8

Non-Newtonian Fluids

In Chapter 6, the linear viscous fluid was discussed as an example of a constitutive equation of an idealized fluid. The mechanical behavior of many real fluids appears to be adequately described under a wide range of circumstances by this constitutive equation which is referred to as the constitutive equation of a Newtonian fluid. Many other real fluids exhibit behaviors which are not accounted for by the theory of Newtonian fluid. Examples of such substances include polymeric solutions, paints, molasses, etc.

For a steady unidirectional laminar flow of water in a circular pipe, the theory of Newtonian fluid gives the experimentally confirmed result that the volume discharge Q is proportional to the (constant) pressure gradient in the axial direction and to the fourth power of the diameter d of the pipe, that is [see Eq. (6.13.6)]

$$Q = \frac{\pi d^4}{128\mu}\left|\frac{dp}{dx}\right| \tag{8.0.1}$$

However, for many polymeric solutions, it was observed that the above equation does not hold. For a fixed d, the Q versus $|dp/dx|$ relation is nonlinear as sketched in Fig. 8.1.

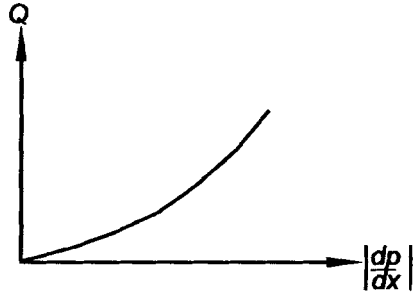

Fig. 8.1

For a steady laminar flow of water placed between two very long coaxial cylinders of radii r_1 and r_2, if the inner cylinder is at rest while the outer one is rotating with an angular velocity Ω, the theory of Newtonian fluid gives the result agreeing with experimental observations that the torque per unit height which must be applied to the cylinders to maintain the flow is proportional to Ω. In fact [see Eq. (6.15.5)]

$$M = \frac{4\pi\mu r_1^2 r_2^2 \Omega}{r_2^2 - r_2^2} \tag{8.0.2}$$

However, for those fluids which do not obey Eq. (8.0.1), it is found that they do not obey Eq. (8.0.2) either. Furthermore, for water in this flow, the normal stress exerted on the outer cylinder is always larger than that on the inner cylinder due to the effect of centrifugal forces. However, for those fluids which do not obey Eq. (8.0.1), the compressive normal stress on the inner cylinder can be larger than that on the outer cylinder. Fig. 8.2 is a schematic diagram showing a higher fluid level in the center tube than in the outer tube for a non-Newtonian fluid in spite of the centrifugal forces due to the rotations of the cylinders. Other manifestations of the non-Newtonian behaviors include the ability of the fluids to store elastic energy and the occurrence of non-zero stress relaxation time when the fluid is suddenly given a constant shear deformation. (For Newtonian fluids, relaxation is instantaneous).

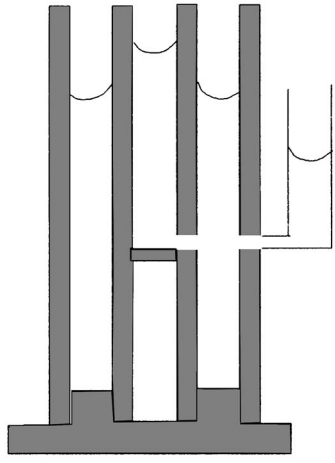

Fig. 8.2

In this chapter, we shall discuss several special constitutive equations and one general one which define idealized viscoelastic fluids exhibiting various characteristics of Non-Newtonian behaviors.

Part A Linear Viscoelastic Fluid

8.1 Linear Maxwell Fluid

The linear Maxwell fluid is defined by the following constitutive equation:

$$\mathbf{T} = -p\mathbf{I} + \boldsymbol{\tau} \tag{8.1.1a}$$

$$\boldsymbol{\tau} + \lambda\frac{\partial \boldsymbol{\tau}}{\partial t} = 2\mu\mathbf{D} \tag{8.1.1b}$$

where $-p\mathbf{I}$ is the isotropic pressure which is constitutively indeterminate due to the incompressibility property of the fluid, $\boldsymbol{\tau}$ is called the "extra stress" which is related to the rate of deformation $\mathbf{D}$ by Eq. (8.1.1b).

In the following example, we show, with the help of a mechanical analogy, that the linear Maxwell fluid possesses elasticity.

Example 8.1.1

Figure 8.3 shows the so-called linear Maxwell element which consists of a spring (an elastic element) with spring constant G, connected in series to a viscous dashpot (viscous element) with a damping coefficient η. The elongation (or strain) of the Maxwell element can be divided into an elastic portion ε_e and a viscous portion ε_v, i.e.,

$$\varepsilon = \varepsilon_e + \varepsilon_v \tag{i}$$

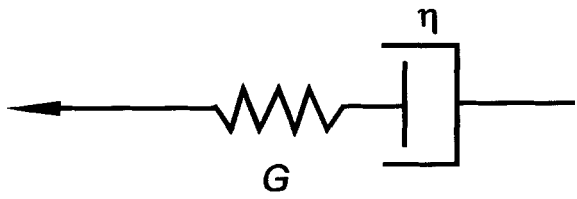

Fig. 8.3

Since the spring and the dashpot are connected in series, the force τ in each is the same (inertia effects are neglected). That is

$$\tau = G\,\varepsilon_e = \eta\,\frac{d\,\varepsilon_v}{dt} \tag{ii}$$

Thus,

$$\frac{d\,\varepsilon_e}{dt} = \frac{1}{G}\frac{d\tau}{dt} \tag{iii}$$

and

$$\frac{d\,\varepsilon_v}{dt} = \frac{1}{\eta}\tau \tag{iv}$$

Taking time derivative of Eq. (i) and using Eqs. (iii) and (iv), we obtain the relation between the rate of strain of the Maxwell element with the force τ as follows:

$$\frac{d\,\varepsilon}{dt} = \frac{1}{G}\frac{d\tau}{dt} + \frac{1}{\eta}\tau \tag{8.1.2}$$

or

$$\tau + \lambda\frac{d\tau}{dt} = \eta\,\frac{d\,\varepsilon}{dt} \tag{8.1.3}$$

where

$$\lambda = \frac{\eta}{G} \tag{8.1.4}$$

has the dimension of time, the physical meaning of which will be discussed below. Equation (8.1.3) is of the same form as Eq. (8.1.1b). Indeed both **D** and d ε/dt (in the right side of these equations) describe rates of deformation. (We note that in a simple shearing flow in the xy plane, the rate of change of shearing strain is given by $2D_{xy}$). Thus, by analogy, we see that the constitutive equation, Eq. (8.1.1b) endows the fluid with "elasticity" through the term $\lambda\partial\tau/\partial t$ with an equivalent elastic modulus G given by Eq. (8.1.4).

Let us consider the following experiment performed on the Maxwell element: Starting at time $t = 0$, a constant force τ_o is applied to the element. We are interested in how, for $t > 0$, the strain changes with time. This is the so-called **creep experiment**. From Eq. (8.1.3), we have, since $d\tau/dt = 0$, for $t > 0$,

$$\frac{d\,\varepsilon}{d\,\tau} = \frac{1}{\eta}\tau_o \qquad \text{for } t > 0 \tag{v}$$

which yields

$$\varepsilon = \frac{\tau_o}{\eta}t + \varepsilon_o \tag{vi}$$

The integration constant ε_o is the instantaneous strain ε of the element at $t = 0+$ from the elastic response of the spring and is therefore given by τ_o/G. Thus

$$\varepsilon = \frac{\tau_o}{\eta} t + \frac{\tau_o}{G} \tag{8.1.5}$$

We see from Eq. (8.1.5) that under the action of a constant force τ_o in creep experiment, the strain of the Maxwell element first has an instantaneous jump from 0 to τ_o/G and then continues to increase with time (i.e. flow) without limit.

We note that there is no contribution to the instantaneous strain from the dashpot because, with $d\,\varepsilon/dt \to \infty$, an infinitely large force is required for the dashpot to do that. On the other hand, there is no contribution to the rate of elongation from the spring because the elastic response is a constant under a constant load.

We may write Eq. (8.1.5) as

$$\frac{\varepsilon}{\tau_o} = \frac{1}{\eta} t + \frac{1}{G} \equiv J \tag{8.1.6}$$

The function $J(t)$ gives the creep history per unit force. It is known as the **creep compliance function** for the linear Maxwell element.

In another experiment, the Maxwell element is given a strain ε_o at $t=0$ which is then maintained at all time. We are interested in how the force τ changes with time. This is the so-called **stress relaxation experiment**. From Eq. (8.1.3), with $d\,\varepsilon/dt = 0$, for $t > 0$, we have

$$\tau + \lambda \frac{d\tau}{dt} = 0 \qquad \text{for } t > 0 \tag{vii}$$

which yields

$$\tau = \tau_o e^{-t/\lambda} \tag{viii}$$

The integration constant τ_o is the instantaneous elastic force which is required to produce the strain ε_o at $t = 0$. That is, $\tau_o = G\,\varepsilon_o$. Thus,

$$\tau = G\,\varepsilon_o e^{-t/\lambda} \tag{8.1.7}$$

Eq. (8.1.7) is the force history for the stress relaxation experiment for the Maxwell element. We may write Eq. (8.1.7) as

$$\phi(t) \equiv \frac{\tau}{\varepsilon_o} = G e^{-t/\lambda} = \frac{\eta}{\lambda} e^{-t/\lambda} \tag{8.1.8}$$

The function $\phi(t)$ gives the stress history per unit strain. It is called the stress **relaxation function**, and the constant λ is known as the relaxation time which is the time for the force to relax to 1/e of the initial value of τ.

It is interesting to consider the limiting cases of the Maxwell element. If $G = \infty$, then the spring element becomes a rigid bar and the element no longer possesses elasticity. That is, it is a purely viscous element. In creep experiment, there will be no instantaneous elongation, the element simply creeps linearly with time (see Eq. (8.1.6)) from the unstretched initial position. In the stress relaxation experiment, an infinitely large force is needed at $t = 0$ to produce the finite jump in elongation (from 0 to 1). The force however is instantaneously returned to zero (i.e., the relaxation time $\lambda = \eta/G \to 0$). We can write the relaxation function for the purely viscous element in the following way

$$\tau = \eta\delta(t) \qquad \text{viscous element only} \tag{8.1.9}$$

where $\delta(t)$ is known as the Dirac delta function which may be defined to be the derivative of the unit step function $H(t)$ defined by:

$$H(t) = \begin{cases} 0 & -\infty < t < 0 \\ 1 & 0 \le t < \infty \end{cases} \tag{8.1.10}$$

Thus,

$$\delta(t) \equiv \frac{dH(t)}{dt} \tag{8.1.11}$$

and

$$\int^{t} \delta(t)dt = H(t) \tag{8.1.12}$$

Example 8.1.2

Consider a linear Maxwell fluid, defined by Eq. (8.1.1), in steady simple shearing flow:

$$v_1 = kx_2, \quad v_2 = v_3 = 0 \tag{i}$$

Find the stress components.

Solution. Since the given velocity field is steady, all field variables are independent of time. Thus, $\frac{\partial \boldsymbol{\tau}}{\partial t} = 0$ and we have

$$\boldsymbol{\tau} = 2\mu\mathbf{D} \tag{ii}$$

Thus, the stress field is exactly the same as that of a Newtonian incompressible fluid and the viscosity is independent of the rate of shear for this fluid.

Example 8.1.3

For a Maxwell fluid, consider the stress relaxation experiment with the displacement field given by

$$u_1 = \varepsilon_o H(t) x_2, \quad u_2 = u_3 = 0 \tag{i}$$

where H(t) is the unit step function defined in Eq. (8.1.10). Neglect inertia effects,

(i) obtain the components of the rate of deformation tensor.

(ii) obtain τ_{12} at $t = 0$.

(iii) obtain the history of the shear stress τ_{12}.

Solution. Differentiate Eq. (i) with respect to time, we get

$$v_1 = \varepsilon_o \delta(t) x_2, \quad v_2 = v_3 = 0 \tag{ii}$$

where $\delta(t)$ is the Dirac delta function defined in Eq. (8.1.11). The only non-zero rate of deformation component is $D_{12} = \dfrac{\varepsilon_o \delta(t)}{2}$. Thus, from the constitutive equation for the linear Maxwell fluid, Eq. (8.1.1b), we obtain

$$\tau_{12} + \lambda \frac{d\tau_{12}}{dt} = \mu \, \varepsilon_o \, \delta(t) \tag{iii}$$

Integrating the above equation from t=0−ε to t=0+ε, we have

$$\int_{o-\varepsilon}^{o+\varepsilon} \tau_{12} \, dt + \lambda \int_{o-\varepsilon}^{o+\varepsilon} \frac{d\,\tau_{12}}{dt} dt = \mu \, \varepsilon_o \int_{o-\varepsilon}^{o+\varepsilon} \delta(t) dt \tag{iv}$$

The integral on the right side of the above equation is equal to 1 [see Eq. (8.1.12)]. As $\varepsilon \to 0$, the first integral on the left side of the above equation approaches zero whereas the second integral becomes:

$$\lambda \, [\tau_{12}(0+) - \tau_{12}(0-)] \tag{v}$$

Thus, since $\tau_{12}(0-) = 0$, from Eq. (iv), we have

$$\tau_{12}(0+) = \frac{\mu \, \varepsilon_o}{\lambda}$$

For $t > 0$, $\delta(t)$=0 so that Eq. (iii) becomes

$$\tau_{12} + \lambda \frac{d\tau_{12}}{dt} = 0 \qquad t > 0 \tag{vi}$$

The solution of the above equation with the initial condition $\tau_{12}(0+) = \dfrac{\mu \, \varepsilon_o}{\lambda}$ is

$$\frac{\tau_{12}}{\varepsilon_o} = \frac{\mu}{\lambda} e^{-t/\lambda} \tag{8.1.13}$$

This is the same relaxation function which we obtained for the spring-dashpot model in Eq.(8.1.7). In arriving at Eq. (8.1.7), we made use of the initial condition $\tau_0 = G\,\varepsilon_o$, which was obtained from considerations of the responses of the elastic element. Here in the present example, the initial condition is obtained by integrating the differential equation, Eq. (iii), over an infinitesimal time interval (from t=0− to t= 0+). By comparing Eq. (8.1.13) here with Eq. (8.1.8) of the mechanical model, we see that $\frac{\mu}{\lambda}$ is the equivalent of the spring constant G of the mechanical model. It gives a measure of the elasticity of the linear Maxwell fluid.

Example 8.1.4

A linear Maxwell fluid is confined between two infinitely large parallel plates. The bottom plate is fixed. The top plate undergoes a one-dimensional oscillation of small amplitude u_o in its own plane. Neglect the inertia effects, find the response of the shear stress.

Solution. The boundary conditions for the displacement components may be written:

$$\text{At } y = h: \quad u_x = u_o\, e^{i\omega t}, \quad u_y = u_z = 0 \tag{i}$$

$$\text{At } y = 0: \quad u_x = u_y = u_z = 0 \tag{ii}$$

where $i = \sqrt{-1}$ and $e^{i\omega t} = \cos\omega t + i\sin\omega t$. We may take the real part of u_x to correspond to our physical problem. That is, in the physical problem, $u_x = u_o\cos\omega t$.

Consider the following displacement field:

$$u_x(y) = u_o e^{i\omega t}\left(\frac{y}{h}\right), \quad u_y = u_z = 0 \tag{iii}$$

Clearly, this displacement field satisfies the boundary conditions (i) and (ii). The velocity field corresponding to Eq. (iii) is given by:

$$v_x(y) = i\omega u_o e^{i\omega t}\left(\frac{y}{h}\right), \quad v_y = v_z = 0 \tag{iii}$$

Thus, the components of the rate of deformation tensor **D** are:

$$D_{12} = \frac{1}{2} i\,\omega\, u_o e^{i\omega t}\left(\frac{1}{h}\right) \quad \text{and all other } D_{ij} = 0 \tag{iv}$$

This is a homogeneous field and it corresponds to a homogeneous stress field. In the absence of inertia forces, every homogeneous stress field satisfies all the momentum equations and is therefore a physically acceptable solution. Let the homogeneous stress component τ_{12} be given by

$$\tau_{12} = \tau_o e^{i\omega t} \tag{v}$$

We wish to obtain the complex number τ_o. Substituting $\tau_{12} = \tau_o e^{i\omega t}$ into the constitutive equation for τ_{12}:

$$\tau_{12} + \lambda \frac{d\tau_{12}}{dt} = 2\mu D_{12} = \mu\, i\, \omega \left(\frac{u_o}{h}\right) e^{i\omega t} \tag{vi}$$

one obtains

$$\frac{\tau_o}{\left(\frac{u_o}{h}\right)} = \frac{\mu i\, \omega}{(1 + \lambda i\, \omega)} = \frac{\lambda\mu\omega^2}{(1 + \lambda^2 \omega^2)} + i \frac{\mu\, \omega}{(1 + \lambda^2 \omega^2)} \tag{vii}$$

The ratio $\dfrac{\tau_o}{\left(\frac{u_o}{h}\right)} \equiv G^*$ is known as the **complex shear modulus**, which can be written as

$$G^* = G'(\omega) + iG''(\omega) \tag{8.1.14a}$$

The real part of this complex modulus is

$$G' = \frac{\lambda\mu\omega^2}{(1 + \lambda^2\omega^2)} \tag{8.1.14b}$$

and the imaginary part is

$$G'' = \frac{\mu\omega}{(1 + \lambda^2\omega^2)} \tag{8.1.14c}$$

If we write $\frac{\mu}{\lambda}$ as G, the spring constant in the spring-dashpot model, we have

$$G' = \frac{\mu^2\omega^2 G}{(G^2 + \mu^2\omega^2)} \tag{viiia}$$

and

$$G'' = \frac{\mu\omega G^2}{(G^2 + \mu^2\omega^2)} \tag{viiib}$$

We note that as limiting cases of the Maxwell model, a purely elastic element has $\mu \to \infty$ so that $G' = G$ and $G'' = 0$ and a purely viscous element has $G \to \infty$ so that $G' = 0$ and $G'' = \mu\omega$. Thus, G' characterizes the extent of elasticity of the fluid which is capable of storing elastic energy whereas G'' characterizes the extent of loss of energy due to

viscous dissipation of the fluid. Thus, G' is called the **storage modulus** and G'' is called the **loss modulus.**

writing

$$G^* = G' + iG'' = |G^*|e^{i\delta} \tag{8.1.15a}$$

where

$$|G^*| = (G'^2 + G''^2)^{1/2} \tag{8.1.15b}$$

and

$$\tan\delta = \frac{G''}{G'} \tag{8.1.15c}$$

we have,

$$G^* e^{i\omega t} = |G^*|e^{i(\omega t + \delta)} \tag{ix}$$

Therefore, taking the real part of Eq. (v), we obtain, with Eq. (ix)

$$\tau_{12} = |G^*|\cos(\omega t + \delta)\left(\frac{u_o}{h}\right) \tag{x}$$

Thus, for a Maxwell fluid, the shear stress response in a sinusoidal oscillatory experiment under the condition that the inertia effects are negligible is

$$T_{12} = \tau_{12} = \left(\frac{u_o}{h}\right)\frac{\mu\,\omega}{\sqrt{(1+\lambda^2\omega^2)}}\cos(\omega t + \tan^{-1}\frac{1}{\lambda\omega}) \tag{xi}$$

The angle δ is known as the **phase angle**. For a purely elastic material in a sinusoidally oscillation, the stress and the strain are oscillating in the same phase ($\delta = 0$) whereas for a purely viscous fluid, the stress is 90° ahead of the strain.

