\theta}{r_o\partial\theta_o}\right) + \left(\frac{\partial z}{\partial r_o}\right)\left(\frac{\partial z}{r_o\partial\theta_o}\right) \tag{3.30.9d}$$

$$C_{r_o z_o} = \left(\frac{\partial r}{\partial r_o}\right)\left(\frac{\partial r}{\partial z_o}\right) + \left(\frac{r\partial\theta}{\partial r_o}\right)\left(\frac{r\partial\theta}{\partial z_o}\right) + \left(\frac{\partial z}{\partial r_o}\right)\left(\frac{\partial z}{\partial z_o}\right) \tag{3.30.9e}$$

$$C_{z_o\theta_o} = \left(\frac{\partial r}{\partial z_o}\right)\left(\frac{\partial r}{r_o\partial\theta_o}\right) + \left(\frac{r\,\partial\theta}{\partial z_o}\right)\left(\frac{r\partial\theta}{r_o\partial\theta_o}\right) + \left(\frac{\partial z}{\partial z_o}\right)\left(\frac{\partial z}{r_o\partial\theta_o}\right) \tag{3.30.9f}$$

Again, the components of $\mathbf{C}^{-1}$ can be obtained by using the equation $d\mathbf{X} = \mathbf{F}^{-1}d\mathbf{x}$ and Eq. (ix). We list here two of the six components. The other four components can be easily written down following the patterns of these two equations.

$$C^{-1}_{r_o r_o} = \left(\frac{\partial r_o}{\partial r}\right)^2 + \left(\frac{\partial r_o}{r\partial\theta}\right)^2 + \left(\frac{\partial r_o}{\partial z}\right)^2 \tag{3.30.9g}$$

$$C^{-1}_{r_o\theta_o} = \left(\frac{\partial r_o}{\partial r}\right)\left(\frac{r_o\partial\theta_o}{\partial r}\right) + \left(\frac{\partial r_o}{r\partial\theta}\right)\left(\frac{r_o\partial\theta_o}{r\partial\theta}\right) + \left(\frac{\partial r_o}{\partial z}\right)\left(\frac{r_o\partial\theta_o}{\partial z}\right) \tag{3.30.9h}$$

(B) Cylindrical coordinates (r,θ,z) for the current configuration and rectangular Cartesian coordinates (X,Y,Z) for the reference configuration.

Let

$$r = r(X,Y,Z,t) \quad \theta = \theta(X,Y,Z,t) \quad z = z(X,Y,Z,t) \tag{xii}$$

describe the motions. Then using the same procedure as described for the case where one single cylindrical coordinate is used, it can be derived that [see Prob.3.76].

$$\mathbf{F}\mathbf{e}_X = \frac{\partial r}{\partial X}\mathbf{e}_r + \frac{r\partial\theta}{\partial X}\mathbf{e}_\theta + \frac{\partial z}{\partial X}\mathbf{e}_z \tag{3.30.10a}$$

$$\mathbf{F}\mathbf{e}_Y = \frac{\partial r}{\partial Y}\mathbf{e}_r + \frac{r\partial\theta}{\partial Y}\mathbf{e}_\theta + \frac{\partial z}{\partial Y}\mathbf{e}_z \tag{3.30.10b}$$

$$\mathbf{F}\mathbf{e}_Z = \frac{\partial r}{\partial Z}\mathbf{e}_r + \frac{r\partial\theta}{\partial Z}\mathbf{e}_\theta + \frac{\partial z}{\partial Z}\mathbf{e}_z \tag{3.30.10c}$$

The matrix

$$[\mathbf{F}] = \begin{bmatrix} \frac{\partial r}{\partial X} & \frac{\partial r}{\partial Y} & \frac{\partial r}{\partial Z} \\ \frac{r\partial\theta}{\partial X} & \frac{r\,\partial\theta}{\partial Y} & \frac{r\,\partial\theta}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{bmatrix} (\mathbf{e}_r \ldots), (\mathbf{e}_X \ldots) \tag{3.30.11}$$

gives the two point components of **F** with respect to the two sets of bases, one at the reference configuration, the other at the current configuration.

The components of the left Cauchy-Green deformation tensor **B** with respect to the basis at the current configuration are given by [see Prob.3.77]

$$B_{rr} = \left(\frac{\partial r}{\partial X}\right)^2 + \left(\frac{\partial r}{\partial Y}\right)^2 + \left(\frac{\partial r}{\partial Z}\right)^2 \tag{3.30.12a}$$

$$B_{\theta\theta} = \left(\frac{r\,\partial\theta}{\partial X}\right)^2 + \left(\frac{r\partial\theta}{\partial Y}\right)^2 + \left(\frac{r\,\partial\theta}{\partial Z}\right)^2 \tag{3.30.12b}$$

$$B_{zz} = \left(\frac{\partial z}{\partial X}\right)^2 + \left(\frac{\partial z}{\partial Y}\right)^2 + \left(\frac{\partial z}{\partial Z}\right)^2 \tag{3.30.12c}$$

$$B_{r\theta} = \left(\frac{\partial r}{\partial X}\right)\left(\frac{r\,\partial\theta}{\partial X}\right) + \left(\frac{\partial r}{\partial Y}\right)\left(\frac{r\,\partial\theta}{\partial Y}\right) + \left(\frac{\partial r}{\partial Z}\right)\left(\frac{r\,\partial\theta}{\partial Z}\right) \tag{3.30.12d}$$

$$B_{rz} = \left(\frac{\partial r}{\partial X}\right)\left(\frac{\partial z}{\partial X}\right) + \left(\frac{\partial r}{\partial Y}\right)\left(\frac{\partial z}{\partial Y}\right) + \left(\frac{\partial r}{\partial Z}\right)\left(\frac{\partial z}{\partial Z}\right) \tag{3.30.12e}$$

$$B_{\theta z} = \left(\frac{r\partial\theta}{\partial X}\right)\left(\frac{\partial z}{\partial X}\right) + \left(\frac{r\,\partial\theta}{\partial Y}\right)\left(\frac{\partial z}{\partial Y}\right) + \left(\frac{r\,\partial\theta}{\partial Z}\right)\left(\frac{\partial z}{\partial Z}\right) \tag{3.30.12f}$$

Again, the components of $\mathbf{B}^{-1}$ can be obtained by using the equation $d\mathbf{X} = \mathbf{F}^{-1}d\mathbf{x}$ and the inverse of Eq. (xii). We list here two of the six components. The other four components can be easily written down following the patterns of these two equations.

$$B_{rr}^{-1} = \left(\frac{\partial X}{\partial r}\right)^2 + \left(\frac{\partial Y}{\partial r}\right)^2 + \left(\frac{\partial Z}{\partial r}\right)^2 \tag{3.30.12g}$$

$$B_{r\theta}^{-1} = \left(\frac{\partial X}{\partial r}\right)\left(\frac{\partial X}{r\partial\theta}\right) + \left(\frac{\partial Y}{\partial r}\right)\left(\frac{\partial Y}{r\partial\theta}\right) + \left(\frac{\partial Z}{\partial r}\right)\left(\frac{\partial Z}{r\partial\theta}\right) \tag{3.30.12h}$$

The components of the right Cauchy-Green deformation tensor **C** with respect to the basis at the reference configuration are given by : [see Prob. 3.78]

$$C_{XX} = \left(\frac{\partial r}{\partial X}\right)^2 + \left(\frac{r\partial\theta}{\partial X}\right)^2 + \left(\frac{\partial z}{\partial X}\right)^2 \tag{3.30.13a}$$

$$C_{YY} = \left(\frac{\partial r}{\partial Y}\right)^2 + \left(\frac{r\partial\theta}{\partial Y}\right)^2 + \left(\frac{\partial z}{\partial Y}\right)^2 \tag{3.30.13b}$$

$$C_{ZZ} = \left(\frac{\partial r}{\partial Z}\right)^2 + \left(\frac{r\partial\theta}{\partial Z}\right)^2 + \left(\frac{\partial z}{\partial Z}\right)^2 \tag{3.30.13c}$$

$$C_{XY} = \left(\frac{\partial r}{\partial X}\right)\left(\frac{\partial r}{\partial Y}\right) + \left(\frac{r\,\partial\theta}{\partial X}\right)\left(\frac{r\,\partial\theta}{\partial Y}\right) + \left(\frac{\partial z}{\partial X}\right)\left(\frac{\partial z}{\partial Y}\right) \tag{3.30.13d}$$

$$C_{XZ} = \left(\frac{\partial r}{\partial X}\right)\left(\frac{\partial r}{\partial Z}\right) + \left(\frac{r\,\partial\theta}{\partial X}\right)\left(\frac{r\,\partial\theta}{\partial Z}\right) + \left(\frac{\partial z}{\partial X}\right)\left(\frac{\partial z}{\partial Z}\right) \tag{3.30.13e}$$

$$C_{YZ} = \left(\frac{\partial r}{\partial Y}\right)\left(\frac{\partial r}{\partial Z}\right) + \left(\frac{r\,\partial\theta}{\partial Y}\right)\left(\frac{r\,\partial\theta}{\partial Z}\right) + \left(\frac{\partial z}{\partial Y}\right)\left(\frac{\partial z}{\partial Z}\right) \tag{3.30.13f}$$

The components of $\mathbf{C}^{-1}$ can be obtained as:

$$C^{-1}_{XX} = \left(\frac{\partial X}{\partial r}\right)^2 + \left(\frac{\partial X}{r\partial\theta}\right)^2 + \left(\frac{\partial X}{\partial z}\right)^2 \tag{3.30.13g}$$

$$C^{-1}_{XY} = \left(\frac{\partial X}{\partial r}\right)\left(\frac{\partial Y}{\partial r}\right) + \left(\frac{\partial X}{r\partial\theta}\right)\left(\frac{\partial Y}{r\partial\theta}\right) + \left(\frac{\partial X}{\partial z}\right)\left(\frac{\partial Y}{\partial z}\right) \tag{3.30.13h}$$

and the other four components can be easily written down following the patterns of these two equation.

(C) Spherical coordinate system for both the reference and the current configuration

Let

$$r = r(r_o, \theta_o, \phi_o, t), \quad \theta = \theta(r_o, \theta_o, \phi_o, t), \qquad \phi = \phi(r_o, \theta_o, \phi_o, t) \tag{xii}$$

be the pathline equations. Then using the same procedure as described for the cylindrical coordinate case, it can derive that the two point components for $\mathbf{F}$ with respect to $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$ at $\mathbf{x}$ and $(\mathbf{e}_{or}, \mathbf{e}_{o\theta}, \mathbf{e}_{o\phi})$ at $\mathbf{X}$ are given by the matrix

$$[\mathbf{F}] = \begin{bmatrix} \dfrac{\partial r}{\partial r_o} & \dfrac{\partial r}{r_o\partial\theta_o} & \dfrac{\partial r}{r_o\sin\theta_o\partial\phi_o} \\ \dfrac{r\partial\theta}{\partial r_o} & \dfrac{r\,\partial\theta}{r_o\partial\theta_o} & \dfrac{r\,\partial\theta}{r_o\sin\theta_o\partial\phi_o} \\ \dfrac{r\sin\theta\partial\phi}{\partial r_o} & \dfrac{r\sin\theta\partial\phi}{r_o\partial\theta_o} & \dfrac{r\sin\theta\partial\phi}{r_o\sin\theta_o\partial\phi_o} \end{bmatrix} (\mathbf{e}_i),\ (\mathbf{e}_{oj}) \tag{3.30.14}$$

The components of the left Cauchy-Green tensor are:

$$B_{rr} = \left(\frac{\partial r}{\partial r_o}\right)^2 + \left(\frac{\partial r}{r_o \partial \theta_o}\right)^2 + \left(\frac{\partial r}{r_o \sin\theta_o \partial \phi_o}\right)^2 \tag{3.30.15a}$$

$$B_{\theta\theta} = \left(\frac{r\partial\theta}{\partial r_o}\right)^2 + \left(\frac{r\partial\theta}{r_o \partial \theta_o}\right)^2 + \left(\frac{r\partial\theta}{r_o \sin\theta_o \partial \phi_o}\right)^2 \tag{3.30.15b}$$

$$B_{\phi\phi} = \left(\frac{r\sin\theta\partial\phi}{\partial r_o}\right)^2 + \left(\frac{r\sin\theta\partial\phi}{r_o \partial \theta_o}\right)^2 + \left(\frac{r\sin\theta\partial\phi}{r_o \sin\theta_o \partial \phi_o}\right)^2 \tag{3.30.15c}$$

$$B_{r\theta} = \left(\frac{\partial r}{\partial r_o}\right)\left(\frac{r\partial\theta}{\partial r_o}\right) + \left(\frac{\partial r}{r_o \partial \theta_o}\right)\left(\frac{r\partial\theta}{r_o \partial \theta_o}\right) + \left(\frac{\partial r}{r_o \sin\theta_o \partial \phi_o}\right)\left(\frac{r\partial\theta}{r_o \sin\theta_o \partial \phi_o}\right) \tag{3.30.15d}$$