8.2 Generalized Linear Maxwell Fluid with Discrete Relaxation Spectra

A linear Maxwell fluid with N discrete relaxation spectra is defined by the following constitutive equation:

$$\boldsymbol{\tau} = \sum_{1}^{N}\boldsymbol{\tau}_n \tag{8.2.1a}$$

where

$$\boldsymbol{\tau}_n + \lambda_n\frac{\partial\boldsymbol{\tau}_n}{\partial t} = 2\mu_n\mathbf{D} \tag{8.2.1b}$$

The mechanical analog for this constitutive equation may be represented by N Maxwell elements connected in parallel. The shear relaxation function is the sum of the N relaxation functions each with a different relaxation time λ_n:

$$\phi(t) = \sum_{1}^{N} \frac{\mu_n}{\lambda_n} e^{-t/\lambda_n} \tag{8.2.2}$$

It can be shown that Eqs. (8.2.1) is equivalent to the following constitutive equation

$$\boldsymbol{\tau} + \sum_{1}^{N} a_n \frac{\partial^n \boldsymbol{\tau}}{\partial t^n} = b_o \mathbf{D} + \sum_{1}^{N-1} b_n \frac{\partial^n \mathbf{D}}{\partial t^n} \tag{8.2.3}$$

We demonstrate this equivalence for the case of $N = 2$ as follows: When $N = 2$,

$$\tau = \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 \tag{8.2.4a}$$

and

$$\boldsymbol{\tau}_1 + \lambda_1 \frac{\partial \boldsymbol{\tau}_1}{\partial t} = 2\mu_1 \mathbf{D} \quad \text{and} \quad \boldsymbol{\tau}_2 + \lambda_2 \frac{\partial \boldsymbol{\tau}_2}{\partial t} = 2\mu_2 \mathbf{D} \tag{8.2.4b}$$

Thus

$$(\lambda_1 + \lambda_2)\frac{\partial \boldsymbol{\tau}}{\partial t} = \lambda_1 \frac{\partial \boldsymbol{\tau}_1}{\partial t} + \lambda_2 \frac{\partial \boldsymbol{\tau}_2}{\partial t} + \lambda_2 \frac{\partial \boldsymbol{\tau}_1}{\partial t} + \lambda_1 \frac{\partial \boldsymbol{\tau}_2}{\partial t} = 2(\mu_1 + \mu_2)\mathbf{D} - (\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2)$$

$$+ \lambda_2 \frac{\partial \boldsymbol{\tau}_1}{\partial t} + \lambda_1 \frac{\partial \boldsymbol{\tau}_2}{\partial t} = (2\mu_1 + 2\mu_2)\mathbf{D} - \boldsymbol{\tau} + \lambda_2 \frac{\partial \boldsymbol{\tau}_1}{\partial t} + \lambda_1 \frac{\partial \boldsymbol{\tau}_2}{\partial t} \tag{i}$$

and

$$\lambda_1 \lambda_2 \frac{\partial^2 \boldsymbol{\tau}}{\partial t^2} = \lambda_2 \lambda_1 \frac{\partial^2 \boldsymbol{\tau}_1}{\partial t^2} + \lambda_1 \lambda_2 \frac{\partial^2 \boldsymbol{\tau}_2}{\partial t^2} = 2(\lambda_2 \mu_1 + \lambda_1 \mu_2)\frac{\partial \mathbf{D}}{\partial t} - \lambda_2 \frac{\partial \boldsymbol{\tau}_1}{\partial t} - \lambda_1 \frac{\partial \boldsymbol{\tau}_2}{\partial t} \tag{ii}$$

Adding Eqs. (i) and (ii), we obtain

$$\boldsymbol{\tau} + (\lambda_1 + \lambda_2)\frac{\partial \boldsymbol{\tau}}{\partial t} + \lambda_1 \lambda_2 \frac{\partial^2 \boldsymbol{\tau}}{\partial t} = 2(\mu_1 + \mu_2)\mathbf{D} + 2(\lambda_2 \mu_1 + \lambda_1 \mu_2)\frac{\partial \mathbf{D}}{\partial t} \tag{iii}$$

Let

$$a_1 = \lambda_1 + \lambda_2,\ a_2 = \lambda_1 \lambda_2,\ b_o = 2(\mu_1 + \mu_2), \text{ and } b_1 = 2(\lambda_2 \mu_1 + \lambda_1 \mu_2) \tag{iv}$$

we have

$$\boldsymbol{\tau} + a_1 \frac{\partial \boldsymbol{\tau}}{\partial t} + a_2 \frac{\partial^2 \boldsymbol{\tau}}{\partial t^2} = b_o \mathbf{D} + b_1 \frac{\partial \mathbf{D}}{\partial t} \tag{8.2.5}$$

In the above equation, if $a_2 = 0$, the equation is sometimes called the **Jeffrey's model.**

8.3 Integral Form of the Linear Maxwell Fluid and of the Generalized Linear Maxwell Fluid with Discrete Relaxation Spectra

Consider the following integral form of constitutive equation:

$$\boldsymbol{\tau} = 2\int_{-\infty}^{t} \phi(t-t')\,\mathbf{D}(t')dt' \tag{8.3.1a}$$

where

$$\phi(t) = \frac{\mu}{\lambda} e^{-t/\lambda} \tag{8.3.1b}$$

is the shear relaxation function for the linear Maxwell fluid defined by Eq. (8.1.1b). If we differentiate Eq. (8.3.1) with respect to time t, we obtain (note that t appears in both the integrand and the integration limit, we need to use the Leibnitz rule of differentiation)

$$\begin{aligned}\frac{\partial \boldsymbol{\tau}}{\partial t} &= \frac{2\mu}{\lambda}\left[\int_{-\infty}^{t} \left(-\frac{1}{\lambda}\right) \exp[-(t-t')/\lambda]\mathbf{D}(t')dt' + \mathbf{D}(t)\right] \\ &= -\left(\frac{1}{\lambda}\right)\boldsymbol{\tau} + \frac{2\mu}{\lambda}\mathbf{D}\end{aligned} \tag{i}$$

That is,

$$\boldsymbol{\tau} + \lambda\frac{\partial \boldsymbol{\tau}}{\partial t} = 2\mu\mathbf{D} \tag{8.1.1b}$$

Thus, the integral form constitutive equation, Eqs. (8.3.1) is the same as the rate form constitutive equation, Eq. (8.1.1b). Of course, Eq. (8.3.1) is nothing but the solution of the linear non-homogeneous ordinary differential equation, Eq. (8.1.1b). [See Prob. 8.6]

It is not difficult to show that the constitutive equation for the generalized linear Maxwell equation with N discrete relaxation spectra, Eq. (8.2.1) is equivalent to the following integral form

$$\boldsymbol{\tau} = 2\int_{-\infty}^{t} \sum_{1}^{N} \frac{\mu_n}{\lambda_n} \exp[-(t-t')/\lambda_n]\,\mathbf{D}(t')dt' \tag{8.3.2}$$

We may write the above equation in the following form:

$$\boldsymbol{\tau} = 2\int_{-\infty}^{t} \phi(t-t')\mathbf{D}(\mathbf{t}')dt' \tag{8.3.3}$$

where the shear relaxation function $\phi(t)$ is given by

$$\phi = \sum_{1}^{N} \frac{\mu_n}{\lambda_n} e^{-t/\lambda_n} \tag{8.3.4}$$

8.4 Generalized Linear Maxwell Fluid with a Continuous Relaxation Spectrum.

The linear Maxwell fluid with a continuous relaxation spectrum is defined by the constitutive equation:

$$\boldsymbol{\tau} = 2\int_{-\infty}^{t} \phi(t-t')\mathbf{D}(t')dt' \tag{8.4.1}$$

where the relaxation function $\phi(t)$ is given by

$$\phi(t) = \int_{o}^{\infty} \frac{H(\lambda)}{\lambda} e^{-t/\lambda} d\lambda \tag{8.4.2a}$$

The function $H(\lambda)/\lambda$ is the relaxation spectrum. Eq. (8.4.2a) can also be written

$$\phi(t) = \int_{o}^{\infty} H(\lambda)\, e^{-t/\lambda}\, d\ln\lambda \tag{8.4.2b}$$

As we shall see later that the linear Maxwell models considered so far are physically acceptable models only if the motion is such that the components of the relative deformation gradient (i.e., deformation gradient measured from the configuration at the current time t, see Section 8.5) are small. When this is the case, the components of rate of deformation tensor $\mathbf{D}$ are also small so that [see Eq. (v), Example 5.2.1]

$$\mathbf{D} \approx \frac{\partial \mathbf{E}}{\partial t} \tag{8.4.3}$$

where $\mathbf{E}$ is the infinitesimal strain measured with respect to the current configuration. Substituting the above approximation in Eq. (8.4.1) and integrating the right hand side by parts, we obtain

$$\boldsymbol{\tau} = 2\int_{-\infty}^{t} \phi(t-t')\frac{\partial \mathbf{E}}{\partial t'}dt' = 2\phi(t-t')\mathbf{E}(t')\Big|_{-\infty}^{t} - 2\int_{-\infty}^{t} \mathbf{E}(t')\frac{d\phi(t-t')}{dt'}dt' \tag{8.4.4}$$

The first term in the right hand side is zero because $\phi(\infty) = 0$ for a fluid and $\mathbf{E}(t)=0$ because the deformation is measured relative to the configuration at time t. Thus,

$$\boldsymbol{\tau} = -2\int_{-\infty}^{t} \frac{d\phi(t-t')}{dt'} \mathbf{E}(t')dt' \tag{8.4.5}$$

Or, letting $t-t' = s$, we can write the above equation as

$$\tau = 2\int_{o}^{\infty} \frac{d\phi(s)}{ds}\mathbf{E}(t-s)ds \tag{8.4.6}$$

Let

$$\frac{d\phi(s)}{ds} \equiv f(s) \tag{8.4.7}$$

we can write Eq. (8.4.6) as

$$\boldsymbol{\tau} = 2\int_{o}^{\infty} f(s)\mathbf{E}(t-s)ds \tag{8.4.8a}$$

or

$$\boldsymbol{\tau} = 2\int_{-\infty}^{t} f(t-t')\mathbf{E}(t')dt' \tag{8.4.8b}$$

The above equation is the integral form of constitutive equation for the linear Maxwell fluid written in terms of the infinitesimal strain tensor **E** (instead of the rate of deformation tensor **D**). The function $f(s)$ in this equation is known as the **memory function**. The relation between the memory function and the relaxation function is given by Eq. (8.4.7).

The constitutive equation given in Eq. (8.4.8) can be viewed as the superposition of all the stresses, weighted by the memory function $f(s)$, caused by the deformation of the fluid particle (relative to the current time) at all the past time ($t' = -\infty$ to the current time t).

For the linear Maxwell fluid with one relaxation time, the memory function is given by

$$f(s) = \frac{d}{ds}\phi(s) = \frac{d}{ds}\left(\frac{\mu}{\lambda}e^{-s/\lambda}\right) = -\frac{\mu}{\lambda^2}e^{-s/\lambda} \tag{8.4.9}$$

For the linear Maxwell fluid with discrete relaxation spectra, the memory function is:

$$f(s) = -\sum_{1}^{N} \frac{\mu_N}{\lambda_N^2}e^{-s/\lambda_N} \tag{8.4.10}$$

and for the Maxwell fluid with a continuous spectrum

$$f(s) = -\int_{o}^{\infty} \frac{H(\lambda)}{\lambda^2}e^{-s/\lambda}d\lambda \tag{8.4.11}$$

Part B Nonlinear Viscoelastic Fluid

8.5 Current Configuration as Reference Configuration

Let $\mathbf{x}$ be the position vector of a particle at current time t, and let $\mathbf{x}'$ be the position vector of the same particle at time τ. Then the equation

$$\mathbf{x}' = \mathbf{x}'_t(\mathbf{x},\tau) \quad \text{with} \quad \mathbf{x} = \mathbf{x}'_t(\mathbf{x},t) \tag{8.5.1}$$

defines the motion of a continuum using the current time t as the reference time. The subscript t in the function $\mathbf{x}'_t(\mathbf{x}, \tau)$ indicates that the current t is the reference time and as such $\mathbf{x}'_t(\mathbf{x}, \tau)$ is also a function of t

For a given velocity field $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$, the velocity at the position $\mathbf{x}'$ at time τ is $\mathbf{v} = \mathbf{v}(\mathbf{x}', \tau)$. On the other hand, for a particular particle (i.e., for fixed $\mathbf{x}$ and t), the velocity at time τ is given by $\left(\frac{\partial \mathbf{x}'_t}{\partial \tau}\right)_{\mathbf{x},t-\text{fixed}}$. Thus,

$$\mathbf{v}(\mathbf{x}',\tau) = \left(\frac{\partial \mathbf{x}'_t}{\partial \tau}\right) \tag{8.5.2}$$

Equation (8.5.2) allows one to obtain the pathline equations from a given velocity field, using the current time t as the reference time.

Example 8.5.1

Given the velocity field of the steady unidirectional flow

$$v_1 = v(x_2), \; v_2 = 0, \; v_3 = 0 \tag{i}$$

describe the motion of the particles by using the current time t as the reference time.

Solution. From the given velocity field, we have, the velocity components at the position (x_1', x_2', x_3') at time τ:

$$v_1 = v(x_2'), \quad v_2 = 0, \quad v_3 = 0 \tag{ii}$$

Thus, with $\mathbf{x}' = x'_i\mathbf{e}_i$, Eq. (8.5.2) gives

$$\frac{\partial x_1'}{\partial \tau} = v(x_2'), \quad \frac{\partial x_2'}{\partial \tau} = 0, \quad \frac{\partial x_3'}{\partial \tau} = 0 \tag{iii}$$

From $\frac{\partial x_2'}{\partial \tau} = 0$, we have

$$x_2' = f(x_1, x_2, x_3, t)$$

But, at $\tau = t, x_2' = x_2$, therefore, for all τ

$$x_2' = x_2 \tag{iv}$$

Similarly, for all τ

$$x_3' = x_3 \tag{v}$$

Since $x_2' = x_2$ for all τ, therefore, from Eq. (ii)

$$\frac{\partial x_1'}{\partial \tau} = v(x_2)$$

Thus

$$x_1' = v(x_2)\tau + g(x_1, x_2, x_3, t)$$

At $\tau = t$, $x_1' = x_1$, therefore

$$x_1 = v(x_2)t + g(x_1, x_2, x_3, t)$$

from which

$$g(x_1, x_2, x_3, t) = x_1 - v(x_2)t$$

and

$$x_1' = v(x_2)\tau + x_1 - v(x_2)t \tag{vi}$$

Thus,

$$x_1' = x_1 + v(x_2)(\tau - t)$$
$$x_2' = x_2$$
$$x_3' = x_3$$

8.6 Relative Deformation Gradient

Let $d\mathbf{x}$ and $d\mathbf{x}'$ be the differential vectors representing the same material element at time t and τ, respectively. Then they are related by

$$d\mathbf{x}' = \mathbf{x}'_t(\mathbf{x}+d\mathbf{x},\tau) - \mathbf{x}'_t(\mathbf{x},\tau) = (\nabla \mathbf{x}'_t)d\mathbf{x} \tag{i}$$

That is

$$d\mathbf{x}' = \mathbf{F}_t d\mathbf{x} \tag{8.6.1}$$

The tensor

$$\mathbf{F}_t \equiv \nabla \mathbf{x}'_t \tag{8.6.2}$$

is known as the **relative deformation gradient.** Here, the adjective "relative " indicates that the deformation gradient is relative to the configuration at the current time. We note that for $\tau = t$, $d\mathbf{x}' = d\mathbf{x}$ so that

$$\mathbf{F}_t(t) = \mathbf{I} \tag{8.6.3}$$

In rectangular Cartesian coordinates,

$$[\mathbf{F}_t] \equiv [\nabla \mathbf{x}'_t] = \begin{bmatrix} \frac{\partial x_1'}{\partial x_1} & \frac{\partial x'_1}{\partial x_2} & \frac{\partial x_1'}{\partial x_3} \\ \frac{\partial x_2'}{\partial x_1} & \frac{\partial x_2'}{\partial x_2} & \frac{\partial x_2'}{\partial x_3} \\ \frac{\partial x_3'}{\partial x_1} & \frac{\partial x_3'}{\partial x_2} & \frac{\partial x_3'}{\partial x_3} \end{bmatrix} \tag{8.6.4}$$