$$B_{r\phi} = \left(\frac{\partial r}{\partial r_o}\right)\left(\frac{r\sin\theta\partial\phi}{\partial r_o}\right) + \left(\frac{\partial r}{r_o \partial \theta_o}\right)\left(\frac{r\sin\theta\partial\phi}{r_o \partial \theta_o}\right) + \left(\frac{\partial r}{r_o \sin\theta_o \partial \phi_o}\right)\left(\frac{r\sin\theta\partial\phi}{r_o \sin\theta_o \partial \phi_o}\right) \tag{3.30.15e}$$

$$B_{\phi\theta} = \left(\frac{r\sin\theta\partial\phi}{\partial r_o}\right)\left(\frac{r\partial\theta}{\partial r_o}\right) + \left(\frac{r\sin\theta\partial\phi}{r_o \partial \theta_o}\right)\left(\frac{r\partial\theta}{r_o \partial \theta_o}\right) + \left(\frac{r\sin\theta\partial\phi}{r_o \sin\theta_o \partial \phi_o}\right)\left(\frac{r\partial\theta}{r_o \sin\theta_o \partial \phi_o}\right) \tag{3.30.15f}$$

The components of $\mathbf{B}^{-1}$ can be written:

$$B^{-1}_{rr} = \left(\frac{\partial r_o}{\partial r}\right)^2 + \left(\frac{r_o\, \partial \theta_o}{\partial r}\right)^2 + \left(\frac{r_o \sin\theta_o \partial \phi_o}{\partial r}\right)^2 \tag{3.30.15g}$$

$$B^{-1}_{rz} = \left(\frac{\partial r_o}{\partial r}\right)\left(\frac{\partial r_o}{r\sin\theta\partial\phi}\right) + \left(\frac{r_o \partial \theta_o}{\partial r}\right)\left(\frac{r_o \partial \theta_o}{r\sin\theta\partial\phi}\right) + \left(\frac{r_o \sin\theta_o \partial \phi_o}{\partial r}\right)\left(\frac{r_o \sin\theta_o \partial \phi_o}{r\sin\theta\partial\phi}\right) \tag{3.30.15h}$$

etc.

The components of $\mathbf{C}$ are:

$$C_{r_o r_o} = \left(\frac{\partial r}{\partial r_o}\right)^2 + \left(\frac{r\, \partial\theta}{\partial r_o}\right)^2 + \left(\frac{r\sin\theta\partial\phi}{\partial r_o}\right)^2 \tag{3.30.16a}$$

$$C_{\theta_o \theta_o} = \left(\frac{\partial r}{r_o \partial \theta_o}\right)^2 + \left(\frac{r\partial\theta}{r_o \partial \theta_o}\right)^2 + \left(\frac{r\sin\theta\partial\phi}{r_o \partial \theta_o}\right)^2 \tag{3.30.16b}$$

$$C_{z_o z_o} = \left(\frac{\partial r}{r_o \sin\theta_o \partial \phi_o}\right)^2 + \left(\frac{r\, \partial\theta}{r_o \sin\theta_o \partial \phi_o}\right)^2 + \left(\frac{r\sin\theta\partial\phi}{r_o \sin\theta_o \partial \phi_o}\right)^2 \tag{3.30.16c}$$

$$C_{r_o \theta_o} = \left(\frac{\partial r}{\partial r_o}\right)\left(\frac{\partial r}{r_o \partial \theta_o}\right) + \left(\frac{r\partial\theta}{\partial r_o}\right)\left(\frac{r\partial\theta}{r_o \partial \theta_o}\right) + \left(\frac{r\sin\theta\partial\phi}{\partial r_o}\right)\left(\frac{r\sin\theta\partial\phi}{r_o \partial \theta_o}\right) \tag{3.30.16d}$$

$$C_{r_o z_o} = \left(\frac{\partial r}{\partial r_o}\right)\left(\frac{\partial r}{r_o \sin\theta_o \partial\phi_o}\right) + \left(\frac{r\partial\theta}{\partial r_o}\right)\left(\frac{r\partial\theta}{r_o \sin\theta_o \partial\phi_o}\right) + \left(\frac{r\sin\theta\partial\phi}{\partial r_o}\right)\left(\frac{r\sin\theta\partial\phi}{r_o \sin\theta_o \partial\phi_o}\right) \tag{3.30.16e}$$

$$C_{z_o\theta_o} = \left(\frac{\partial r}{r_o \sin\theta_o \partial\phi_o}\right)\left(\frac{\partial r}{r_o \partial\theta_o}\right) + \left(\frac{r\,\partial\theta}{r_o \sin\theta_o \partial\phi_o}\right)\left(\frac{r\partial\theta}{r_o \partial\theta_o}\right) + \left(\frac{r\sin\theta\partial\phi}{r_o \sin\theta_o \partial\phi_o}\right)\left(\frac{\partial z}{r_o \partial\theta_o}\right) \tag{3.30.16f}$$

The components of $\mathbf{C}^{-1}$ can be written:

$$C^{-1}_{r_o r_o} = \left(\frac{\partial r_o}{\partial r}\right)^2 + \left(\frac{\partial r_o}{r\partial\theta}\right)^2 + \left(\frac{\partial r_o}{r\sin\theta\partial\phi}\right)^2 \tag{3.30.16g}$$

$$C^{-1}_{r_o\theta_o} = \left(\frac{\partial r_o}{\partial r}\right)\left(\frac{r_o\partial\theta_o}{\partial r}\right) + \left(\frac{\partial r_o}{r\partial\theta}\right)\left(\frac{r_o\partial\theta_o}{r\partial\theta}\right) + \left(\frac{\partial r_o}{r\sin\theta\partial\phi}\right)\left(\frac{r_o\partial\theta_o}{r\sin\theta\partial\phi}\right) \tag{3.30.16h}$$

etc.

3.31 Current Configuration as the Reference Configuration

Let $\mathbf{x}'$ be the position at time τ of a material point which is at the spatial position $\mathbf{x}$ at time t, then the kinematic equations of motion (the pathline equations) take the form of

$$\mathbf{x}' = \mathbf{x}_t'(\mathbf{x},\tau), \quad \text{with} \quad \mathbf{x} = \mathbf{x}_t'(\mathbf{x},t) \tag{3.31.1}$$

Equations (3.31.1) describe the motion using the current configuration as the reference configuration. The subscript t in $\mathbf{x}_t'$ indicates that the current time t is the reference time and as such in addition to $\mathbf{x}$ and τ, it is also a function of t.

Example 3.31.1

Given the velocity field

$$v_1 = kx_2, \quad v_2 = v_3 = 0 \tag{i}$$

Find the pathline equations using the current configuration as the reference configuration.

Solution. Let $x'_1(\mathbf{x},\tau)$, $x'_2(\mathbf{x},\tau)$, $x'_3(\mathbf{x},\tau)$ be the position at time τ then

$$\frac{dx_1'}{d\tau} = kx_2', \quad \frac{dx_2'}{d\tau} = \frac{dx_3'}{d\tau} = 0 \tag{ii}$$

The second and the third equation state that both x'_2 and x'_3 are constants. Since they must be x_2 and x_3 at time t, therefore,

$$x_2' = x_2, \quad x_3' = x_3$$

Now from the first equation, since $x'_2 = x_2$, we have

$$x_1' = kx_2\tau + C$$

so that

$$x_1' = x_1 + kx_2(\tau - t) \tag{ii}$$

When the current configuration is used as the reference, it is customary also to denote tensors based on such a reference with a subscript t, e.g.,

$$\mathbf{F}_t \equiv \nabla_{\mathbf{x}}\, \mathbf{x}_t' \quad \text{(relative deformation gradient)}$$

$$\mathbf{C}_t \equiv \mathbf{F}_t^T\mathbf{F}_t \quad \text{(relative right Cauchy-Green Tensor)}$$

$$\mathbf{B}_t \equiv \mathbf{F}_t\mathbf{F}_t^T \quad \text{(relative left Cauchy-Green Tensor)}$$

etc. All the formulas derived earlier, based on a fixed reference configuration, can be easily rewritten for the case where the current configuration is used as the reference. For example, let (r', θ', z', τ) denote the cylindrical coordinates for the position $\mathbf{x}'$ at time τ for a material point which is at (r, θ, z) at time t i.e.,

$$r' = r'(r, \theta, z, \tau), \quad \theta' = \theta'(r, \theta, z, \tau), \quad z' = z'(r, \theta, z, \tau)$$

then, with respect to the current bases $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$

$$(C_t)_{rr} = \left(\frac{\partial r'}{\partial r}\right)^2 + \left(\frac{r'\,\partial\theta'}{\partial r}\right)^2 + \left(\frac{\partial z'}{\partial r}\right)^2 \tag{3.31.1a}$$

$$(C_t)_{\theta\theta} = \left(\frac{\partial r'}{r\partial\theta}\right)^2 + \left(\frac{r'\partial\theta'}{r\partial\theta}\right)^2 + \left(\frac{\partial z'}{r\partial\theta}\right)^2 \tag{3.31.1b}$$

$$(C_t)_{zz} = \left(\frac{\partial r'}{\partial z}\right)^2 + \left(\frac{r'\,\partial\theta'}{\partial z}\right)^2 + \left(\frac{\partial z'}{\partial z}\right)^2 \tag{3.31.1c}$$

$$(C_t)_{r\theta} = \left(\frac{\partial r'}{\partial r}\right)\left(\frac{\partial r'}{r\partial\theta}\right) + \left(\frac{r'\partial\theta'}{\partial r}\right)\left(\frac{r'\,\partial\theta'}{r\partial\theta}\right) + \left(\frac{\partial z'}{\partial r}\right)\left(\frac{\partial z'}{r\partial\theta}\right) \tag{3.31.1d}$$

$$(C_t)_{rz} = \left(\frac{\partial r'}{\partial r}\right)\left(\frac{\partial r'}{\partial z}\right) + \left(\frac{r'\,\partial\theta'}{\partial r}\right)\left(\frac{r'\,\partial\theta'}{\partial z}\right) + \left(\frac{\partial z'}{\partial r}\right)\left(\frac{\partial z'}{\partial z}\right) \tag{3.31.1e}$$

$$(C_t)_{z\theta} = \left(\frac{\partial r'}{\partial z}\right)\left(\frac{\partial r'}{r\partial\theta}\right) + \left(\frac{r'\,\partial\theta'}{\partial z}\right)\left(\frac{r'\,\partial\theta'}{r\partial\theta}\right) + \left(\frac{\partial z'}{\partial z}\right)\left(\frac{\partial z'}{r\partial\theta}\right) \tag{3.31.1f}$$

We will have more to say about relative deformation tensors in Chapter 8 where we shall discuss the constitutive equations for Non-Newtonian fluids.

PROBLEMS

3.1. Consider the motion

$$x_1 = kt + X_1$$
$$x_2 = X_2$$
$$x_3 = X_3$$

where the material coordinates X_i designate the position of a particle at $t = 0$.

(a) Determine the velocity and acceleration of a particle in both a material and spatial description.

(b) If in a spatial description, there is a temperature field $\theta = Ax_1$, find the material derivative $D\theta/Dt$.

(c) Do part (b) if the temperature field is given by $\theta = Bx_2$.

3.2. Consider the motion

$$x_1 = X_1$$
$$x_2 = kX_1^2t^2 + X_2$$
$$x_3 = X_3$$

where X_i are the material coordinates.

(a) At $t = 0$ the corners of a unit square are at $A(0,0,0)$, $B(0,1,0)$, $C(1,1,0)$ and $D(1,0,0)$. Determine the position of A, B, C, D at $t = 1$, and sketch the new shape of the square.

(b) Find the velocity $\mathbf{v}$ and the acceleration $D\mathbf{v}/Dt$ in a material description.

(c) Show that the spatial velocity field is given by

$$v_1 = v_3 = 0, \quad v_2 = 2kx_1^2t.$$

3.3. Consider the motion

$$x_1 = kX_2^2t^2 + X_1$$
$$x_2 = kX_2t + X_2$$
$$x_3 = X_3$$

(a)At $t = 0$, the corners of a unit square are at $A(0,0,0)$, $B(0,1,0)$, $C(1,1,0)$, and $D(1,0,0)$. Sketch the deformed shape of the square at $t = 2$.

(b) Obtain the spatial description of the velocity field.

(c) Obtain the spatial description of the acceleration field.

3.4. Consider the motion

$$x_1 = (k + X_1)t + X_1$$
$$x_2 = X_2$$

$$x_3 = X_3$$

(a) For this motion, repeat part (a) of the previous problem.

(b) Find the velocity and acceleration as a function of time of a particle that is initially at the origin.

(c)Find the velocity and acceleration as a function of time of the particles that are passing through the origin.

3.5. The position at time t of a particle initially at (X_1,X_2,X_3) is given by

$$x_1 = X_1 - 2X_2^2t^2, \quad x_2 = X_2 - X_3t, \quad x_3 = X_3$$

(a) Sketch the deformed shape, at time $t = 1$ of the material line OA which was a straight line at $t = 0$ with O at (0,0,0) and A at (0,1,0).