In cylindrical coordinates, with pathline equations given by

$$r' = r'(r, \theta, z, \tau), \quad \theta' = \theta'(r, \theta, z, \tau), \quad z' = z'(r, \theta, z, \tau) \tag{8.6.5}$$

the two point components of $\mathbf{F}_t$, with respect to $(\mathbf{e}'_r, \mathbf{e}'_\theta, \mathbf{e}'_z)$ at time τ and $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ at the current time t are given by the matrix

$$[\mathbf{F}_t] = \begin{bmatrix} \frac{\partial r'}{\partial r} & \frac{1}{r}\frac{\partial r'}{\partial \theta} & \frac{\partial r'}{\partial z} \\ \frac{r'\partial \theta'}{\partial r} & \frac{r'}{r}\frac{\partial \theta'}{\partial \theta} & \frac{r'\partial \theta'}{\partial z} \\ \frac{\partial z'}{\partial r} & \frac{1}{r}\frac{\partial z'}{\partial \theta} & \frac{\partial z'}{\partial z} \end{bmatrix}_{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z, \mathbf{e}_r', \mathbf{e}_\theta', \mathbf{e}_z'} \tag{8.6.6}$$

In spherical coordinates, with pathline equations given by

$$r' = r'(r, \theta, \phi, \tau), \quad \theta' = \theta'(r, \theta, \phi, \tau), \quad \phi' = \phi'(r, \theta, \phi, \tau) \tag{8.6.7}$$

the two point components of $\mathbf{F}_t$, with respect to $(\mathbf{e}'_r, \mathbf{e}'_\theta, \mathbf{e}'_\phi)$ at time τ and $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$ at the current time t are given by the matrix

$$[\mathbf{F}_t] = \begin{bmatrix} \frac{\partial r'}{\partial r} & \frac{1}{r}\frac{\partial r'}{\partial \theta} & \frac{\partial r'}{r\sin\theta\partial\phi} \\ \frac{r'\partial \theta'}{\partial r} & \frac{r'}{r}\frac{\partial \theta'}{\partial \theta} & \frac{r'\partial \theta'}{r\sin\theta\partial\phi} \\ \frac{r'\sin\theta'\partial\phi'}{\partial r} & \frac{r'\sin\theta'\partial\phi'}{r\partial\theta} & \frac{\partial z'}{r\sin\theta\partial\phi} \end{bmatrix}_{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi, \mathbf{e}_r', \mathbf{e}_\theta', \mathbf{e}_\phi'} \tag{8.6.8}$$

8.7 Relative Deformation Tensors

The descriptions of the relative deformation tensors (using the current time t as reference time) are similar to those of the deformation tensors using a fixed reference time. [See Chapter 3, Section 3. 18 to 3.29]. Indeed by polar decomposition theorem (Section 3.21)

$$\mathbf{F}_t = \mathbf{R}_t \mathbf{U}_t = \mathbf{V}_t \mathbf{R}_t \tag{8.7.1}$$

where $\mathbf{U}_t$ and $\mathbf{V}_t$ are relative right and left stretch tensor respectively and $\mathbf{R}_t$ is the relative rotation tensor. Note

$$\mathbf{F}_t(t) = \mathbf{U}_t(t) = \mathbf{V}_t(t) = \mathbf{R}_t(t) = \mathbf{I} \tag{8.7.2}$$

From Eq. (8.7.1) , we clearly also have

$$\mathbf{V}_t = \mathbf{R}_t\,\mathbf{U}_t\,\mathbf{R}_t^T \tag{8.7.3}$$

and

$$\mathbf{U}_t = \mathbf{R}_t^T\,\mathbf{V}_t\,\mathbf{R}_t \tag{8.7.4}$$

The relative right Cauchy-Green deformation tensor $\mathbf{C}_t$is defined by

$$\mathbf{C}_t = \mathbf{U}_t^2 = \mathbf{F}_t^T\mathbf{F}_t \tag{8.7.5}$$

and the relative left Cauchy-Green deformation tensor $\mathbf{B}_t$ is defined by

$$\mathbf{B}_t = \mathbf{V}_t^2 = \mathbf{F}_t\mathbf{F}_t^T \tag{8.7.6}$$

and these two tensors are related by

$$\mathbf{B}_t = \mathbf{R}_t\,\mathbf{C}_t\,\mathbf{R}_t^T \text{ and } \mathbf{C}_t = \mathbf{R}_t^T\,\mathbf{B}_t\,\mathbf{R}_t \tag{8.7.7}$$

The tensors $\mathbf{C}_t^{-1}$ and $\mathbf{B}_t^{-1}$ are often encountered in the literature. They are known as the **relative Finger deformation tensor** and the **relative Piola deformation tensor** respectively.

We note that

$$\mathbf{C}_t(\mathbf{x},t) = \mathbf{B}_t(\mathbf{x},t) = \mathbf{C}_t^{-1}(\mathbf{x},t) = \mathbf{B}_t^{-1}(\mathbf{x},t) = \mathbf{I}$$

Example 8.7.1

Show that if $d\mathbf{x}^{(1)}$ and $d\mathbf{x}^{(2)}$ are two material elements emanating from a point P at time t and $\mathbf{dx}'^{(1)}$ and $\mathbf{dx}'^{(2)}$ are the corresponding elements at time τ, then

$$d\mathbf{x}'^{(1)}\cdot d\mathbf{x}'^{(2)} = d\mathbf{x}^{(1)}\cdot\mathbf{C}_t d\mathbf{x}^{(2)} \tag{8.7.8}$$

and

$$d\mathbf{x}^{(1)}\cdot d\mathbf{x}^{(2)} = d\mathbf{x}'^{(1)}\cdot\mathbf{B}_t^{-1}d\mathbf{x}'^{(2)} \tag{8.7.9}$$

Solution. From Eq. (8.6.1), we have

$$d\mathbf{x}'^{(1)}\cdot d\mathbf{x}'^{(2)} = [(\mathbf{F}_t)d\mathbf{x}^{(1)}]\cdot[(\mathbf{F}_t)d\mathbf{x}^{(2)}] \tag{i}$$

By the definition of the transpose

$$d\mathbf{x}'^{(1)}\cdot d\mathbf{x}'^{(2)} = d\mathbf{x}^{(1)}\cdot(\mathbf{F}_t)^T(\mathbf{F}_t)d\mathbf{x}^{(2)} \tag{ii}$$

Thus ,

$$d\mathbf{x}'^{(1)} \cdot d\mathbf{x}'^{(2)} = d\mathbf{x}^{(1)} \cdot \mathbf{C}_t d\mathbf{x}^{(2)} \tag{iii}$$

Also, since

$$d\mathbf{x} = \mathbf{F}_t^{-1} d\mathbf{x}' \tag{8.7.10}$$

therefore,

$$d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} = [(\mathbf{F}_t^{-1}) d\mathbf{x}'^{(1)}] \cdot [(\mathbf{F}_t^{-1}) d\mathbf{x}'^{(2)}] = d\mathbf{x}'^{(1)} \cdot \mathbf{B}_t^{-1} d\mathbf{x}'^{(2)} \tag{iv}$$

Let $d\mathbf{x} = ds\mathbf{e}_1$ be a material element at the current time t and $d\mathbf{x}' = ds'\mathbf{n}$ be the same material element at time τ , (where $\mathbf{e}_1$ is a unit vector in a coordinate direction and $\mathbf{n}$ is a unit vector), then Eq. (8.7.8) gives

$$(C_t)_{11} = \mathbf{e}_1 \cdot \mathbf{C}_t \, \mathbf{e}_1 = \left(\frac{ds'}{ds}\right)^2 \tag{8.7.11}$$

On the other hand, if $d\mathbf{x}' = ds'\mathbf{e}_1$ is a material element at time τ and $d\mathbf{x} = ds\mathbf{n}$ is the same material element at current time t, then Eq. (8.7.9) gives

$$(B_t^{-1})_{11} = \mathbf{e}_1 \cdot \mathbf{B}_t^{-1} \, \mathbf{e}_1 = \left(\frac{ds}{ds'}\right)^2 \tag{8.7.12}$$

The meaning of the other components can be obtained using Eq. (8.7.8) and (8.7.9). [See also Sections 3.23 to 3.26 on finite deformation tensors in Chapter 3. However, care must be taken in comparing equations in those sections with those in this chapter because of the difference in reference configurations.

We note that

$$\mathbf{C}_t(\mathbf{x},t) = \mathbf{B}_t(\mathbf{x},t) = \mathbf{C}_t^{-1}(\mathbf{x},t) = \mathbf{B}_t^{-1}(\mathbf{x},t) = \mathbf{I} \tag{8.7.13}$$

8.8 Calculations of the Relative Deformation Tensor

(A)Rectangular Coordinates

With the motion given by:

$$x_1' = x_1'(x_1, x_2, x_3, \tau), \quad x_2' = x_2'(x_1, x_2, x_3, \tau), \quad x_3' = x_3'(x_1, x_2, x_3, \tau) \tag{8.8.1}$$

Equations (8.7.5) and (8.6.4) give

$$(C_t)_{11} = \left(\frac{\partial x_1'}{\partial x_1}\right)^2 + \left(\frac{\partial x_2'}{\partial x_1}\right)^2 + \left(\frac{\partial x_3'}{\partial x_1}\right)^2 \tag{8.8.2a}$$

$$(C_t)_{22} = \left(\frac{\partial x_1'}{\partial x_2}\right)^2 + \left(\frac{\partial x_2'}{\partial x_2}\right)^2 + \left(\frac{\partial x_3'}{\partial x_2}\right)^2 \tag{8.8.2.b}$$

$$(C_t)_{33} = \left(\frac{\partial x_1'}{\partial x_3}\right)^2 + \left(\frac{\partial x_2'}{\partial x_3}\right)^2 + \left(\frac{\partial x_3'}{\partial x_3}\right)^2 \tag{8.8.2.c}$$

$$(C_t)_{12} = \left(\frac{\partial x_1'}{\partial x_1}\right)\left(\frac{\partial x_1'}{\partial x_2}\right) + \left(\frac{\partial x'_2}{\partial x_1}\right)\left(\frac{\partial x_2'}{\partial x_2}\right) + \left(\frac{\partial x_3'}{\partial x_1}\right)\left(\frac{\partial x_3'}{\partial x_2}\right) \tag{8.8.2.d}$$

$$(C_t)_{23} = \left(\frac{\partial x_1'}{\partial x_2}\right)\left(\frac{\partial x_1'}{\partial x_3}\right) + \left(\frac{\partial x_2'}{\partial x_2}\right)\left(\frac{\partial x'_2}{\partial x_3}\right) + \left(\frac{\partial x_3'}{\partial x_2}\right)\left(\frac{\partial x_3'}{\partial x_3}\right) \tag{8.8.2.e}$$

$$(C_t)_{13} = \left(\frac{\partial x_1'}{\partial x_1}\right)\left(\frac{\partial x_1'}{\partial x_3}\right) + \left(\frac{\partial x_2'}{\partial x_1}\right)\left(\frac{\partial x'_2}{\partial x_3}\right) + \left(\frac{\partial x_3'}{\partial x_1}\right)\left(\frac{\partial x_3'}{\partial x_3}\right) \tag{8.8.2.f}$$

To obtain the components of $\mathbf{C}_t^{-1}$, one can either invert the symmetric matrix whose components are given by Eqs. (8.8.2), or one can obtain them from the inverse functions of Eq. (8.8.1), i.e.,

$$x_1 = x_1(x_1', x_2', x_3', \tau), \quad x_2 = x_2(x_1', x_2', x_3', \tau), \quad x_3 = x_3(x_1', x_2', x_3', \tau) \tag{8.8.3}$$

Indeed, it can be obtained

$$(C_t^{-1})_{11} = \left(\frac{\partial x_1}{\partial x_1'}\right)^2 + \left(\frac{\partial x_1}{\partial x_2'}\right)^2 + \left(\frac{\partial x_1}{\partial x_3'}\right)^2 \tag{8.8.4a}$$

$$(C_t^{-1})_{22} = \left(\frac{\partial x_2}{\partial x_1'}\right)^2 + \left(\frac{\partial x_2}{\partial x_2'}\right)^2 + \left(\frac{\partial x_2}{\partial x_3'}\right)^2 \tag{8.8.4.b}$$

$$(C_t^{-1})_{12} = \left(\frac{\partial x_1}{\partial x_1'}\right)\left(\frac{\partial x_2}{\partial x_1'}\right) + \left(\frac{\partial x_1}{\partial x_2'}\right)\left(\frac{\partial x_2}{\partial x_2'}\right) + \left(\frac{\partial x_1}{\partial x_3'}\right)\left(\frac{\partial x_2}{\partial x_3'}\right) \tag{8.8.4.c}$$

etc.

Example 8.8.1

Find the relative right Cauchy-Green deformation tensor and its inverse for the velocity field given in Example 8.5.1

Solution. Since

$$x_1' = x_1 + v(x_2)(\tau - t), \quad x_2' = x_2, \quad x_3' = x_3, \tag{i}$$

we have, with $k \equiv dv/dx_2$,

$$[\mathbf{F}_t] = \begin{bmatrix} 1 & k(\tau-t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{ii}$$

and

$$[\mathbf{C}_t] = \begin{bmatrix} 1 & 0 & 0 \\ k(\tau-t) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & k(\tau-t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & k(\tau-t) & 0 \\ k(\tau-t) & k^2(\tau-t)^2+1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{iii}$$

The inverse of Eqs.(i) are

$$x_1 = x_1' - v(x_2)(\tau-t), \quad x_2 = x_2', \quad x_3 = x_3', \tag{iv}$$

$$[\mathbf{F}_t^{-1}] = \begin{bmatrix} 1 & -k(\tau-t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{vii}$$

$$[\mathbf{C}_t^{-1}] = \begin{bmatrix} 1 & -k(\tau-t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -k(\tau-t) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+k^2(\tau-t)^2 & -k(\tau-t) & 0 \\ -k(\tau-t) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{viii}$$

(B) Cylindrical Coordinates

The procedures described below for obtaining the formulas for computing the components for the relative right Cauchy-Green tensors, are the same as those used in Section 3.30 of Chapter 3.

We have

$$d\mathbf{x}' = dr'\mathbf{e}'_r + r'd\theta'\mathbf{e}'_\theta + dz'\mathbf{e}'_z, \quad d\mathbf{x} = dr\mathbf{e}_r + rd\theta\mathbf{e}_\theta + dz\mathbf{e}_z \tag{8.8.5}$$

Thus, from $d\mathbf{x}' = \mathbf{F}_t d\mathbf{x}$,

$$dr' = dr(\mathbf{e}'_r \cdot \mathbf{F}_t\mathbf{e}_r) + rd\theta(\mathbf{e}'_r \cdot \mathbf{F}_t\mathbf{e}_\theta) + dz(\mathbf{e}'_r \cdot \mathbf{F}_t\mathbf{e}_z) \tag{i}$$

But from calculus

$$dr' = \frac{\partial r'}{\partial r}dr + \frac{\partial r'}{\partial \theta}d\theta + \frac{\partial r'}{\partial z}dz \tag{ii}$$

Thus, we have

$$\mathbf{e}'_r \cdot \mathbf{F}_t\mathbf{e}_r = \frac{\partial r'}{\partial r} \quad \mathbf{e}'_r \cdot \mathbf{F}_t\mathbf{e}_\theta = \frac{1}{r}\frac{\partial r'}{\partial \theta} \quad \mathbf{e}'_r \cdot \mathbf{F}_t\mathbf{e}_z = \frac{\partial r'}{\partial z} \tag{iii}$$

Similarly, one can obtain

$$\mathbf{e}'_\theta \cdot \mathbf{F}_t\mathbf{e}_r = r'\frac{\partial \theta'}{\partial r} \quad \mathbf{e}'_\theta \cdot \mathbf{F}_t\mathbf{e}_\theta = \frac{r'}{r}\frac{\partial \theta'}{\partial \theta} \quad \mathbf{e}'_\theta \cdot \mathbf{F}_t\mathbf{e}_z = r'\frac{\partial \theta'}{\partial z} \tag{iv}$$

and

$$\mathbf{e}'_z \cdot \mathbf{F}_t \mathbf{e}_r = \frac{\partial z'}{\partial r} \quad \mathbf{e}'_z \cdot \mathbf{F}_t \mathbf{e}_\theta = \frac{1}{r}\frac{\partial z'}{\partial \theta} \quad \mathbf{e}'_z \cdot \mathbf{F}_t \mathbf{e}_z = \frac{\partial z'}{\partial z} \tag{v}$$

Equations (iii) to (v) are equivalent to the following equations:

$$\mathbf{F}_t \mathbf{e}_r = \frac{\partial r'}{\partial r}\mathbf{e}'_r + r'\frac{\partial \theta'}{\partial r}\mathbf{e}'_\theta + \frac{\partial z'}{\partial r}\mathbf{e}'_z \tag{8.8.6a}$$

$$\mathbf{F}_t \mathbf{e}_\theta = \frac{1}{r}\frac{\partial r'}{\partial \theta}\mathbf{e}'_r + \frac{r'}{r}\frac{\partial \theta'}{\partial \theta}\mathbf{e}'_\theta + \frac{1}{r}\frac{\partial z'}{\partial \theta}\mathbf{e}'_z \tag{8.8.6b}$$

$$\mathbf{F}_t \mathbf{e}_z = \frac{\partial r'}{\partial z}\mathbf{e}'_r + r'\frac{\partial \theta'}{\partial z}\mathbf{e}'_\theta + \frac{\partial z'}{\partial z}\mathbf{e}'_z \tag{8.8.6c}$$