(b) Find the velocity at $t = 2$, of the particle which is at (1,3,1) at $t = 0$.

(c) Find the velocity of a particle which is at (1,3,1) at $t = 2$.

3.6. The position at time t of a particle initially at (X_1,X_2,X_3), is given by

$$x_1 = X_1 + (X_1 + X_2)t, \quad x_2 = X_2 + (X_1 + X_2)t, \quad x_3 = X_3$$

(a) Find the velocity at $t = 2$ for the particle which was at (1,1,0) at the reference time.

(b) Find the velocity at $t = 2$ for the particle which is at the position (1,1,0) at $t = 2$.

3.7. Consider the motion

$$x_1 = \frac{1+t}{1+t_o}X_1, \quad x_2 = X_2, \quad x_3 = X_3$$

(a) Show that reference time is $t = t_o$.

(b) Find the velocity field in spatial coordinates.

(c) Show that the velocity field is identical to that of the following motion

$$x_1 = (1+t)X_1, \quad x_2 = X_2, \quad x_3 = X_3.$$

3.8. The position at time t of a particle initially at (X_1,X_2,X_3) is given by

$$x_1 = X_1 + X_2^2t^2, \quad x_2 = X_2 + X_2t, \quad x_3 = X_3$$

(a) For the particle which was initially at (1,1,0), what are its positions in the following instants of time: $t = 0$, $t = 1$, $t = 2$.

(b) Find the initial position for a particle which is at (1,3,2) at $t = 2$.

(c) Find the acceleration at $t = 2$ of the particle which was initially at (1,3,2).

(d) Find the acceleration of a particle which is at (1,3,2) at $t = 2$.

3.9. (a)Show that the velocity field

$$v_i = \frac{x_i}{1+t}$$

corresponds to the motion

$$x_i = X_i(1+t)$$

(b) Find the acceleration of this motion in the material description.

3.10. Given the two-dimensional velocity field

$$v_x = -2y, \quad v_y = 2x$$

(a) Obtain the acceleration field.

(b) Obtain the pathline equations.

3.11. Given the two-dimensional velocity field

$$v_x = kx, \quad v_y = -ky$$

(a) Obtain the acceleration field.

(b) Obtain the pathline equations.

3.12. Given the two-dimensional velocity field,

$$v_x = x^2 - y^2, \quad v_y = -2xy$$

Obtain the acceleration field.

3.13. In a spatial description the equation to evaluate the acceleration

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + (\nabla \mathbf{v})\mathbf{v}$$

is nonlinear. That is, if we consider two velocity fields $\mathbf{v}^A$ and $\mathbf{v}^B$, then

$$\mathbf{a}^A + \mathbf{a}^B \neq \mathbf{a}^{A+B}$$

where $\mathbf{a}^A$ and $\mathbf{a}^B$ denote respectively the acceleration fields corresponding to the velocity fields $\mathbf{v}^A$ and $\mathbf{v}^B$ each existing alone, $\mathbf{a}^{A+B}$ denotes the acceleration field corresponding to the combined velocity field $\mathbf{v}^A + \mathbf{v}^B$. Verify this inequality for the velocity fields

$$\mathbf{v}^A = -2x_2\mathbf{e}_1 + 2x_1\mathbf{e}_2$$

$$\mathbf{v}^B = 2x_2\mathbf{e}_1 - 2x_1\mathbf{e}_2$$

3.14. Consider the motion

$$x_1 = X_1$$

$$x_2 = X_2 + (\sin \pi t)(\sin \pi X_1)$$

$$x_3 = X_3$$

(a) At $t = 0$ a material filament coincides with the straight line that extends from (0,0,0) to (1,0,0). Sketch the deformed shape of this filament at $t = 1/2$, $t = 1$, and $t = 3/2$.

(b) Find the velocity and acceleration in a material and a spatial description.

3.15. Consider the following velocity and temperature fields:

$$\mathbf{v} = \frac{x_1\mathbf{e}_1 + x_2\mathbf{e}_2}{x_1^2 + x_2^2}, \quad \theta = k(x_1^2 + x_2^2)$$

(a) Determine the velocity at several positions and indicate the general nature of this velocity field. What do the isotherms look like?

(b) At the point $A(1,1,0)$, determine the acceleration and the material derivative of the temperature field.

3.16. Do the previous problem for the temperature and velocity fields:

$$\mathbf{v} = \frac{-x_2\mathbf{e}_1 + x_1\mathbf{e}_2}{x_1^2 + x_2^2}, \quad \theta = k(x_1^2 + x_2^2).$$

3.17. Consider the motion $\mathbf{x} = \mathbf{X} + X_1 k\mathbf{e}_1$ and let $d\mathbf{X}^{(1)} = (dS_1/\sqrt{2})(\mathbf{e}_1 + \mathbf{e}_2)$ and $d\mathbf{X}^{(2)} = (dS_2/\sqrt{2})(-\mathbf{e}_1 + \mathbf{e}_2)$ be differential material elements in the undeformed configuration.

(a) Find the deformed elements $d\mathbf{x}^{(1)}$ and $d\mathbf{x}^{(2)}$.

(b) Evaluate the stretches of these elements, ds_1/dS_1 and ds_2/dS_2, and the change in the angle between them.

(c)Do part (b) for $k = 1$ and $k = 10^{-2}$.

(d) Compare the results of part(c) to that predicted by the small strain tensor **E**.

3.18. The motion of a continuum from initial position **X** to current position **x** is given by

$$\mathbf{x} = (\mathbf{I} + \mathbf{B})\mathbf{X}$$

where **I** is the identity tensor and **B** is a tensor whose components B_{ij} are constants and small compared to unity. If the components of **x** are x_i and those of **X** are X_i, find

(a) the components of the displacement vector **u**, and

(b) the small strain tensor **E**.

3.19. At time t, the position of a particle initially at (X_1,X_2,X_3) is defined by

$$x_1 = X_1 + kX_3$$
$$x_2 = X_2 + kX_2$$
$$x_3 = X_3$$

where $k = 10^{-5}$.

(a) Find the components of the strain tensor.

(b) Find the unit elongation of an element initially in the direction of $\mathbf{e}_1 + \mathbf{e}_2$.

3.20. Consider the displacement field

$$u_1 = k(2X_1^2 + X_1 X_2), \quad u_2 = kX_2^2, \quad u_3 = 0,$$

where $k = 10^{-4}$.