As already noted in the previous section, the matrix

$$[\mathbf{F}_t] = \begin{bmatrix} \dfrac{\partial r'}{\partial r} & \dfrac{1}{r}\dfrac{\partial r'}{\partial \theta} & \dfrac{\partial r'}{\partial z} \\ r'\dfrac{\partial \theta'}{\partial r} & \dfrac{r'}{r}\dfrac{\partial \theta'}{\partial \theta} & r'\dfrac{\partial \theta'}{\partial z} \\ \dfrac{\partial z'}{\partial r} & \dfrac{1}{r}\dfrac{\partial z'}{\partial \theta} & \dfrac{\partial z'}{\partial z} \end{bmatrix}_{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z, \mathbf{e}_r', \mathbf{e}_\theta', \mathbf{e}_z'} \tag{8.8.7}$$

being obtained using bases at two different points, give the **two point components** of the tensor $\mathbf{F}_t$. Now, from

$$\mathbf{C}_t = \mathbf{F}_t^T \mathbf{F}_t$$

we have

$$\begin{aligned}
(\mathbf{C}_t)_{rr} &= \mathbf{e}_r \cdot \mathbf{F}_t^T \mathbf{F}_t \mathbf{e}_r = \mathbf{e}_r \cdot \mathbf{F}_t^T \left[\frac{\partial r'}{\partial r}\mathbf{e}'_r + r'\frac{\partial \theta'}{\partial r}\mathbf{e}'_\theta + \frac{\partial z'}{\partial r}\mathbf{e}'_z\right] \\
&= \frac{\partial r'}{\partial r}(\mathbf{e}_r \cdot \mathbf{F}_t^T \mathbf{e}'_r) + r'\frac{\partial \theta'}{\partial r}(\mathbf{e}_r \cdot \mathbf{F}_t^T \mathbf{e}'_\theta) + \frac{\partial z'}{\partial r}(\mathbf{e}_r \cdot \mathbf{F}_t^T \mathbf{e}'_z) \\
&= \frac{\partial r'}{\partial r}(\mathbf{e}'_r \cdot \mathbf{F}_t \mathbf{e}_r) + r'\frac{\partial \theta'}{\partial r}(\mathbf{e}'_\theta \cdot \mathbf{F}_t \mathbf{e}_r) + \frac{\partial z'}{\partial r}(\mathbf{e}'_z \cdot \mathbf{F}_t \mathbf{e}_r) \\
&= \left(\frac{\partial r'}{\partial r}\right)^2 + \left(r'\frac{\partial \theta'}{\partial r}\right)^2 + \left(\frac{\partial z'}{\partial r}\right)^2
\end{aligned} \tag{vi}$$

$$\begin{aligned}
(\mathbf{C}_t)_{r\theta} &= \mathbf{e}_r \cdot \mathbf{F}_t^T \mathbf{F}_t \mathbf{e}_\theta = \mathbf{e}_r \cdot \mathbf{F}_t^T \left[\frac{1}{r}\frac{\partial r'}{\partial \theta}\mathbf{e}'_r + \frac{r'}{r}\frac{\partial \theta'}{\partial \theta}\mathbf{e}'_\theta + \frac{1}{r}\frac{\partial z'}{\partial \theta}\mathbf{e}'_z\right] \\
&= \frac{1}{r}\frac{\partial r'}{\partial \theta}(\mathbf{e}_r \cdot \mathbf{F}_t^T \mathbf{e}'_r) + \frac{r'}{r}\frac{\partial \theta'}{\partial \theta}(\mathbf{e}_r \cdot \mathbf{F}_t^T \mathbf{e}'_\theta) + \frac{1}{r}\frac{\partial z'}{\partial \theta}(\mathbf{e}_r \cdot \mathbf{F}_t^T \mathbf{e}'_z)
\end{aligned}$$

$$= \frac{1}{r}\frac{\partial r'}{\partial \theta}(\mathbf{e}'_r \cdot \mathbf{F}_t \mathbf{e}_r) + \frac{r'}{r}\frac{\partial \theta'}{\partial \theta}(\mathbf{e}'_\theta \cdot \mathbf{F}_t \mathbf{e}_r) + \frac{1}{r}\frac{\partial z'}{\partial \theta}(\mathbf{e}'_z \cdot \mathbf{F}_t \mathbf{e}_r)$$

$$= \left(\frac{1}{r}\frac{\partial r'}{\partial \theta}\right)\left(\frac{\partial r'}{\partial r}\right) + \frac{r'}{r}\left(\frac{\partial \theta'}{\partial \theta}\right)\left(\frac{r'\partial \theta'}{\partial r}\right) + \left(\frac{1}{r}\frac{\partial z'}{\partial \theta}\right)\left(\frac{\partial z'}{\partial r}\right) \tag{vii}$$

Other components can be derived similarly. Thus, with the pathline equations given by

$$r' = r'(r, \theta, z, \tau), \quad \theta' = \theta'(r, \theta, z, \tau), \quad z' = z'(r, \theta, z, \tau) \tag{8.8.8}$$

the components of $\mathbf{C}_t$ with respect to the bases $\mathbf{e}_r, \mathbf{e}_\theta$ and $\mathbf{e}_z$ are:

$$(\mathbf{C}_t)_{rr} = \left(\frac{\partial r'}{\partial r}\right)^2 + \left(r'\frac{\partial \theta'}{\partial r}\right)^2 + \left(\frac{\partial z'}{\partial r}\right)^2 \tag{8.8.9a}$$

$$(\mathbf{C}_t)_{\theta\theta} = \frac{1}{r^2}\left[\left(\frac{\partial r'}{\partial \theta}\right)^2 + \left(r'\frac{\partial \theta'}{\partial \theta}\right)^2 + \left(\frac{\partial z'}{\partial \theta}\right)^2\right] \tag{8.8.9b}$$

$$(\mathbf{C}_t)_{zz} = \left(\frac{\partial r'}{\partial z}\right)^2 + \left(r'\frac{\partial \theta'}{\partial z}\right)^2 + \left(\frac{\partial z'}{\partial z}\right)^2 \tag{8.8.9c}$$

$$(\mathbf{C}_t)_{r\theta} = \frac{1}{r}\left[\left(\frac{\partial r'}{\partial r}\right)\left(\frac{\partial r'}{\partial \theta}\right) + r'^2\left(\frac{\partial \theta'}{\partial r}\right)\left(\frac{\partial \theta'}{\partial \theta}\right) + \left(\frac{\partial z'}{\partial r}\right)\left(\frac{\partial z'}{\partial \theta}\right)\right] \tag{8.8.9d}$$

$$(\mathbf{C}_t)_{rz} = \left[\left(\frac{\partial r'}{\partial r}\right)\left(\frac{\partial r'}{\partial z}\right) + r'^2\left(\frac{\partial \theta'}{\partial r}\right)\left(\frac{\partial \theta'}{\partial z}\right) + \left(\frac{\partial z'}{\partial r}\right)\left(\frac{\partial z'}{\partial z}\right)\right] \tag{8.8.9e}$$

$$(\mathbf{C}_t)_{\theta z} = \frac{1}{r}\left[\left(\frac{\partial r'}{\partial \theta}\right)\left(\frac{\partial r'}{\partial z}\right) + r'^2\left(\frac{\partial \theta'}{\partial \theta}\right)\left(\frac{\partial \theta'}{\partial z}\right) + \left(\frac{\partial z'}{\partial \theta}\right)\left(\frac{\partial z'}{\partial z}\right)\right] \tag{8.8.9f}$$

To obtain the components of $\mathbf{C}_t^{-1}$, one can either invert the symmetric matrix whose components are given by Eqs. (8.8.9), or one can obtain them from the inverse functions of Eq. (8.8.8), i.e.,

$$r = r(r', \theta', z', \tau), \quad \theta = \theta(r', \theta', z', \tau), \quad z = z(r', \theta', z', \tau) \tag{8.8.10}$$

In fact, from $d\mathbf{x} = \mathbf{F}_t^{-1} d\mathbf{x}'$, we obtain

$$dr = \mathbf{e}_r \cdot d\mathbf{x} = dr'(\mathbf{e}_r \cdot \mathbf{F}_t^{-1}\mathbf{e}'_r) + r'd\theta'(\mathbf{e}_r \cdot \mathbf{F}_t^{-1}\mathbf{e}'_\theta) + dz'(\mathbf{e}_r \cdot \mathbf{F}_t^{-1}\mathbf{e}'_z) \tag{viii}$$

$$rd\theta = \mathbf{e}_\theta \cdot d\mathbf{x} = dr'(\mathbf{e}_\theta \cdot \mathbf{F}_t^{-1}\mathbf{e}'_r) + r'd\theta'(\mathbf{e}_\theta \cdot \mathbf{F}_t^{-1}\mathbf{e}'_\theta) + dz'(\mathbf{e}_\theta \cdot \mathbf{F}_t^{-1}\mathbf{e}'_z) \tag{ix}$$

etc. Thus,

$$\mathbf{e}_r \cdot \mathbf{F}_t^{-1}\mathbf{e}'_r = \mathbf{e}'_r \cdot \mathbf{F}_t^{-1T}\mathbf{e}_r = \frac{\partial r}{\partial r'}, \qquad \mathbf{e}_r \cdot \mathbf{F}_t^{-1}\mathbf{e}'_\theta = \mathbf{e}'_\theta \cdot \mathbf{F}_t^{-1T}\mathbf{e}_r = \frac{\partial r}{r'\partial \theta'} \tag{x}$$

$$\mathbf{e}_\theta \cdot \mathbf{F}_t^{-1}\mathbf{e}'_r = \mathbf{e}'_r \cdot \mathbf{F}_t^{-1T}\mathbf{e}_\theta = \frac{r\partial\theta}{\partial r'}, \qquad \mathbf{e}_\theta \cdot \mathbf{F}_t^{-1}\mathbf{e}'_\theta = \mathbf{e}'_\theta \cdot \mathbf{F}_t^{-1T}\mathbf{e}_\theta = \frac{r\partial\theta}{r'\partial\theta'} \tag{xi}$$

etc. These equations are equivalent to the following equations:

$$\mathbf{F}_t^{-1}\mathbf{e}'_r = \frac{\partial r}{\partial r'}\mathbf{e}_r + \frac{r\partial\theta}{\partial r'}\mathbf{e}_\theta + \frac{\partial z}{\partial r'}\mathbf{e}_z \tag{8.8.11a}$$

$$\mathbf{F}_t^{-1}\mathbf{e}'_\theta = \frac{\partial r}{r'\partial\theta'}\mathbf{e}_r + \frac{r\partial\theta}{r'\partial\theta'}\mathbf{e}_\theta + \frac{\partial z}{r'\partial\theta'}\mathbf{e}_z \tag{8.8.11b}$$

$$\mathbf{F}_t^{-1}\mathbf{e}'_z = \frac{\partial r}{\partial z'}\mathbf{e}_r + \frac{r\partial\theta}{\partial z'}\mathbf{e}_\theta + \frac{\partial z}{\partial z'}\mathbf{e}_z \tag{8.8.11c}$$

and

$$\mathbf{F}_t^{-1T}\mathbf{e}_r = \frac{\partial r}{\partial r'}\mathbf{e}'_r + \frac{\partial r}{r'\partial\theta'}\mathbf{e}'_\theta + \frac{\partial r}{\partial z'}\mathbf{e}'_z \tag{8.8.12a}$$

$$\mathbf{F}_t^{-1T}\mathbf{e}_\theta = \frac{r\partial\theta}{\partial r'}\mathbf{e}'_r + \frac{r\partial\theta}{r'\partial\theta'}\mathbf{e}'_\theta + \frac{r\partial\theta}{\partial z'}\mathbf{e}'_z \tag{8.8.12b}$$

$$\mathbf{F}_t^{-1T}\mathbf{e}_z = \frac{\partial z}{\partial r'}\mathbf{e}'_r + \frac{\partial z}{r'\partial\theta'}\mathbf{e}'_\theta + \frac{\partial z}{\partial z'}\mathbf{e}'_z \tag{8.8.12c}$$

From

$$(C_t^{-1})_{rr} = \mathbf{e}_r \cdot \mathbf{F}_t^{-1}\,\mathbf{F}_t^{-1T}\mathbf{e}_r\,, \quad (C_t^{-1})_{r\theta} = \mathbf{e}_r \cdot \mathbf{F}_t^{-1}\,\mathbf{F}_t^{-1T}\mathbf{e}_\theta \tag{8.8.13}$$

etc., we obtain, with the help of Eqs. (8.8.11) and (8.8.12),

$$(\mathbf{C}_t^{-1})_{rr} = \left(\frac{\partial r}{\partial r'}\right)^2 + \left(\frac{\partial r}{r'\partial\theta'}\right)^2 + \left(\frac{\partial r}{\partial z'}\right)^2 \tag{8.8.14a}$$

$$(\mathbf{C}_t^{-1})_{\theta\theta} = \left[\left(\frac{r\partial\theta}{\partial r'}\right)^2 + \left(\frac{r\partial\theta}{r'\partial\theta'}\right)^2 + \left(\frac{r\partial\theta}{\partial z'}\right)^2\right] \tag{8.8.14b}$$

$$(\mathbf{C}_t^{-1})_{r\theta} = \left[\left(\frac{\partial r}{\partial r'}\right)\left(\frac{r\partial\theta}{\partial r'}\right) + \left(\frac{\partial r}{r'\partial\theta'}\right)\left(\frac{r\partial\theta}{r'\partial\theta'}\right) + \left(\frac{\partial r}{\partial z'}\right)\left(\frac{r\partial\theta}{\partial z'}\right)\right] \tag{8.8.14c}$$

The other components can be easily written down following the patterns given in the above equations..

(C) Spherical coordinates

With path line equations given by

$$r' = r'(\tau, r, \theta, \phi, t), \quad \theta' = \theta'(\tau, r, \theta, \phi, t), \quad \phi' = \phi'(\tau, r, \theta, \phi, t) \tag{8.8.15}$$

the components of $\mathbf{C}_t$ with respect to $\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_\theta$ can be obtained to be

$$(\mathbf{C}_t)_{rr} = \left(\frac{\partial r'}{\partial r}\right)^2 + \left(r'\frac{\partial \theta'}{\partial r}\right)^2 + \left(\frac{r'\sin\theta'\partial\phi'}{\partial r}\right)^2 \tag{8.8.16a}$$

$$(\mathbf{C}_t)_{\theta\theta} = \frac{1}{r^2}\left[\left(\frac{\partial r'}{\partial \theta}\right)^2 + \left(r'\frac{\partial \theta'}{\partial \theta}\right)^2 + \left(\frac{r'\sin\theta'\partial\phi'}{\partial \theta}\right)^2\right] \tag{8.8.16b}$$

$$(\mathbf{C}_t)_{\phi\phi} = \frac{1}{(r\sin\theta)^2}\left[\left(\frac{\partial r'}{\partial \phi}\right)^2 + \left(r'\frac{\partial \theta'}{\partial \phi}\right)^2 + \left(\frac{r'\sin\theta'\partial\phi'}{\partial \phi}\right)^2\right] \tag{8.8.16c}$$

$$(\mathbf{C}_t)_{r\theta} = \frac{1}{r}\left[\left(\frac{\partial r'}{\partial r}\right)\left(\frac{\partial r'}{\partial \theta}\right) + r'^2\left(\frac{\partial \theta'}{\partial r}\right)\left(\frac{\partial \theta'}{\partial \theta}\right) + (r'\sin\theta')^2\left(\frac{\partial \phi'}{\partial r}\right)\left(\frac{\partial \phi'}{\partial \theta}\right)\right] \tag{8.8.16d}$$

$$(\mathbf{C}_t)_{\theta\phi} = \frac{1}{r^2\sin\theta}\left[\left(\frac{\partial r'}{\partial \theta}\right)\left(\frac{\partial r'}{\partial \phi}\right) + r'^2\left(\frac{\partial \theta'}{\partial \theta}\right)\left(\frac{\partial \theta'}{\partial \phi}\right) + (r'\sin\theta')^2\left(\frac{\partial \phi'}{\partial \theta}\right)\left(\frac{\partial \phi'}{\partial \phi}\right)\right] \tag{8.8.16e}$$

$$(\mathbf{C}_t)_{\phi r} = \frac{1}{r\sin\theta}\left[\left(\frac{\partial r'}{\partial r}\right)\left(\frac{\partial r'}{\partial \phi}\right) + r'^2\left(\frac{\partial \theta'}{\partial r}\right)\left(\frac{\partial \theta'}{\partial \phi}\right) + (r'\sin\theta')^2\left(\frac{\partial \phi'}{\partial r}\right)\left(\frac{\partial \phi'}{\partial \phi}\right)\right] \tag{8.8.16f}$$

Again, using the inverse functions of Eqs. (8.8.15), we can obtain the following components

$$(\mathbf{C}_t^{-1})_{rr} = \left(\frac{\partial r}{\partial r'}\right)^2 + \left(\frac{\partial r}{r'\partial \theta'}\right)^2 + \left(\frac{\partial r}{r'\sin\theta'\partial\phi'}\right)^2 \tag{8.8.17a}$$

$$(\mathbf{C}_t^{-1})_{\theta\theta} = \left(\frac{r\partial \theta}{\partial r'}\right)^2 + \left(\frac{r\partial \theta}{r'\partial \theta'}\right)^2 + \left(\frac{r\partial \theta}{r'\sin\theta'\partial\phi'}\right)^2 \tag{8.8.17b}$$

$$(\mathbf{C}_t^{-1})_{r\theta} = \left(\frac{\partial r}{\partial r'}\right)\left(\frac{r\partial \theta}{\partial r'}\right) + \left(\frac{\partial r}{r'\partial \theta'}\right)\left(\frac{r\partial \theta}{r'\partial \theta}\right) + \left(\frac{\partial r}{r'\sin\theta'\partial\phi'}\right)\left(\frac{r\partial \theta}{r'\sin\theta'\partial\phi'}\right) \tag{8.8.17c}$$

The other components can be easily written down following the patterns given in the above equations.