(a) Find the unit elongations and the change of angle for two material elements $d\mathbf{X}^{(1)} = dX_1\mathbf{e}_1$ and $d\mathbf{X}^{(2)} = dX_2\mathbf{e}_2$ that emanate from a particle designated by $\mathbf{X} = \mathbf{e}_1 + \mathbf{e}_2$.

(b) Find the deformed shape of these two elements.

3.21. For the displacement field of Example 3.8.3, determine the increase in length for the diagonal element of the cube in the direction of $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ (a) by using the strain tensor and (b) by geometry.

3.22. With reference to a rectangular Cartesian coordinate system, the state of strain at a point is given by the matrix

$$[\mathbf{E}] = \begin{bmatrix} 5 & 3 & 0 \\ 3 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \times 10^{-4}$$

(a) What is the unit elongation in the direction $2\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3$?

(b) What is the change of angle between two perpendicular lines (in the undeformed state) emanating from the point and in the directions of $2\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3$ and $3\mathbf{e}_1 - 6\mathbf{e}_3$?

3.23. Do the previous problem for (a) the unit elongation in the direction $3\mathbf{e}_1 - 4\mathbf{e}_2$, (b) the change in angle between two elements in the direction $3\mathbf{e}_1 - 4\mathbf{e}_3$ and $4\mathbf{e}_1 + 3\mathbf{e}_3$.

3.24. (a)For Prob.3.22, determine the principal scalar invariants of the strain tensor.

(b) Show that the following matrix

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix} \times 10^{-4}$$

cannot represent the same state of strain of Prob.3.22.

3.25. For the displacement field

$$u_1 = kX_1^2, \quad u_2 = kX_2X_3, \quad u_3 = k(2X_1X_3 + X_1^2), \quad k = 10^{-6}$$

find the maximum unit elongation for an element that is initially at (1,0,0).

3.26. Given the matrix of an infinitesimal strain field

$$[\mathbf{E}] = \begin{bmatrix} k_1X_2 & 0 & 0 \\ 0 & -k_2X_2 & 0 \\ 0 & 0 & -k_2X_2 \end{bmatrix}$$

(a) Find the location of the particle that does not undergo any volume change.

(b) What should be the relation between k_1 and k_2 be such that no element changes volume?

3.27. The displacement components for a body are

$$u_1 = k(X_1^2 + X_2), \quad u_2 = k(4X_3^2 - X_1), \quad u_3 = 0, \qquad k = 10^{-4}$$

(a) Find the strain tensor.

(b) Find the change of length per unit length for an element which was at (1,2,1) and in the direction of $\mathbf{e}_1 + \mathbf{e}_2$.

(c) What is the maximum unit elongation at the same point (1,2,1)?

(d) What is the change of volume for the unit cube with a corner at the origin and with three of its edges along the positive coordinate axes.

3.28. For any motion the mass of a particle (material volume) remains constant. Consider the mass to be a product of its volume times its mass density and show that (a)for infinitesimal deformation $\rho(1 + E_{kk}) = \rho_o$, where ρ_o denotes the initial density and ρ the current density. (b) Use the smallness of E_{kk} to show that the current density is given by

$$\rho = \rho_o(1 - E_{kk})$$

3.29. True or false: At any point in a body, there always exist two mutually perpendicular material elements which do not suffer any change of angle in an arbitrary small deformation of the body. Give reasons.

3.30. Given the following strain components at a point in a continuum:

$$E_{11} = E_{12} = E_{22} = k, \quad E_{33} = 3k, \quad E_{13} = E_{23} = 0 \qquad k>0$$

Does there exist a material element at the point which decreases in length under the deformation? Explain your answer.

3.31. The unit elongations at a certain point on the surface of a body are measured experimentally by means of strain gages that are arranged 45° apart (called the 45° strain rosette) in the directions $\mathbf{e}_1$, $(\sqrt{2}/2)(\mathbf{e}_1 + \mathbf{e}_2)$ and $\mathbf{e}_2$. If these unit elongations are designated by a,b,c respectively, what are the strain components E_{11},E_{22},E_{12}.

3.32. (a) Do Problem 3.31 if the measured strains are 200×10^{-6}, 50×10^{-6}, 100×10^{-6}, respectively.

(b) If $E_{33} = E_{32} = E_{31} = 0$, find the principal strains and directions of part (a).

(c) How will the result of part (b) be altered if $E_{33} \neq 0$?

3.33. Repeat Problem 3.32 except that $a = b = c = 1000\times10^{-6}$.

3.34. The unit elongations at a certain point on the surface of a body are measured experimentally by means of strain gages that are arranged 60° apart (called the 60° strain rosette) in the directions $\mathbf{e}_1$, $\frac{1}{2}(\mathbf{e}_1 + \sqrt{3}\mathbf{e}_2)$, and $\frac{1}{2}(-\mathbf{e}_1 + \sqrt{3}\mathbf{e}_2)$. If these elongations are designated by a,b,c respectively, what are the strain components E_{11},E_{22},E_{12}?

3.35. Do Problem 3.34 if the strain rosette measurements give $a = 2\times10^{-6}$, $b = 1\times10^{-6}$, $c = 1.5\times10^{-6}$.

3.36. Do Problem 3.35 except that $a = b = c = 2000\times10^{-6}$.

3.37. For the velocity field, $\mathbf{v} = (kx_2^2)\mathbf{e}_1$

(a) Find the rate of deformation and spin tensors.

(b) Find the rate of extensions of a material element $d\mathbf{x} = (ds)\mathbf{n}$ where

$$\mathbf{n} = (\sqrt{2}/2)(\mathbf{e}_1 + \mathbf{e}_2) \quad \text{at } \mathbf{x} = 5\mathbf{e}_1 + 3\mathbf{e}_2.$$

3.38. For the velocity field

$$\mathbf{v} = \left(\frac{t+k}{1+x_1}\right)\mathbf{e}_1$$

find the rates of extension for the following material elements: $d\mathbf{x}^{(1)} = ds_1\mathbf{e}_1$ and $d\mathbf{x}^{(2)} = (ds_2/\sqrt{2})(\mathbf{e}_1 + \mathbf{e}_2)$ at the origin at time $t = 1$.

3.39. (a) Find the rate of deformation and spin tensors for the velocity field $\mathbf{v} = (\cos t)(\sin \pi x_1)\mathbf{e}_2$.

(b) For the velocity field of part (a), find the rates of extension of the elements $d\mathbf{x}^{(1)} = (ds_1)\mathbf{e}_1$, $d\mathbf{x}^{(2)} = (ds_2)\mathbf{e}_2$, $d\mathbf{x}^{(3)} = ds_3/\sqrt{2}(\mathbf{e}_1 + \mathbf{e}_2)$ at the origin at t = 0.