8.9 History of the Relative Deformation Tensor. Rivlin-Ericksen Tensors

The tensor $\mathbf{C}_t(\mathbf{x},\tau)$ describes the deformation at time τ of the element which is at $\mathbf{x}$ at time t. Thus, as one varies τ from $\tau = -\infty$ to $\tau = t$ in the function $\mathbf{C}_t(\mathbf{x},\tau)$, one gets the whole history of the deformation from infinitely long time ago to the present time t.

If we assume that we can expand the components of $\mathbf{C}_t$ *in Taylor series about* $\tau = t$, we have,

$$\mathbf{C}_t(\mathbf{x},\tau) = \mathbf{C}_t(\mathbf{x},t) + \left(\frac{\partial \mathbf{C}_t}{\partial \tau}\right)_{\tau=t}(\tau - t) + \frac{1}{2}\left(\frac{\partial^2 \mathbf{C}_t}{\partial \tau^2}\right)_{\tau=t}(\tau - t)^2 + \ldots \tag{8.9.1}$$

Let

$$\mathbf{A}_1 = \left(\frac{\partial \mathbf{C}_t}{\partial \tau}\right)_{\tau = t} \tag{8.9.2a}$$

$$\mathbf{A}_2 = \left(\frac{\partial^2 \mathbf{C}_t}{\partial \tau^2}\right)_{\tau = t} \tag{8.9.2b}$$

and

$$\mathbf{A}_N = \left(\frac{\partial^N \mathbf{C}_t}{\partial \tau^N}\right)_{\tau = t}, \quad N=3,4,... \tag{8.9.2c}$$

We have, (note, $\mathbf{C}_t(\mathbf{x},t) = \mathbf{I}$)

$$\mathbf{C}_t(\mathbf{x},\tau) = \mathbf{I}+(\tau-t)\mathbf{A}_1+\frac{(\tau-t)^2}{2}\mathbf{A}_2+\ldots \tag{8.9.3}$$

The tensor $\mathbf{A}_1,\mathbf{A}_2,\ldots\mathbf{A}_n$ are known as **Rivlin-Ericksen tensors.**

We see from the above equation that provided the Taylor series expansion is valid , the $\mathbf{A}_n$*'s* determine the history of relative deformation. It should be noted however, that not all histories of relative deformation can be expanded in Taylor series; For example, the stress relaxation test in which a sudden jump in deformation is imposed on the fluid, has a history of deformation which is not representable by a Taylor series.

Example 8.9.1

Find the Rivlin-Ericksen tensor for the uni-directional flows of Example 8.8.1.

Solution. We have, from Example 8.8.1

$$[\mathbf{C}_t(\mathbf{x},\tau)] = \begin{bmatrix} 1 & k(\tau-t) & 0 \\ k(\tau-t) & 1+k^2(\tau-t)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}(\tau-t)+\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\frac{(\tau-t)^2}{2} \tag{i}$$

Thus, (see Eq. (8.9.3))

$$[\mathbf{A}_1] = \begin{bmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{ii}$$

$$[\mathbf{A}_2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{iii}$$

$$[\mathbf{A}_n] = 0 \text{ for all } n \geq 3$$

where $k = dv/dx_2$.

Example 8.9.2

Given an axisymmetric velocity field in cylindrical coordinates:

$$v_r = 0, \quad v_\theta = 0, \quad v_z = v(r) \tag{i}$$

(a) Obtain the motion using current time t as reference

(b) Compute the relative deformation tensor $\mathbf{C}_t$

(c) Find the Rivlin-Ericksen tensors.

Solution. (a) Let the motion be

$$r' = r'(r, \theta, z, \tau), \quad \theta' = \theta'(r, \theta, z, \tau), \quad z' = z'(r, \theta, z, \tau) \tag{ii}$$

then, from the given velocity field, we have

$$\frac{dr'}{d\tau} = 0, \quad \frac{d\theta'}{d\tau} = 0, \quad \frac{dz'}{d\tau} = v(r') \tag{iii}$$

Integration of these equations with the conditions that at $\tau = t$, $r' = r$, $\theta' = \theta$ and $z' = z$, we obtain

$$r' = r, \qquad \theta' = \theta, \qquad z' = z + v(r)(\tau - t) \tag{iv}$$

(b) Using Eq. (8.8.9), we obtain, with $k(r) \equiv dv/dr$

$$[\mathbf{C}_t] = \begin{bmatrix} 1+k^2(\tau-t)^2 & 0 & k(\tau-t) \\ 0 & 1 & 0 \\ k(\tau-t) & 0 & 1 \end{bmatrix} \tag{v}$$

(c)From the result of part (b) , we have

$$[\mathbf{C}_t] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & k(r) \\ 0 & 0 & 0 \\ k(r) & 0 & 0 \end{bmatrix} (\tau - t) + \begin{bmatrix} 2k^2(r) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{(\tau-t)^2}{2} \tag{vi}$$

Thus, the Rivlin-Ericksen tensors are [see Eq. (8.9.3)]

$$[\mathbf{A}_1] = \begin{bmatrix} 0 & 0 & k(r) \\ 0 & 0 & 0 \\ k(r) & 0 & 0 \end{bmatrix} \tag{v}$$

$$[\mathbf{A}_2] = \begin{bmatrix} 2k^2(r) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{vi}$$

$$[\mathbf{A}_n] = 0 \quad \text{for } n \geq 3 \tag{vii}$$

Example 8.9.3

Consider the Couette flow with a velocity field given in cylindrical coordinates as

$$v_r = 0, \quad v_\theta = v(r), \quad v_z = 0 \tag{i}$$

(a) Obtain the motion using current time t as reference.

(b) Compute the relative deformation tensor $\mathbf{C}_t$.

(c) Find the Rivlin-Ericksen tensors.

Solution. (a) From the given velocity field, one has

$$\frac{dr'}{d\tau} = 0, \quad r'\frac{d\theta'}{d\tau} = v(r'), \quad \frac{dz'}{d\tau} = 0 \tag{ii}$$

Integration of the above equation gives the pathline equations to be:

$$r' = r, \quad \theta' = \theta + \frac{v(r)}{r}(\tau - t), \quad z' = z \tag{iii}$$

(b) Using Eqs. (8.8.9), one easily obtain the relative right Cauchy-Green deformation tensor to be

$$[\mathbf{C}_t(\tau)] = \begin{bmatrix} 1 + k^2(r)(\tau-t)^2 & k(r)(\tau-t) & 0 \\ k(r)(\tau-t) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [\mathbf{I}] + (\tau - t)\begin{bmatrix} 0 & k(r) & 0 \\ k(r) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{(\tau-t)^2}{2}\begin{bmatrix} 2k^2(r) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{iv}$$

$$k(r) = \left(\frac{dv}{dr} - \frac{v}{r}\right) \tag{v}$$

(c) The nonzero Rivlin Ericksen tensors are

$$[\mathbf{A}_1]_{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z} = \begin{bmatrix} 0 & k(r) & 0 \\ k(r) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{vi}$$

$$[\mathbf{A}_2]_{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z} = \begin{bmatrix} 2k^2(r) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{vii}$$

Example 8.9.4

Given the velocity field of a sink flow in spherical coordinates:

$$v_r = -\frac{a}{r^2}, \quad v_\theta = 0, \quad v_\phi = 0 \tag{i}$$

(a) Obtain the motion using current time t as reference

(b) Compute the relative deformation tensor $\mathbf{C}_t$

(c) Find the Rivlin-Ericksen tensors.

Solution. (a) Let the motion be

$$r' = r'(r,\theta,\phi,\tau), \quad \theta' = \theta'(r,\theta,\phi,\tau), \quad \phi' = \phi'(r,\theta,\phi,\tau) \tag{ii}$$

then, from the given velocity field, we have

$$\frac{dr'}{d\tau} = -\frac{a}{r'^2}, \quad \frac{d\theta'}{d\tau} = 0, \quad \frac{d\phi'}{d\tau} = 0 \tag{iii}$$

Integration of these equations with the conditions that at $\tau = t$, $r' = r$, $\theta' = \theta$ and $\phi' = \phi$, we obtain

$$r'^3 = r^3 + 3a(t-\tau), \qquad \theta' = \theta, \qquad \phi' = \phi \tag{iv}$$

(b) Using Eq. (8.8.16), we obtain, with $k \equiv dv/dr$

$$[\mathbf{C}_t] = \begin{bmatrix} r^4[r^3 + 3a(t-\tau)]^{-4/3} & 0 & 0 \\ 0 & \dfrac{[r^3 + 3a(t-\tau)]^{2/3}}{r^2} & 0 \\ 0 & 0 & \dfrac{[r^3 + 3a(t-\tau)]^{2/3}}{r^2} \end{bmatrix} \tag{v}$$

(c)From the result of part (b) , we have

$$\frac{d(C_t)_{rr}}{d\tau} = \frac{4a}{r^3}\left[1 + \frac{3a(t-\tau)}{r^3}\right]^{-\frac{7}{3}},$$

$$\frac{d^2(C_t)_{rr}}{d\tau^2} = \frac{28a^2}{r^6}\left[1 + \frac{3a(t-\tau)}{r^3}\right]^{-\frac{10}{3}},$$

$$\frac{d(C_t)_{\theta\theta}}{d\tau} = \frac{d(C_t)_{\phi\phi}}{d\tau} = -\frac{2a}{r^3}\left[1 + \frac{3a(t-\tau)}{r^3}\right]^{-\frac{1}{3}},$$

$$\frac{d^2(C_t)_{\theta\theta}}{d\tau^2} = \frac{d^2(C_t)_{\phi\phi}}{d\tau^2} = -\frac{2a^2}{r^6}\left[1 + \frac{3a(t-\tau)}{r^3}\right]^{-\frac{4}{3}},$$

.....

.....

etc.

Thus, from Eqs. (8.9.2)

$$[\mathbf{A}_1] = \begin{bmatrix} \frac{4a}{r^3} & 0 & 0 \\ 0 & \frac{-2a}{r^3} & 0 \\ 0) & 0 & \frac{-2a}{r^3} \end{bmatrix} \qquad \text{(vi)}$$

$$[\mathbf{A}_2] = \begin{bmatrix} \frac{28a^2}{r^6} & 0 & 0 \\ 0 & \frac{-2a^2}{r^6} & 0 \\ 0 & 0 & \frac{-2a^2}{r^6} \end{bmatrix} \qquad \text{(vii)} \qquad \text{(vii)}$$

$[\mathbf{A}_3]$, $[\mathbf{A}_4]$ ..... etc. can be obtained by computing the higher derivatives of the components of $\mathbf{C}_t$ and evaluating them at $\tau = t$.

8.10 Rivlin-Ericksen Tensor in Terms of Velocity Gradients - The Recursive Formulas

In this section, we show that

$$\mathbf{A}_1 = 2\mathbf{D} = \nabla\mathbf{v}+(\nabla\mathbf{v})^T \tag{8.10.1}$$

$$\mathbf{A}_2 = \frac{D\mathbf{A}_1}{Dt}+\mathbf{A}_1(\nabla\mathbf{v})+(\nabla\mathbf{v})^T\mathbf{A}_1 \tag{8.10.2}$$

and

$$\mathbf{A}_{N+1} = \frac{D\mathbf{A}_N}{Dt}+\mathbf{A}_N(\nabla\mathbf{v})+(\nabla\mathbf{v})^T\mathbf{A}_N, \qquad N=1,2,3.... \tag{8.10.3}$$

where $\nabla\mathbf{v}$ is the velocity gradient and $\mathbf{D}$ is the rate of deformation tensor.

We have, at any time τ

$$ds^{2\prime} = d\mathbf{x}'(\tau)\cdot d\mathbf{x}'(\tau) = d\mathbf{x}\cdot\mathbf{C}_t d\mathbf{x} \tag{i}$$

$$\frac{D}{Dt}(ds^2) \equiv \left[\frac{\partial}{\partial\tau}(ds^{2\prime})\right]_{x_i-\text{fixed},\,\tau=t} = d\mathbf{x}\cdot\left(\frac{\partial\mathbf{C}_t}{\partial\tau}\right)_{\tau=t} d\mathbf{x} \tag{ii}$$

That is, [see Eqs. (8.9.2)]

$$\frac{D}{Dt}(ds^2) = d\mathbf{x}\cdot\mathbf{A}_1\, d\mathbf{x} \tag{8.10.4}$$

clearly, we also have

$$\frac{D^2}{Dt^2}(ds^2) = d\mathbf{x}\cdot\mathbf{A}_2\, d\mathbf{x} \tag{8.10.5}$$

and

$$\frac{D^N}{Dt^N}(ds^2) = d\mathbf{x}\cdot\mathbf{A}_N\, d\mathbf{x} \tag{8.10.6}$$

We now recall from Section 3.13, Eq. (3.13.6a), that

$$\frac{D}{Dt}(ds^2) = 2d\mathbf{x}\cdot\mathbf{D}d\mathbf{x}$$

where $\mathbf{D} = \frac{1}{2}[\nabla\mathbf{v} + (\nabla\mathbf{v})^T]$ is the rate of deformation tensor. Thus,

$$\mathbf{A}_1 = 2\mathbf{D}$$

Next, from Eq. (8.10.4),

$$\frac{D^2}{Dt^2}(ds^2) = \left(\frac{D}{Dt}d\mathbf{x}\right)\cdot\mathbf{A}_1 d\mathbf{x} + d\mathbf{x}\cdot\frac{D}{Dt}\mathbf{A}_1\, d\mathbf{x} + d\mathbf{x}\cdot\mathbf{A}_1\frac{D}{Dt}d\mathbf{x} \tag{iii}$$

But

$$\frac{D}{Dt}d\mathbf{x} = (\nabla\mathbf{v})d\mathbf{x} \qquad \text{[see Eq. (3.12.4)]}$$

Therefore,

$$\frac{D^2}{Dt^2}(ds^2) = (\nabla\mathbf{v})d\mathbf{x}\cdot\mathbf{A}_1\,d\mathbf{x} + d\mathbf{x}\cdot\frac{D\mathbf{A}_1}{Dt}\,d\mathbf{x} + d\mathbf{x}\cdot\mathbf{A}_1\,(\nabla\mathbf{v})d\mathbf{x} \tag{iv}$$

From the definition of transpose

$$(\nabla\mathbf{v})d\mathbf{x}\cdot\mathbf{A}_1\,d\mathbf{x} = d\mathbf{x}\cdot\,(\nabla\mathbf{v})^T\,\mathbf{A}_1\,d\mathbf{x} \tag{v}$$

$$\frac{D^2}{Dt^2}(ds^2) = d\mathbf{x}\cdot\left[\frac{D\mathbf{A}_1}{Dt} + (\nabla\mathbf{v})^T\mathbf{A}_1 + \mathbf{A}_1(\nabla\mathbf{v})\right]d\mathbf{x} \tag{vi}$$

Thus, from Eq. (8.10.5),

$$\mathbf{A}_2 = \frac{D\mathbf{A}_1}{Dt} + \mathbf{A}_1(\nabla\mathbf{v}) + (\nabla\mathbf{v})^T\mathbf{A}_1$$

Equation (8.10.3) can be similarly proved.

8.11 Relation Between Velocity Gradient and Deformation Gradient

From

$$d\mathbf{x}'(\tau) = \mathbf{x}_t'(\mathbf{x}+d\mathbf{x},\tau) - \mathbf{x}_t'(\mathbf{x},\tau) \tag{8.11.1}$$

we have

$$d\mathbf{x}' = \mathbf{F}_t(\mathbf{x},\tau)\,d\mathbf{x} \qquad \text{and} \tag{8.11.2}$$

$$\frac{D}{D\tau}d\mathbf{x}'(\tau) = \mathbf{v}'(\mathbf{x}+d\mathbf{x},\tau) - \mathbf{v}'(\mathbf{x},\tau) = \nabla_{\mathbf{x}}\mathbf{v}'(\mathbf{x},\tau)d\mathbf{x} \tag{8.11.3}$$

From Eq. (8.11.2)

$$\frac{Dd\mathbf{x}'}{D\tau} = \left(\frac{D\mathbf{F}_t}{D\tau}\right)d\mathbf{x} \tag{8.11.4}$$

Comparing Eqs. (8.11.3) and (8.11.4), we have

$$\frac{D\mathbf{F}_t}{D\tau} = \nabla_{\mathbf{x}}\mathbf{v}'(\mathbf{x},\tau) \tag{8.11.5}$$

and from which

$$\left[\frac{D\mathbf{F}_t}{D\tau}\right]_{\tau=t} = \nabla_{\mathbf{x}}\mathbf{v}(\mathbf{x},t) \tag{8.11.6}$$

Using this relations, we can obtain the following relations between the rate of deformation tensor $\mathbf{D}$ and the relative stretch tensor $\mathbf{U}_t$ as well as the relation between the spin tensor $\mathbf{W}$ and the relative rotation tensor $\mathbf{R}_t$. In fact, from the polar decomposition theorem

$$\mathbf{F}_t(\tau) = \mathbf{R}_t(\tau)\mathbf{U}_t(\tau) \tag{8.11.7}$$

we have

$$\frac{D\mathbf{F}_t(\tau)}{D\tau} = \frac{D\mathbf{R}_t(\tau)}{D\tau}\mathbf{U}_t(\tau) + \mathbf{R}_t(\tau)\frac{D\mathbf{U}_t(\tau)}{D\tau} \tag{8.11.8}$$

Evaluating the above equation at $\tau = t$, using Eq. (8.11.6) and noting that $\mathbf{U}_t(t) = \mathbf{R}_t(t) = \mathbf{I}$, we obtain

$$\nabla_{\mathbf{x}}\mathbf{v} = \left[\frac{D\mathbf{R}_t(\tau)}{D\tau}\right]_{\tau=t} + \left[\frac{D\mathbf{U}_t(\tau)}{D\tau}\right]_{\tau=t} \tag{8.11.9}$$