3.40. Show that the following velocity components correspond to a rigid body motion:

$$v_1 = x_2 - x_3, \quad v_2 = -x_1 + x_3, \quad v_3 = x_1 - x_2$$

3.41. For the velocity field of Prob.3.15

(a) Find the rate of deformation and spin tensors.

(b) Find the rate of extension of a radial material line element.

3.42. Given the two-dimensional velocity field in cylindrical coordinates

$$v_r = 0, \quad v_\theta = 2r + \frac{4}{r}$$

(a) Find the acceleration at $r = 2$.

(b) Find the rate of deformation tensor at $r = 2$.

3.43. Given the velocity field in spherical coordinates

$$v_r = 0, \quad v_\theta = 0, \quad v_\phi = \left(Ar + \frac{B}{r^2}\right)\sin\theta$$

(a) Find the acceleration field.

(b) Find the rate of deformation field.

3.44. A motion is said to be irrotational if the spin tensor vanishes. Show that the velocity field of Prob.3.16 describes an irrotational motion.

3.45. (a) Let $d\mathbf{x}^{(1)} = (ds_1)\mathbf{n}$, and $d\mathbf{x}^{(2)} = (ds_2)\mathbf{m}$ be two material elements that emanate from a particle P which at present has a rate of deformation $\mathbf{D}$. Consider $(D/Dt)(d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)})$ and show that

$$\left(\frac{1}{ds_1}\frac{D(ds_1)}{Dt} + \frac{1}{ds_2}\frac{D(ds_2)}{Dt}\right)\cos\theta - (\sin\theta)\frac{D\theta}{Dt} = 2\mathbf{m}\cdot\mathbf{D}\mathbf{n}.$$

where θ is the angle between $\mathbf{m}$ and $\mathbf{n}$.

(b) Consider the special cases (i) $d\mathbf{x}^{(1)} = d\mathbf{x}^{(2)}$ and (ii) $\theta = \pi/2$. Show that the above expression reduces to the results of Section 3.13.

3.46. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and D_1, D_2, D_3 be the principal directions and values of the rate of deformation tensor $\mathbf{D}$. Further, let

$$d\mathbf{x}^{(1)} = (ds_1)\mathbf{e}_1, \; d\mathbf{x}^{(2)} = (ds_2)\mathbf{e}_2, \; d\mathbf{x}^{(3)} = (ds_3)\mathbf{e}_3$$

be three material line elements. Consider the material derivative $(D/Dt)[d\mathbf{x}^{(1)} \cdot (d\mathbf{x}^{(2)} \times d\mathbf{x}^{(3)})]$ and show that

$$\frac{1}{dV}\frac{D(dV)}{Dt} = D_1 + D_2 + D_3$$

where the infinitesimal volume $dV = (ds_1)(ds_2)(ds_3)$.

3.47. Consider a material element $d\mathbf{x} = ds\mathbf{n}$

(a) Show that

$$(D/Dt)(\mathbf{n}) = \mathbf{D}\mathbf{n} + \mathbf{W}\mathbf{n} - (\mathbf{n}\cdot\mathbf{D}\mathbf{n})\mathbf{n}$$

(b) Show that if $\mathbf{n}$ is an eigenvector of $\mathbf{D}$ then

$$\frac{D\mathbf{n}}{Dt} = \mathbf{W}\mathbf{n} = \boldsymbol{\omega}\times\mathbf{n}$$

where $\boldsymbol{\omega}$ is the axial vector of $\mathbf{W}$.

3.48. Given the following velocity field

$$v_1 = k(x_2 - 2)^2 x_3,$$
$$v_2 = -x_1 x_2 \quad ,$$

$$v_3 = kx_1x_3 \quad ,$$

for an incompressible fluid, determine k such that the equation of mass conservation is satisfied.

3.49. Given the velocity field in cylindrical coordinates

$$v_r = f(r,\theta), \quad v_\theta = 0, \quad v_z = 0$$

For an incompressible material, from the conservation of mass principle, obtain the most general form of the function $f(r, \theta)$.

3.50. An incompressible fluid undergoes a two-dimensional motion with

$$v_r = \frac{k\cos\theta}{\sqrt{r}}$$

find v_θ if $v_\theta = 0$ at $\theta = 0$.

3.51. Are the fluid motions described in (a) Prob.3.15 and (b) Prob.3.16 incompressible?

3.52. In a spatial description, the density of an incompressible fluid is given by $\rho = kx_2$. Find the permissible form for the velocity field with $v_3 = 0$, so that the conservation of mass equation is satisfied.

3.53. Consider the velocity field

$$\mathbf{v} = \left(\frac{x_1}{1+t}\right)\mathbf{e}_1$$

(a) Find the density if it is independent of spatial position, i.e., $\rho = \rho(t)$.

(b) Find the density if it is a function x_1 alone.

3.54. Given the velocity field

$$\mathbf{v} = x_1 t\, \mathbf{e}_1 + x_2 t\, \mathbf{e}_2,$$

determine how the fluid density varies with time, if in a spatial description it is a function of time only.

3.55. Check whether or not the following distribution of the state of strain satisfies the compatibility conditions:

$$[\mathbf{E}] = k\begin{bmatrix} X_1 + X_2 & X_1 & X_2 \\ X_1 & X_2 + X_3 & X_3 \\ X_2 & X_3 & X_1 + X_3 \end{bmatrix}$$

where $k = 10^{-4}$.

3.56. Check whether or not the following distribution of the state of strain satisfies the compatibility conditions:

$$[\mathbf{E}] = k \begin{bmatrix} X_1^2 & X_2^2 + X_3^2 & X_1X_3 \\ X_2^2 + X_3^2 & 0 & X_1 \\ X_1X_3 & X_1 & X_2^2 \end{bmatrix}$$

where $k = 10^{-4}$.

3.57. Does the displacement field

$$u_1 = \sin X_1, \quad u_2 = X_1^3 X_2, \quad u_3 = \cos X_3$$

correspond to a compatible strain field?

3.58. Given the strain field

$$E_{12} = E_{21} = k X_1 X_2$$

where $k = 10^{-4}$ and all other $E_{ij} = 0$.

(a) Check the equations of compatibility for this strain field.

(b) By attempting to integrate the strain field, show that there does not exist a continuous displacement field for this strain field.

3.59. The strain components are given by

$$E_{11} = \frac{1}{\alpha} f(X_2, X_3)$$

$$E_{22} = E_{33} = -\frac{\nu}{\alpha} f(X_2, X_3)$$

$$E_{12} = E_{13} = E_{23} = 0$$

Show that for the strains to be compatible $f(X_2, X_3)$ must be linear.

3.60. In cylindrical coordinates (r, θ, z), consider a differential volume bounded by the three pairs of faces $r = r_o, r = r_o + dr$; $\theta = \theta_o, \theta = \theta_o + d\theta$; $z = z_o, z = z_o + dz$. The rate at which mass is flowing into the volume across the face $r = r_o$ is given by $(\rho v_r)(r_o d\theta)(dz)$ and similar expressions for other faces. By demanding that the net rate of inflow of mass must be equal to the rate of increase of mass inside the volume, obtain the equation of conservation of mass in cylindrical coordinates as that given in Eq.(3.15.5).

3.61. Given the following deformation in rectangular Cartesian coordinates

$$x_1 = 3X_3$$

$$x_2 = -X_1$$

$$x_3 = -2X_2$$

Determine (a) the deformation gradient **F**, (b) the right Cauchy-Green deformation tensor **C**, (c) the left Cauchy-Green deformation tensor **B**, (d) the rotation tensor **R**, (e) the Lagrangian strain tensor, (f) the Euler strain tensor, (g) ratio of deformed volume to the initial

volume, (h) the deformed area (magnitude and its normal) for the area whose normal was in the direction of $\mathbf{e}_2$ and whose magnitude was unity in the undeformed state.

3.62. Do Prob. 3.61 for the following deformation

$$x_1 = 2X_2$$
$$x_2 = 3X_3$$
$$x_3 = X_1$$

3.63. Do Prob. 3.61 for the following deformation

$$x_1 = X_1$$
$$x_2 = 3X_3$$
$$x_3 = -2X_2$$

3.64. Do Prob. 3.61 for the following deformation

$$x_1 = 2X_2$$
$$x_2 = -X_1$$
$$x_3 = 3X_3$$

3.65. Given

$$x_1 = X_1 + 3X_2, \quad x_2 = X_2, \quad x_3 = X_3$$

Obtain

(a)$\mathbf{F}$, $\mathbf{C}$.

(b) the eigenvalues and eigenvectors of $\mathbf{C}$.

(c) the matrix of $\mathbf{U}$ and $\mathbf{U}^{-1}$ using the eigenvectors of $\mathbf{C}$ as basis.

(d) the matrix of $\mathbf{U}$ and $\mathbf{U}^{-1}$ with respect to the $\mathbf{e}_i$ basis.

(d) the rotation tensor $\mathbf{R}$ with respect to the $\mathbf{e}_i$ basis.

You may check your results with the formulas given in the next problem.

3.66. Verify that with respect to rectangular Cartesian base vectors, the right stretch tensor $\mathbf{U}$ and the rotation tensor $\mathbf{R}$ for the simple shear deformation

$$x_1 = X_1 + kX_2, \; x_2 = X_2, \; x_3 = X_3$$

are given by

$$[\mathbf{U}] = \begin{bmatrix} f & \frac{k}{2}f & 0 \\ \frac{k}{2}f & \left(1+\frac{k^2}{2}\right)f & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad [\mathbf{R}] = \begin{bmatrix} f & \frac{k}{2}f & 0 \\ -\frac{k}{2}f & f & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $f = \left(1 + \frac{k^2}{4}\right)^{-\frac{1}{2}}$.

3.67. Let $d\mathbf{X}^{(1)} = dS_1\mathbf{N}^{(1)}$ and $d\mathbf{X}^{(2)} = dS_2\mathbf{N}^{(2)}$ be two material elements at a point P. Show that if θ denotes the angle between their respective deformed elements $d\mathbf{x}^{(1)}$ and $d\mathbf{x}^{(2)}$, then

$$\cos\theta = \frac{C_{\alpha\beta}N_\alpha^{(1)}N_\beta^{(1)}}{\lambda_1\lambda_2}$$

where $\mathbf{N}^{(1)} = N_\alpha^{(1)}\mathbf{e}_\alpha$, $\mathbf{N}^{(2)} = N_\alpha^{(2)}\mathbf{e}_\alpha$, $\lambda_1 = \frac{ds_1}{dS_1}$ and $\lambda_2 = \frac{ds_2}{dS_2}$.

3.68. Given the following right Cauchy-Green deformation tensor at a point

$$[\mathbf{C}] = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0.36 \end{bmatrix}$$

(a) Find the stretch for the material elements which were in the direction of $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ directions.

(b) Find the stretch for the material element which was in the direction of $\mathbf{e}_1 + \mathbf{e}_2$

(c) Find $\cos\theta$, where θ is the angle between $d\mathbf{x}^{(1)}$ and $d\mathbf{x}^{(2)}$.

3.69. Show that for any tensor $\mathbf{A}(X_1,X_2,X_3)$

$$\frac{\partial}{\partial X_m}\det\mathbf{A} = (\det\mathbf{A})(A^{-1})_{nj}\frac{\partial A_{jn}}{\partial X_m}$$

3.70. Given

$$r = r_o, \quad \theta = \theta_o + kz_o, \quad z = z_o$$

where (r, θ, z) and (r_o, θ_o, z_o) are cylindrical coordinates for the current and reference configuration respectively.

(a) Obtain the components of the left Cauchy-Green tensor $\mathbf{B}$ with respect to the basis at the current configuration.

(b) Obtain the components of the right Cauchy-Green tensor $\mathbf{C}$ with respect to the basis at the reference configuration.

3.71. Given

$$r = (2aX + b)^{1/2}, \quad \theta = \frac{Y}{a}, \quad z = Z$$

where (r, θ, z) is cylindrical coordinates for the current configuration and (X, Y, Z) are rectangular Cartesian coordinates for the reference configuration.

(a) Calculate the change of volume.

(b) Obtain the components of the left Cauchy-Green tensor **B** with respect to the basis at the current configuration.

3.72. Given

$$r = f(X), \quad \theta = g(Y), \quad z = h(Z)$$

where (r, θ, z) and (X, Y, Z) are cylindrical coordinates and rectangular Cartesian coordinates for the current and reference configuration, respectively. Obtain the components of the right Cauchy-Green tensor **C** with respect to the basis at the reference configuration

3.73. From Eqs.(3.30.4a), obtain Eqs.(3.30.5).

3.74. Verify Eq.(3.30.8b) and (3.30.8d).

3.75. Verify Eq.(3.30.9b) and (3.30.9d).

3.76. Derive Eqs.(3.30.10).

3.77. Using Eqs.(3.30.10) derive Eqs.(3.30.12a) and (3.30.12d).

3.78. Verify Eqs. (3.30.13 a) and (3.30.13d).