Clearly $\left[\frac{D\mathbf{U}_t(\tau)}{D\tau}\right]_{\tau=t}$ is a symmetric tensor and it can be easily shown that $\left[\frac{D\mathbf{R}_t(\tau)}{D\tau}\right]_{\tau=t}$ is an antisymmetric tensor. Thus, in view of the fact that the decomposition of a tensor into a symmetric and an antisymmetric tensor is unique, therefore,

$$\left[\frac{D\mathbf{U}_t(\tau)}{D\tau}\right]_{\tau=t} = \mathbf{D}(t) \tag{8.11.10}$$

$$\left[\frac{D\mathbf{R}_t(\tau)}{D\tau}\right]_{\tau=t} = \mathbf{W}(t) \tag{8.11.11}$$

8.12 Transformation Laws for the Relative Deformation Tensors under a Change of Frame

The concept of objectivity was discussed in Chapter 5, Section 5.31. We recall that a change of frame, from $\mathbf{x}$ to $\mathbf{x}^*$, is defined by the transformation

$$x^*(t) = \mathbf{c}(t) + \mathbf{Q}(t)(\mathbf{x} - \mathbf{x}_o) \tag{8.12.1a}$$

$$t^* = t - a \tag{8.12.1b}$$

and if a tensor $\mathbf{A}$, in the un-starred frame, transforms to $\mathbf{A}^*$ in the starred frame in accordance with the relation

$$\mathbf{A}^* = \mathbf{Q}(t)\mathbf{A}\mathbf{Q}^T(t) \tag{8.12.2}$$

then, the tensor **A** is said to be objective, or frame indifferent (i.e., independent of observers). From Eq. (8.12.1), we have

$$d\mathbf{x}^*(t) = Q(t)d\mathbf{x}(t) \tag{i}$$

and

$$d\mathbf{x}'^*(\tau) = Q(\tau)d\mathbf{x}'(\tau) \tag{ii}$$

Since $d\mathbf{x}'(\tau) = \mathbf{F}_t(\tau)d\mathbf{x}(t)$ and $d\mathbf{x}'^*(\tau) = \mathbf{F}_t^*(\tau)d\mathbf{x}^*(t)$, therefore from Eq. (ii), we have

$$\mathbf{F}_t^*(\tau)d\mathbf{x}^*(t) = \mathbf{Q}(\tau)d\mathbf{x}'(\tau) = \mathbf{Q}(\tau)\mathbf{F}_t(\tau)d\mathbf{x}(t) \tag{iii}$$

Now, use the inverse of Eq. (i), we get

$$\mathbf{F}_t^*(\tau)d\mathbf{x}^*(t) = \mathbf{Q}(\tau)\mathbf{F}_t(\tau)\mathbf{Q}^T(t)d\mathbf{x}^*(t) \tag{iv}$$

Thus,

$$\mathbf{F}_t^*(\tau) = \mathbf{Q}(\tau)\mathbf{F}_t(\tau)\mathbf{Q}^T(t) \tag{8.12.3}$$

This is the transformation law for $\mathbf{F}_t(\tau)$ under a change of frame. We see that this tensor is not objective.

Since $\mathbf{F}_t^* = \mathbf{R}_t^*\mathbf{U}_t^*$ and $\mathbf{F}_t = \mathbf{R}_t\mathbf{U}_t$, therefore, from Eq. (8.12.3)

$$\mathbf{R}_t^*\,\mathbf{U}_t^* = \mathbf{Q}(\tau)\mathbf{R}_t\mathbf{U}_t\mathbf{Q}^T(t) \tag{v}$$

This equation can be written:

$$\mathbf{F}_t^* = \mathbf{R}_t^*\,\mathbf{U}_t^* = [\mathbf{Q}(\tau)\mathbf{R}_t\mathbf{Q}^T(t)][\mathbf{Q}(t)\mathbf{U}_t\mathbf{Q}^T(t)] \tag{vi}$$

In the above equation, the tensor inside the first bracket is an orthogonal tensor and the tensor inside the second bracket is a symmetric tensor. Since the polar decomposition for $\mathbf{F}_t^*$ is unique, therefore, we have

$$\mathbf{R}_t^* = \mathbf{Q}(\tau)\mathbf{R}_t\mathbf{Q}^T(t) \tag{8.12.4}$$

$$\mathbf{U}_t^* = \mathbf{Q}(t)\mathbf{U}_t\,\mathbf{Q}^T(t) \tag{8.12.5}$$

It is a simple matter to show the following transformation laws under a change of frames:

$$\mathbf{C}_t^* = \mathbf{Q}(t)\mathbf{C}_t\mathbf{Q}^T(t) \tag{8.12.6}$$

$$\mathbf{C}_t^{*^{-1}} = \mathbf{Q}(t)\mathbf{C}_t^{-1}\mathbf{Q}^T(t) \tag{8.12.7}$$

$$\mathbf{V}_t^* = \mathbf{Q}(\tau)\mathbf{V}_t\mathbf{Q}^T(\tau) \tag{8.12.8}$$

$$\mathbf{B}_t^* = \mathbf{Q}(\tau)\mathbf{B}_t\mathbf{Q}^T(\tau) \tag{8.12.9}$$

$$\mathbf{B}_t^{*^{-1}} = \mathbf{Q}(\tau)\mathbf{B}_t^{-1}\mathbf{Q}^T(\tau) \tag{8.12.10}$$

Equations (8.12.5),(8.12.6) and (8.12.7) show that the relative right stretch tensor, the relative right Cauchy-Green deformation tensor $\mathbf{C}_t$ and its inverse $\mathbf{C}_t^{-1}$ (the relative Finger tensor) are objective. On the other hand, $\mathbf{V}_t$, $\mathbf{B}_t$ and $\mathbf{B}_t^{-1}$ are nonobjective. We note, this situation is different from that of the deformation tensor using a fixed reference configuration [See Section 5.31].

From Eq. (8.12.4) and (8.11.11), one can also show that in a change of frame

$$\mathbf{W}^* = \mathbf{Q}(t)\mathbf{W}(t)\mathbf{Q}^T(t) + \frac{d\mathbf{Q}(t)}{dt}\mathbf{Q}^T(t) \tag{8.12.11}$$

which shows, as expected that the spin tensor is non-objective.

Using Eq. (8.12.11), one can derive for any objective tensor $\mathbf{T}$ (i.e., $\mathbf{T}^* = \mathbf{Q}(t)\mathbf{T}\mathbf{Q}^T(t)$) that

$$\frac{D\mathbf{T}}{Dt} + \mathbf{TW} - \mathbf{WT} \tag{8.12.12}$$

is objective, that is, [see Prob. 8.22]

$$\frac{D\mathbf{T}^*}{Dt} + \mathbf{T}^*\mathbf{W}^* - \mathbf{W}^*\mathbf{T}^* = \mathbf{Q}(t)\left[\frac{D\mathbf{T}}{Dt} + \mathbf{TW} - \mathbf{WT}\right]\mathbf{Q}^T(t) \tag{8.12.13}$$

The expression given in (8.12.12) is known as the **Jaumann derivative** of $\mathbf{T}$ which will be discussed further in a later section.

8.13 Transformation law for the Rivlin-Ericksen Tensors under a Change of Frame

From Eq. (8.12.6)

$$\mathbf{C}_t^*(\tau) = \mathbf{Q}(t)\mathbf{C}_t(\tau)\mathbf{Q}^T(t) \tag{8.13.1}$$

we obtain

$$\frac{D}{D\tau}\mathbf{C}_t^*(\tau) = \mathbf{Q}(t)\frac{D}{D\tau}\mathbf{C}_t(\tau)\mathbf{Q}^T(t) \tag{8.13.2}$$

and in fact, for all N,

$$\frac{D^N}{D\tau^N}\mathbf{C}_t^*(\tau) = \mathbf{Q}(t)\frac{D^N}{D\tau^N}\mathbf{C}_t(\tau)\mathbf{Q}^T(t) \tag{8.13.3}$$

Thus, from Eqs. (8.9.2), we have, for all N

$$\mathbf{A}_N^*(t) = \mathbf{Q}(t)\mathbf{A}_N(\tau)\mathbf{Q}^T(t) \tag{8.13.4}$$

We see therefore that all $\mathbf{A}_N$ are objective. This is quite to be expected because these tensors characterize the rate and the higher rates of changes of length of material elements at time t which are independent of the observers.

8.14 Incompressible Simple Fluid

An incompressible simple fluid is an isotropic ideal material having the following constitutive equation

$$\mathbf{T} = -p\mathbf{I}+\boldsymbol{\tau} \tag{8.14.1}$$

where $\boldsymbol{\tau}$ depends on the past histories up to the current time t of the relative deformation tensor $\mathbf{C}_t$. In other words, a simple fluid is defined by

$$\mathbf{T} = -p\mathbf{I}+\mathbf{H}\,[\mathbf{C}_t(\mathbf{x},\tau);-\infty<\tau\leq t] \tag{8.14.2}$$

where the index $\tau=-\infty$ to t indicates that the values of the functional $\mathbf{H}$ depends on all $\mathbf{C}_t$ from $\mathbf{C}_t(\mathbf{x},-\infty)$ to $\mathbf{C}_t(\mathbf{x},t)$. We note that such a fluid is called "simple" because it depends only on the history of the relative deformation gradient $\mathbf{F}_t(\tau) = \nabla\mathbf{x}'$ tensor (from which $\mathbf{C}_t(\tau)$ is obtained), and not on the history of the higher gradient of the relative deformation tensor (e.g., $\nabla\nabla\mathbf{x}'$).

Obviously, the functional $\mathbf{H}$ in Eq. (8.14.2) is to be the same for all observers (i.e., $\mathbf{H}^*=\mathbf{H}$). However, it can not be arbitrary, because it must satisfy the frame indifference requirement. That is, in a change of frame,

$$\mathbf{H}[\mathbf{C}_t^*(\tau)] = \mathbf{Q}(t)\mathbf{H}[\mathbf{C}_t(\tau)]\mathbf{Q}^T(t) \tag{8.14.3}$$

Since $\mathbf{C}_t(\tau)$ transforms in a change of frame as [see Eq. (8.12.6)]

$$\mathbf{C}_t^* = \mathbf{Q}(t)\mathbf{C}_t\mathbf{Q}^T(t) \tag{8.14.4}$$

therefore, the functional $\mathbf{H}\,[\mathbf{C}_t(\mathbf{x},\tau);-\infty<\tau\leq t]$ must satisfy the condition

$$\mathbf{H}[\mathbf{Q}(t)\mathbf{C}_t\mathbf{Q}^T(t)] = \mathbf{Q}(t)\mathbf{H}[\mathbf{C}_t]\mathbf{Q}^T(t) \tag{8.14.5}$$

We note that Eq. (8.14.5) also states that the fluid defined by Eq. (8.14.2) is an isotropic fluid.

Any function or functional which obeys the condition given by Eq. (8.14.5) is known as an **isotropic function** or **isotropic functional**.

The relationship between stress and deformation given by Eq. (8.14.2) for a simple fluid is completely general. In fact, it includes Newtonian incompressible fluid and Maxwell fluids as

special cases. In this most general form, only very special flow problems can be solved. A class of such flows, called the viscometric flow, will be considered in Part C, using this general form of constitutive equation. However, in the following few sections, we shall first discuss some special constitutive equations. Some of these constitutive equations have been shown to be approximations to the general constitutive equation given in Eq. (8.14.2) under certain conditions (slow flow and/or fading memory). They can also be considered simply as special fluids. For example, a Newtonian incompressible fluid can be considered either as a special fluid by itself or as an approximation to the general simple fluid when it has no memory of its past history of deformation and is under slow flow condition relative to its relaxation time (which is zero).

8.15 Special Single Integral Type Nonlinear Constitutive Equations

In Section 8.4, we see that the constitutive equation for the linear Maxwell fluids is defined by

$$\boldsymbol{\tau} = 2\int_{o}^{\infty} f(s)\,\mathbf{E}(t-s)ds \tag{8.15.1}$$

where $\mathbf{E}$ is the infinitesimal strain tensor measured with respect to the configuration at time t. It can be shown that for small deformations, (see Example 8.15.2 below)

$$\mathbf{C}_t - \mathbf{I} = \mathbf{I} - \mathbf{C}_t^{-1} = 2\,\mathbf{E} \tag{i}$$

Thus, the following two nonlinear viscoelastic fluids represent natural generalizations of the linear Maxwell fluid in that they reduce to Eq. (8.15.1) under small deformation conditions.

$$\boldsymbol{\tau} = \int_{o}^{\infty} f_1(s)[\mathbf{C}_t\,(t-s) - \mathbf{I}]ds \tag{8.15.2}$$

and

$$\boldsymbol{\tau} = \int_{o}^{\infty} f_2(s)[\mathbf{I} - \mathbf{C}_t^{-1}\,(t-s)]ds \tag{8.15.3}$$

where

$$f_1(s) = f_2(s) = f(s) \tag{8.15.4}$$

and $f(s)$ may be given by any one of Eqs. (8.4.9), (8.4.10) and (8.4.11).

We note that since $\mathbf{C}_t(\tau)$ is an objective tensor, therefore the constitutive equations defined by Eqs. (8.15.2) and (8.15.3) are frame indifferent. We note also that even though the fluids defined by Eqs. (8.15.2) and (8.15.3), with $f_1 = f_2$ have the same behaviors at small deformation, they are two different nonlinear viscoelastic fluids, behaving differently at large deformation. Furthermore, if we treat $f_1(s)$ and $f_2(s)$ as two different memory functions, then

Eqs. (8.15.2) and (8.15.3) define two nonlinear viscoelastic fluids whose behavior at small deformations are also different.

Example 8.15.1

For the nonlinear viscoelastic fluid defined by Eq. (8.15.2), find the stress components when the fluid is under steady shearing flow defined by the velocity field:

$$v_1 = kx_2, \qquad v_2 = v_3 = 0 \tag{i}$$

Solution. The relative Cauchy-Green deformation tensor corresponding to this flow was obtained in Example 8.8.1 as:

$$[\mathbf{C}_t(\tau)] = \begin{bmatrix} 1 & k(\tau-t) & 0 \\ k(\tau-t) & k^2(\tau-t)^2+1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{ii}$$

Thus,

$$[\mathbf{C}_t(t-s)-\mathbf{I}] = \begin{bmatrix} 0 & -ks & 0 \\ -ks & k^2s^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{iii}$$

Thus, from Eq. (8.15.2)

$$\tau_{11} = \tau_{13} = \tau_{23} = \tau_{33} = 0 \tag{iv}$$

$$\tau_{12} = -k \int_o^\infty s f_1(s) ds \tag{8.15.5}$$

$$\tau_{22} = k^2 \int_o^\infty s^2 f_1(s) ds \tag{8.15.6}$$

We see that for this fluid, the viscosity is given by

$$\mu \equiv \tau_{12}/k = -\int_o^\infty s f_1(s)\, ds \tag{8.15.7}$$

We also note that the normal stresses are not equal in the simple shearing flow. In fact,

$$T_{11} = -p + \tau_{11} = -p \tag{v}$$

$$T_{22} = -p + \tau_{22} = -p + k^2 \int_o^\infty s^2 f_1(s) ds \tag{vi}$$

$$T_{33} = -p + \tau_{33} = -p \tag{vii}$$

We see from this example that for the nonlinear viscoelastic fluid defined by Eq. (15.2), i.e.,

$$\boldsymbol{\tau} = 2\int_o^\infty f_1(s)\,[\mathbf{C}_t(t-s)-\mathbf{I}]ds \tag{8.15.2}$$

the **viscosity function** $\mu(k)$ is given by

$$\mu(k) \equiv \tau_{12} = -k\int_o^\infty sf_1(s)ds \tag{8.15.8}$$

and the two **normal stress functions** are given either by

$$\sigma_1(k) \equiv T_{11}-T_{22} = -k^2\int_o^\infty s^2 f_1(s)ds \tag{8.15.9a}$$

$$\sigma_2(k) \equiv T_{22}-T_{33} = k^2\int_o^\infty s^2 f_1(s)ds \tag{8.15.10a}$$

or

$$\bar{\sigma}_1(k) \equiv T_{22}-T_{33} = k^2\int_o^\infty s^2 f_1(s)ds \tag{8.15.9b}$$

$$\bar{\sigma}_2(k) \equiv T_{11}-T_{33} = 0 \tag{8.15.10b}$$

The shear stress function, and the two normal stress functions (either σ_1 and σ_2, or $\bar{\sigma}_1$ and $\bar{\sigma}_2$) completely describe the material properties of this nonlinear viscoelastic fluid in the simple shearing flow. In part C we will show that these three material functions completely describe the material properties of every simple fluid, of which the present nonlinear fluid is a special case, in viscometric flows, of which the simple shearing flow is a special case.

Similarly, for the nonlinear viscoelastic fluid defined by Eq. (8.15.3),

$$\boldsymbol{\tau} = \int_o^\infty f_2(s)[\mathbf{I}-\mathbf{C}_t^{-1}(t-s)]ds \tag{8.15.3}$$

the viscosity function and the two normal stress functions can be obtained to be

$$\mu(k) = -k\int_o^\infty sf_2(s)ds \tag{8.15.11}$$

$$\sigma_1(k) = -k^2\int_o^\infty s^2 f_2(s)ds \tag{8.15.12}$$

$$\sigma_2(k) = 0 \tag{8.15.13}$$

A special nonlinear viscoelastic fluid defined by Eq. (8.15.3) with a memory function dependent on the second invariant I_2 of the tensor $\mathbf{C}_t$ in the following way

$$f_2(s) = f(s) \qquad \text{when } I_2 \geq B^2+3 \tag{8.15.14}$$

$$f_2(s) = 0 \qquad \text{when } I_2 < B^2+3$$

is known as Tanner and Simmons network model fluid. The function $f(s)$ is given by Eq. (8.4.9). For this fluid, the network "breaks" when a scalar measure of the deformation I_2 reaches a limiting value $B^2 + 3$, where B is called the "strength" of the network.

Example 8.15.2

Show that for small deformations relative to the configuration at time t

$$\mathbf{C}_t - \mathbf{I} \approx \mathbf{I} - \mathbf{C}_t^{-1} \approx 2\,\mathbf{E} \tag{8.15.15}$$

where $\mathbf{E}$ is the infinitesimal strain tensor.

Solution. Let $\mathbf{u}$ denote the displacement vector measured from the configuration at time t. Then

$$\mathbf{x}'(\tau) = \mathbf{x} + \mathbf{u}(\mathbf{x},\tau) \tag{i}$$

Thus,

$$\mathbf{F}_t = \nabla\mathbf{x}' = \mathbf{I} + \nabla\mathbf{u} \tag{ii}$$

Now, if $\mathbf{u}$ is infinitesimal, then

$$\mathbf{C}_t = \mathbf{F}_t^T\,\mathbf{F}_t = [\mathbf{I} + (\nabla\mathbf{u})^T][\mathbf{I} + \nabla\mathbf{u}] \approx \mathbf{I} + 2\frac{(\nabla\mathbf{u})^T + (\nabla\mathbf{u})}{2} = \mathbf{I} + 2\mathbf{E} \tag{iii}$$

Also

$$\mathbf{C}_t^{-1} = \mathbf{F}_t^{-1}\,\mathbf{F}_t^{-1T} \approx [\mathbf{I} - (\nabla\mathbf{u})][\mathbf{I} - (\nabla\mathbf{u})^T] = \mathbf{I} - 2\frac{(\nabla\mathbf{u})^T + (\nabla\mathbf{u})}{2} = \mathbf{I} - 2\mathbf{E} \tag{iv}$$

Thus

$$\mathbf{C}_t - \mathbf{I} \approx 2\mathbf{E} \qquad \text{and } \mathbf{I} - \mathbf{C}_t^{-1} \approx 2\mathbf{E}$$

Example 15.3

Show that any polynomial function of a real symmetric tensor **A** can be represented by

$$\mathbf{F}(\mathbf{A}) = f_o\mathbf{I} + f_1\mathbf{A} + f_2\mathbf{A}^{-1} \tag{8.15.16}$$

where f_i are real valued functions of the scalar invariants of the symmetric tensor **A**.

Solution. Let

$$\mathbf{F}(\mathbf{A}) = a_o\mathbf{I} + a_1\mathbf{A} + a_2\mathbf{A}^2 + \ldots a_N\mathbf{A}^N \tag{i}$$

Since **A** satisfies its own characteristic equation:

$$\mathbf{A}^3 - I_1\mathbf{A}^2 + I_2\mathbf{A} - I_3\mathbf{I} = 0 \tag{ii}$$

therefore,

$$\mathbf{A}^3 = I_1\mathbf{A}^2 - I_2\mathbf{A} + I_3\mathbf{I} \tag{iii}$$

$$\mathbf{A}^4 = I_1\mathbf{A}^3 - I_2\mathbf{A}^2 + I_3\mathbf{A} = I_1(I_1\mathbf{A}^2 - I_2\mathbf{A} + I_3\mathbf{I}) - I_2\mathbf{A}^2 + I_3\mathbf{A} \tag{iv}$$

etc., Thus, every $\mathbf{A}^N$ for $N \geq 3$ can be expressed as a sum of **A**, $\mathbf{A}^2$ and **I** with coefficients being functions of the scalar invariants of **A**. Substituting these expressions in Eq. (i), one obtains

$$\mathbf{F}(\mathbf{A}) = b_o(I_i)\mathbf{I} + b_1(I_i)\mathbf{A} + b_2(I_i)\mathbf{A}^2 \tag{v}$$

Now, from Eq. (iii), we can obtain

$$\mathbf{A}^2 = I_1\mathbf{A} - I_2\mathbf{I} + I_3\mathbf{A}^{-1} \tag{vi}$$

therefore, Eq. (v) can also be written as

$$\mathbf{F}(\mathbf{A}) = f_o(I_i)\mathbf{I} + f_1(I_i)\mathbf{A} + f_2(I_i)\mathbf{A}^{-1} \tag{vii}$$

which is Eq. (8.15.16). Actually the representation of **F**(**A**) given in this example can be shown to be true under the more general condition that the symmetric function **F** of the symmetric tensor **A** is an isotropic function (of which the polynomial function of **A** is a special case). An isotropic function **F** is a function which satisfies the condition

$$\mathbf{F}(\mathbf{Q}\mathbf{A}\mathbf{Q}^{\mathbf{T}}) = \mathbf{Q}\mathbf{F}(\mathbf{A})\mathbf{Q}^T \tag{viii}$$

for any orthogonal tensor **Q**. Now, let us identify **A** with $\mathbf{C}_t$ and I_i with the scalar invariants of $\mathbf{C}_t$ (note however that $I_3 = 1$ for incompressible fluid), then the most general representation of $\mathbf{F}(\mathbf{C}_t)$ is [we recall that $\mathbf{F}(\mathbf{C}_t)$ is required to satisfy Eq. (viii) for frame indifference, see Eq. (8.14.5) also],

$$\mathbf{F}(\mathbf{C}_t) = f_1(I_1,I_2)\mathbf{C}_t + f_2(I_1,I_2)\mathbf{C}_t^{-1} \tag{ix}$$

8.16 General Single Integral Type Nonlinear Constitutive Equations

From the discussions given in the previous example, we see that the most general single integral type nonlinear constitutive equation for an compressible fluid is defined by

$$\boldsymbol{\tau} = \int_{o}^{\infty} [f_1(s,I_1,I_2)\mathbf{C}_t + f_2(s,I_1,I_2)\mathbf{C}_t^{-1}]ds \tag{8.16.1}$$

A special nonlinear viscoelastic fluid, known as the BKZ fluid, is defined by Eq. (8.16.1) with the functions $f_1(s)$ and $f_2(s)$ given by

$$f_1(s)=2\frac{\partial U}{\partial I_1} \tag{8.16.2a}$$

$$f_2(s) = -2\frac{\partial U}{\partial I_2} \tag{8.16.2b}$$

where

$$-U = \frac{\alpha}{2}(I_1 - 3)^2 + \frac{9}{2}\beta \ln\frac{I_1+I_2+3}{9} + 24(\beta-c)\ln\frac{I_1 + 15}{I_2 + 15} + c(I_1-3) \tag{8.16.2c}$$

where α, β and c are constants.

8.17 Differential Type Constitutive Equations for Incompressible Fluids

We see in Section 8.9 that under the assumption that the Taylor series expansion of the history of the deformation tensor $\mathbf{C}_t(\mathbf{x},\tau)$ is justified, the Rivlin-Ericksen tensor $\mathbf{A}_n(n = 1,2,...\infty)$ determines the history of $\mathbf{C}_t(\mathbf{x},\tau)$. Thus we may write Eq. (8.14.2) as

$$\mathbf{T} = -p\mathbf{I} + \mathbf{f}(\mathbf{A}_1,\mathbf{A}_2,...\mathbf{A}_n,...) \tag{8.17.1}$$

where $\mathbf{f}(\mathbf{A}_1,\mathbf{A}_2,...\mathbf{A}_n,...)$ is a function of the Rivlin-Ericksen tensor and $\mathrm{tr}\mathbf{A}_1 = 0$ which follows from the equation of conservation of mass for an incompressible fluid.

In order to satisfy the frame-indifference condition, the function **f** cannot be arbitrary but must satisfy the relation

$$\mathbf{Q}\mathbf{f}(\mathbf{A}_1,\mathbf{A}_2,...\mathbf{A}_n)\mathbf{Q}^T = \mathbf{f}(\mathbf{Q}\mathbf{A}_1\mathbf{Q}^T,\mathbf{Q}\mathbf{A}_2\mathbf{Q}^T,...\mathbf{Q}\mathbf{A}_n\mathbf{Q}^T) \tag{8.17.2}$$

for any orthogaonal tensor **Q**. We note again, Equation (8.17.2) makes "isotropy of material property" as part of the definition of a simple fluid. Equation.(8.17.2) is obtained in the same way as Eq. (8.14.5) is.

The following are special constitutive equations of this type

(A) *Rivlin-Ericksen incompressible fluid of complexity n*

$$\mathbf{T} = -p\,\mathbf{I} + \mathbf{f}(\mathbf{A}_1,\mathbf{A}_2,...\mathbf{A}_n) \tag{8.17.3}$$

In particular, a Rivlin-Ericksen liquid of complexity 2 is given by:

$$\begin{aligned}\mathbf{T} = -p\,\mathbf{I} + \mu_1\mathbf{A}_1 + \mu_2\mathbf{A}_1^2 + \mu_3\mathbf{A}_2 + \mu_4\mathbf{A}_2{}^2 \\ +\mu_5(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \mu_6(\mathbf{A}_1\mathbf{A}_2^2 + \mathbf{A}_2^2\mathbf{A}_1) \\ +\mu_7(\mathbf{A}_1^2\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1^2) + \mu_8(\mathbf{A}_1^2\mathbf{A}_2^2 + \mathbf{A}_2^2\mathbf{A}_1^2)\end{aligned} \tag{8.17.4}$$

where $\mu_1, \mu_2, \ldots \mu_8$ are scalar material functions of the following scalar invariants:

$$\begin{aligned}tr\mathbf{A}_1^2,\ tr\mathbf{A}_1^3,\ tr\mathbf{A}_2,\ tr\mathbf{A}_2^2,\ tr\mathbf{A}_2^3 \\ tr\mathbf{A}_1\mathbf{A}_2,\ tr\mathbf{A}_1^2\mathbf{A}_2,\ tr\mathbf{A}_1\mathbf{A}_2^2,\ tr\mathbf{A}_1^2\mathbf{A}_2^2\end{aligned} \tag{8.17.5}$$

We note that if $\mu_2 = \mu_3 \ldots = \mu_n = 0$ and μ_1 = a constant, Eq. (8.17.4) reduces to the constitutive equation for a Newtonian liquid with viscosity μ_1.

(B) *Second Order Fluid*

$$\mathbf{T} = -p\,\mathbf{I} + \mu_1\mathbf{A}_1 + \mu_2\mathbf{A}_1^2 + \mu_3\mathbf{A}_2 \tag{8.17.6}$$

where μ_1, μ_2, and μ_3 are material constants. The second order fluid may be regarded as a special case of the Rivlin-Ericksen fluid. However, it has also been shown that under the assumption of fading memory, small deformation and slow flow, Eq. (8.17.6) provides the second-order approximation whereas the Newtonian fluid provides the first order approximation and the inviscid fluid, the zeroth order approximation.

Example 8.17.1

For a second order fluid, compute the stress components in a simple shearing flow given by the velocity field

$$v_1 = kx_2,\ v_2 = v_3 = 0 \tag{i}$$

Solution. From Example 8.9.1, we have for the simple shearing flow,

$$[\mathbf{A}_1] = \begin{bmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{ii}$$

$$[\mathbf{A}_2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{iii}$$

and

$$\mathbf{A}_3 = \mathbf{A}_4 = \ldots = 0 \tag{iv}$$

Now,

$$[\mathbf{A}_1^2] = [\mathbf{A}_1][\mathbf{A}_1] = \begin{bmatrix} k^2 & 0 & 0 \\ 0 & k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{v}$$

therefore, Eq. (8.17.6) gives

$$T_{11} = -p + \mu_2 k^2, \qquad T_{22} = -p + \mu_2 k^2 + 2\mu_3 k^2, \quad T_{33} = -p \tag{vi}$$

$$T_{12} = \mu_1 k, \qquad T_{13} = T_{23} = 0 \tag{vii}$$

We see that because of the presence of μ_2 and μ_3, normal stresses, in excess of p on the planes x_1 =constant and x_2 = constant are necessary to maintain the shearing flow. Furthermore, these normal stress components are not equal. The normal stress difference

$$\sigma_1(k) \equiv T_{11} - T_{22} \tag{viii}$$

and

$$\sigma_2(k) \equiv T_{22} - T_{33} \tag{ix}$$

are given by

$$\sigma_1 = -2\mu_3 k^2, \quad \sigma_2 = \mu_2 k^2 + 2\mu_3 k^2 \tag{x}$$

By measuring the normal stress differences and the shearing stress components T_{12}, the three material constants can be determined.

Example 8.17.2

For the simple shearing flow, compute the scalar invariants of Eq. (8.17.5).

Solution. Since

$$[\mathbf{A}_1] = \begin{bmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad [\mathbf{A}_2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad [\mathbf{A}_1^2] = \begin{bmatrix} k^2 & 0 & 0 \\ 0 & k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad [\mathbf{A}_1^3] = \begin{bmatrix} 0 & k^3 & 0 \\ k^3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{i}$$

$$[\mathbf{A}_2^2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4k^4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad [\mathbf{A}_2^3] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 8k^6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{ii}$$

$$[\mathbf{A}_1][\mathbf{A}_2] = \begin{bmatrix} 0 & 2k^3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad [\mathbf{A}_2][\mathbf{A}_1] = \begin{bmatrix} 0 & 0 & 0 \\ 2k^3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad [\mathbf{A}_1^2][\mathbf{A}_2] = [\mathbf{A}_2][\mathbf{A}_1^2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2k^4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{iii}$$

$$[\mathbf{A}_1][\mathbf{A}_2^2] = \begin{bmatrix} 0 & 4k^5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad [\mathbf{A}_2^2][\mathbf{A}_1] = \begin{bmatrix} 0 & 0 & 0 \\ 4k^5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad [\mathbf{A}_1^2][\mathbf{A}_2^2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4k^6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{iv}$$

Therefore,

$$\mathrm{tr}\mathbf{A}_1^2 = 2k^2, \quad \mathrm{tr}\mathbf{A}_1^3 = 0, \quad \mathrm{tr}\mathbf{A}_2 = 2k^2,$$
$$\mathrm{tr}\mathbf{A}_2^2 = 4k^4, \quad \mathrm{tr}\mathbf{A}_2^3 = 8k^6, \quad \mathrm{tr}\mathbf{A}_1\mathbf{A}_2 = 0,$$
$$\mathrm{tr}\mathbf{A}_1^2\mathbf{A}_2 = 2k^4, \quad \mathrm{tr}\mathbf{A}_1\mathbf{A}_2^2 = 0, \quad \mathrm{tr}\mathbf{A}_1^2\mathbf{A}_2^2 = 4k^6$$

Example 8.17.3

In a simple shearing flow, compute the stress components for the Rivlin -Ericksen liquid.

Solution. From Eqs. (8.17.4) and the results of the previous example, we have (note $\mathbf{A}_3 = \mathbf{A}_4 = \ldots = 0$)

$$T_{11} = -p + k^2[\mu_2(k^2)]$$

$$T_{22} = -p + 2k^2[\mu_3(k^2) + \frac{1}{2}\mu_2(k^2) + 2(\mu_4(k^2) + \mu_7(k^2))k^2 + 4k^4\mu_8(k^2)]$$

$$T_{33} = -p$$

$$T_{12} = k[\mu_1(k^2) + 2k^2\mu_5(k^2) + 4k^4\mu_6(k^2)] = k\eta(k^2) \equiv s(k)$$

$$T_{23} = T_{13} = 0$$

where $\mu_1(k^2)$ indicates that μ_1 is a function of k^2, etc. The normal stress differences $T_{11} - T_{22}$ and $T_{22} - T_{33}$ are even functions of k (= rate of shear), whereas the shear stress function $s(k)$ is an odd function of k.

8.18 Objective Rate of Stress

The stress tensor $\mathbf{T}$ is objective, therefore in a change of frame

$$\mathbf{T}^* = \mathbf{Q}(t)\mathbf{T}\mathbf{Q}^T(t) \tag{8.18.1}$$

Taking material derivative of the above equation, we obtain [note $D/Dt^* = D/Dt$]

$$\frac{D\mathbf{T}^*}{Dt} = \frac{d\mathbf{Q}}{dt}\mathbf{T}\mathbf{Q}^T + \mathbf{Q}\frac{D\mathbf{T}}{Dt}\mathbf{Q}^T + \mathbf{Q}\mathbf{T}\left(\frac{d\mathbf{Q}}{dt}\right)^T \tag{8.18.2}$$

The above equation shows that the material derivative of stress tensor $\mathbf{T}$ is not objective.

That the stress rate $D\,\mathbf{T}/Dt$ is not objective is physically quite clear. Consider the case of a time-independent uni-axial state of stress with respect to the first observer. With respect to this observer, the stress rate $D\mathbf{T}/Dt$ is identically zero. Consider a second observer who rotates with respect to the first observer. To the second observer, the given stress state is rotating respect to him and therefore, to him, the stress rate $D\mathbf{T}^*/Dt$ is not zero.

In the following we shall present several stress rates at time t which are objective

(A) *Jaumann derivative of stress*

Let us consider the tensor

$$\mathbf{J}(\tau) = \mathbf{R}_t^T(\tau)\mathbf{T}(\tau)\mathbf{R}_t(\tau) \tag{8.18.3}$$

We note that since $\mathbf{R}_t(t) = \mathbf{R}_t^T(t) = \mathbf{I}$, therefore, the tensor $\mathbf{J}$ and the tensor $\mathbf{T}$ are the same at time t. That is

$$\mathbf{J}(t) = \mathbf{T}(t) \tag{i}$$

However, while $D\mathbf{T}/Dt$ is not an objective stress rate , we will show that

$$\left[\frac{D\mathbf{J}(\tau)}{D\tau}\right]_{\tau=t} \tag{ii}$$

is an objective stress rate. To show this, we note that in Sect.8.12 , we obtained , in a change of frame

$$\mathbf{R}_t^*(\tau) = \mathbf{Q}(\tau)\mathbf{R}_t(\tau)\mathbf{Q}^T(t) \tag{iii}$$

Thus,

$$\mathbf{J}^*(\tau) = \mathbf{R}_t^{*T}(\tau)\mathbf{T}^*(\tau)\mathbf{R}_t^*(\tau) = \mathbf{Q}(t)\mathbf{R}_t^T(\tau)\mathbf{Q}^T(\tau)\mathbf{Q}(\tau)\mathbf{T}(\tau)\mathbf{Q}^T(\tau)\mathbf{Q}(\tau)\mathbf{R}_t(\tau)\mathbf{Q}^T(t) \tag{iv}$$

$$= \mathbf{Q}(t)[\mathbf{R}_t^T(\tau)\mathbf{T}(\tau)\mathbf{R}_t(\tau)]\mathbf{Q}^T(t) \tag{v}$$

Thus,

$$\mathbf{J}^*(\tau) = \mathbf{Q}(t)\mathbf{J}(\tau)\mathbf{Q}^T(t) \tag{8.18.4}$$

and

$$\left[\frac{D^N \mathbf{J}^*(\tau)}{D\tau^N}\right]_{\tau=t} = \mathbf{Q}(t)\left[\frac{D^N \mathbf{J}(\tau)}{D^N\tau}\right]_{\tau=t}\mathbf{Q}^T(t), \qquad N=1,2,3... \tag{8.18.5}$$

That is, the tensor $\mathbf{J}(\tau)$ as well as its material derivatives evaluated at time t, is objective. The derivative

$$\frac{D_{cr}\mathbf{T}}{Dt} \equiv \left[\frac{D\mathbf{J}(\tau)}{D\tau}\right]_{\tau=t} \equiv \left[\frac{D[\mathbf{R}_t(\tau)\mathbf{T}(\tau)\mathbf{R}_t^T(\tau)]}{D\tau}\right]_{\tau=t} \tag{8.18.6}$$

is called the **first Jaumann derivative** of $\mathbf{T}$ and the corresponding Nth derivatives are called the **Nth Jaumann derivatives**. They are also called the **co-rotational derivatives**, because they are the derivatives of $\mathbf{T}$ at time t as seen by an observer who rotates with the material element (whose rotation tensor is $\mathbf{R}$).

We shall now show that

$$\frac{D_{cr}\mathbf{T}}{Dt} = \frac{D\mathbf{T}}{Dt} + \mathbf{T}(t)\mathbf{W}(t) - \mathbf{W}(t)\mathbf{T}(t) \tag{8.18.7}$$

where $\mathbf{W}(t)$ is the spin tensor of the element. The right side of Eq. (8.18.6) is

$$\frac{D[(\mathbf{R}_t(\tau)\mathbf{T}(\tau)\mathbf{R}_t^T(\tau)]}{D\tau} = \frac{D\mathbf{R}_t(\tau)}{D\tau}\mathbf{T}(\tau)\mathbf{R}_t^T(\tau) + \mathbf{R}_t(\tau)\frac{D\mathbf{T}(\tau)}{D\tau}\mathbf{R}_t^T(\tau) + \mathbf{R}_t(\tau)\mathbf{T}(\tau)\frac{D\mathbf{R}_t^T(\tau)}{D\tau} \tag{vi}$$

Evaluating the above equation at $\tau=t$ and noting that

$$\mathbf{R}_t(t) = \mathbf{R}_t^T(t) = I$$

and

$$\left[\frac{D\mathbf{R}_t(\tau)}{D\tau}\right]_{\tau=t} = \mathbf{W}(t) \tag{vii}$$

$$\left[\frac{D\mathbf{R}_t^T(\tau)}{D\tau}\right]_{\tau=t} = \mathbf{W}^T(t) = -\mathbf{W}(t) \tag{viii}$$

we obtain immediately

$$\frac{D_{cr}\mathbf{T}}{Dt} = \frac{D\mathbf{T}}{Dt} + \mathbf{T}(t)\mathbf{W}(t) - \mathbf{W}(t)\mathbf{T}(t) \tag{ix}$$

(B) *Oldroyd lower convected derivative*

Let us consider the tensor

$$\mathbf{J}_L(\tau) = \mathbf{F}_t^T(\tau)\mathbf{T}(\tau)\mathbf{F}_t(\tau) \tag{8.18.8}$$

Again, as in (A),

$$\mathbf{J}_L(t) = \mathbf{T}(t) \tag{x}$$

and

$$\left[\frac{D\mathbf{J}_L(\tau)}{D\tau}\right]_{\tau=t} \tag{xi}$$

is an objective stress rate. To show this, we note that in Sect. 8.12 , we obtained , in a change of frame

$$\mathbf{F}_t^*(\tau) = \mathbf{Q}(\tau)\mathbf{F}_t(\tau)\mathbf{Q}^T(t) \tag{xii}$$

Thus,

$$\begin{aligned}\mathbf{J}_L^*(\tau) = \mathbf{F}_t^{*T}(\tau)\mathbf{T}^*(\tau)\mathbf{F}_t^*(\tau) &= \mathbf{Q}(t)\mathbf{F}_t^T(\tau)\mathbf{Q}^T(\tau)\mathbf{Q}(\tau)\mathbf{T}(\tau)\mathbf{Q}^T(\tau)\mathbf{Q}(\tau)\mathbf{F}_t(\tau)\mathbf{Q}^T(t)\\ &= \mathbf{Q}(t)[\mathbf{F}_t^T(\tau)\mathbf{T}(\tau)\mathbf{F}_t(\tau)]\mathbf{Q}^T(t)\end{aligned} \tag{xiii}$$

Thus,

$$\mathbf{J}_L^*(\tau) = \mathbf{Q}(t)\mathbf{J}_L(\tau)\mathbf{Q}^T(t) \tag{8.18.9}$$

and

$$\left[\frac{D^N\mathbf{J}_L^*(\tau)}{D\tau^N}\right]_{\tau=t} = \mathbf{Q}(t)\left[\frac{D^N\mathbf{J}_L(\tau)}{D\tau^N}\right]_{\tau=t}\mathbf{Q}^T(t), \quad N=1,2,3... \tag{8.18.10}$$

That is , the tensor $\mathbf{J}_L(\tau)$ as well as its material derivative evaluated at time t , is objective. The derivative

$$\frac{D_{Lc}\mathbf{T}}{Dt} \equiv \left[\frac{D\mathbf{J}_L(\tau)}{D\tau}\right]_{\tau=t} \equiv \left[\frac{D[\mathbf{F}_t(\tau)\mathbf{T}(\tau)\mathbf{F}_t^T(\tau)]}{D\tau}\right]_{\tau=t} \tag{8.18.11}$$

is called the **first Oldroyd lower convected derivative.** The Nth derivatives of $\mathbf{J}_L$ are called the **Nth Oldroyd lower derivatives.** Noting that

$$\mathbf{R}_t(t) = \mathbf{R}_t^T(t) = \mathbf{I} \tag{xiv}$$

and

$$\left[\frac{D\mathbf{F}_t(\tau)}{D\tau}\right]_{\tau=t} = (\nabla\mathbf{v})(t) \tag{xv}$$

it can be shown that

$$\frac{D_{Lc}\mathbf{T}}{Dt} = \frac{D\mathbf{T}}{Dt} + \mathbf{T}(\nabla\mathbf{v}) + (\nabla\mathbf{v})^T\mathbf{T} \tag{8.18.12}$$

and

$$\frac{D_{Lc}^{N+1}\mathbf{T}}{Dt} = \left(\frac{D}{Dt}\right)\left(\frac{D_{Lc}^{N}\mathbf{T}}{Dt}\right) + \left(\frac{D_{Lc}^{N}\mathbf{T}}{Dt}\right)(\nabla\mathbf{v}) + (\nabla\mathbf{v})^T\left(\frac{D_{Lc}^{N}\mathbf{T}}{Dt}\right) \tag{8.18.13}$$

Further, since

$$(\nabla\mathbf{v}) \equiv \mathbf{D} + \mathbf{W} \tag{xvi}$$

Equation (8.18.12) can also be written as

$$\frac{D_{Lc}\mathbf{T}}{Dt} = \frac{\mathbf{D}_{cr}T}{Dt} + \mathbf{TD} + \mathbf{DT} \tag{8.18.14}$$

where the first term in the right hand side is the co-rotational derivative of $\mathbf{T}$ given by Eq. (8.18.7).

(C) *Oldroyd upper convected derivative*

Let us consider the tensor

$$\mathbf{J}_u(\tau) = \mathbf{F}_t^{-1}(\tau)\mathbf{T}(\tau)\mathbf{F}_t^{-1T}(\tau) \tag{8.18.15}$$

Again, as in (A) and (B),

$$\mathbf{J}_u(t) = \mathbf{T}(t) \tag{xvii}$$

and the derivatives

$$\left[\frac{D^N\mathbf{J}_u(\tau)}{D^N\tau}\right]_{\tau=t}, \quad N=1,2,3\ldots \tag{xviii}$$

can be shown to be objective stress rates. [See Prob. 23] These are called the **Oldroyd upper convected derivatives.**

Let

$$\frac{D_{uc}\mathbf{T}}{Dt} \equiv \left[\frac{D\mathbf{J}_u(\tau)}{D\tau}\right]_{\tau=t} \equiv \left[\frac{D[\mathbf{F}_t^{-1}(\tau)\mathbf{T}(\tau)\mathbf{F}_t^{-1T}(\tau)]}{D\tau}\right]_{\tau=t} \tag{8.18.16}$$

and note that

$$\frac{D\mathbf{F}_t^{-1}(\tau)}{D\tau} = -\mathbf{F}_t^{-1}(\tau)\frac{D\mathbf{F}(\tau)}{D\tau}\mathbf{F}_t^{-1}(\tau) \tag{8.18.17}$$

one can derive

$$\frac{D_{uc}\mathbf{T}}{Dt}=\frac{D\mathbf{T}}{Dt}-\mathbf{T}(\nabla\mathbf{v})^T-(\nabla\mathbf{v})\mathbf{T} \tag{8.18.18}$$

or more generally

$$\frac{D_{uc}^{N+1}\mathbf{T}}{Dt}=\left(\frac{D}{Dt}\right)\left(\frac{D_{uc}^{N}\mathbf{T}}{Dt}\right)-\left(\frac{D_{uc}^{N}\mathbf{T}}{Dt}\right)(\nabla\mathbf{v})^T-(\nabla\mathbf{v})\left(\frac{D_{uc}^{N}\mathbf{T}}{Dt}\right) \tag{8.18.19}$$

Again, using Eq. (xvi), Eq. (8.18.18) can also be written

$$\frac{D_{uc}\mathbf{T}}{Dt}=\frac{D_{cr}\mathbf{T}}{Dt}-(\mathbf{TD}+\mathbf{DT}) \tag{8.18.20}$$

where the first term in the right hand side is the co-rotational derivative of $\mathbf{T}$ given by Eq. (8.18.7).

(D) Other objective stress rates

The stress rates given in (A)(B)(C) are not the only ones that are objectives. Indeed there are infinitely many. For example, the addition of any term or terms that is (are) objective to any of the above derivatives will give a new objective stress rate. In particular, the derivative

$$\frac{D_{cr}\mathbf{T}}{Dt}+a(\mathbf{TD}+\mathbf{DT}) \tag{8.18.21}$$

is objective for any value a. We note that For $a = +1$, it is the Oldroyd lower convected derivative and for $a = -1$, the Oldroyd upper convected derivative.

8.19 The Rate Type Constitutive Equations

Constitutive equations of the following form are known as the rate type nonlinear constitutive equations:

$$\mathbf{T} = -p\,\mathbf{I}+\boldsymbol{\tau} \tag{8.19.1a}$$

$$\boldsymbol{\tau}+\lambda_1\frac{D_*\boldsymbol{\tau}}{Dt}+\lambda_2\frac{D_*^2\boldsymbol{\tau}}{Dt^2}+\ldots=2\mu_1\mathbf{D}+\mu_2\frac{D_*\mathbf{D}}{Dt}+\ldots \tag{8.19.1b}$$

where D_*/Dt, D_*^2/Dt^2 etc., denote some objective time derivative and objective higher time derivatives, $\boldsymbol{\tau}$ is the extra stress and $\mathbf{D}$ is rate of deformation tensor. Equation (8.19.1) may be regarded as a generalization of the generalized linear Maxwell fluid defined in Sect. 8.2.

The following are some examples:

(a) *The convected Maxwell fluid*

The convected Maxwell fluid is defined by the constitutive equation

$$\boldsymbol{\tau} + \lambda \frac{D_{cr}\boldsymbol{\tau}}{Dt} = 2\mu \mathbf{D} \tag{8.19.2}$$

where $\dfrac{D_{cr}}{Dt}$ is the corotational derivative. That is

$$\frac{D_{cr}\boldsymbol{\tau}}{Dt} = \frac{D\boldsymbol{\tau}}{Dt} + \boldsymbol{\tau}\mathbf{W} - \mathbf{W}\boldsymbol{\tau} \tag{8.19.3}$$

Example 8.19.1

Obtain the stress components for the convected Maxwell fluid in a simple shearing flow.

Solution. With the velocity field for a simple shearing flow given by

$$v_1 = k x_2, \quad v_2 = v_3 = 0 \tag{i}$$

the rate of deformation tensor and the spin tensor are given by

$$[\mathbf{D}] = \begin{bmatrix} 0 & \frac{k}{2} & 0 \\ \frac{k}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad [\mathbf{W}] = \begin{bmatrix} 0 & \frac{k}{2} & 0 \\ -\frac{k}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{ii}$$

Thus,

$$[\boldsymbol{\tau}\mathbf{W}] = \begin{bmatrix} -\frac{k}{2}\tau_{12} & \frac{k}{2}\tau_{11} & 0 \\ -\frac{k}{2}\tau_{22} & \frac{k}{2}\tau_{12} & 0 \\ -\frac{k}{2}\tau_{23} & \frac{k}{2}\tau_{13} & 0 \end{bmatrix} \tag{iii}$$

$$[\mathbf{W}\boldsymbol{\tau}] = \begin{bmatrix} \frac{k}{2}\tau_{12} & \frac{k}{2}\tau_{22} & \frac{k}{2}\tau_{23} \\ -\frac{k}{2}\tau_{11} & -\frac{k}{2}\tau_{12} & -\frac{k}{2}\tau_{13} \\ 0 & 0 & 0 \end{bmatrix} \tag{iv}$$

$$[\boldsymbol{\tau}\mathbf{W}-\mathbf{W}\boldsymbol{\tau}] = \begin{bmatrix} -k\tau_{12} & \frac{k}{2}(\tau_{11}-\tau_{22}) & -\frac{k}{2}\tau_{23} \\ \frac{k}{2}(\tau_{11}-\tau_{22}) & k\tau_{12} & \frac{k}{2}\tau_{13} \\ -\frac{k}{2}\tau_{23} & \frac{k}{2}\tau_{13} & 0 \end{bmatrix} \tag{v}$$

Since the flow is steady and the rate of deformation is a constant independent of position, therefore, the stress field is also independent of time and position. Thus, the material derivative $D\boldsymbol{\tau}/Dt$ is zero so that Eq. (v) is the corotational derivative of $\boldsymbol{\tau}$(see Eq. (8.19.3). Substituting this equation into the constitutive equation, we obtain

$$\tau_{11}-k\lambda\tau_{12} = 0 \tag{vi}$$

$$\tau_{12} + \frac{k\lambda}{2}(\tau_{11} - \tau_{22}) = \mu k \tag{vii}$$

$$\tau_{13}-\frac{k\lambda}{2}\tau_{23} = 0 \tag{viii}$$

$$\tau_{22} + k\lambda\tau_{12} = 0 \tag{ix}$$

$$\tau_{23} + \frac{k\lambda}{2}\tau_{13} = 0 \tag{x}$$

$$\tau_{33} = 0 \tag{xi}$$

From Eqs. (viii) and (x), we obtain,

$$\tau_{13} = \tau_{23} = 0 \tag{xii}$$

From Eqs. (vi) and (ix), respectively

$$\tau_{11} = k\lambda\tau_{12} \tag{xiii}$$

and

$$\tau_{22} = -k\lambda\tau_{12} \tag{xiv}$$

Using the above two equations, we obtain from Eq. (vii) the shear stress function $\tau(k)$

$$\tau(k) = \tau_{12} = \frac{\mu k}{1 + k^2\lambda^2} \tag{xv}$$

The apparent viscosity η is

$$\eta(k) \equiv \frac{\tau(k)}{k} = \frac{\mu}{1 + k^2\lambda^2} \tag{xi}$$

The normal stress functions are

$$\sigma_1 \equiv T_{11} - T_{22} = \frac{2\mu k^2 \lambda}{1 + k^2\lambda^2} \tag{xii}$$

$$\sigma_2 \equiv T_{22} - T_{33} = \frac{-\mu k^2 \lambda}{1 + k^2\lambda^2} \tag{xiii}$$

(b) *The Corotational Jeffrey Fluid*

The corotational Jeffrey Fluid is defined by the constitutive equation

$$\mathbf{T} = -p\,\mathbf{I} + \boldsymbol{\tau} \tag{8.19.4a}$$

$$\boldsymbol{\tau} + \lambda_1 \frac{D_{cr}\boldsymbol{\tau}}{Dt} = 2\mu\left(\mathbf{D} + \lambda_2 \frac{D_{cr}\mathbf{D}}{Dt}\right) \tag{8.19.4b}$$

Example 8.19.2

Obtain the stress components for the corotational Jeffrey fluid in simple shearing flow.

Solution. The corotational derivative of the extra stress is the same as the previous example, thus,

$$\left[\frac{D_{cr}\boldsymbol{\tau}}{DT}\right] = [\mathbf{0}\,] + [\boldsymbol{\tau}\mathbf{W} - \mathbf{W}\boldsymbol{\tau}] = \begin{bmatrix} -k\tau_{12} & \frac{k}{2}(\tau_{11}-\tau_{22}) & -\frac{k}{2}\tau_{23} \\ \frac{k}{2}(\tau_{11}-\tau_{22}) & k\tau_{12} & \frac{k}{2}\tau_{13} \\ -\frac{k}{2}\tau_{23} & \frac{k}{2}\tau_{13} & 0 \end{bmatrix} \tag{i}$$

$$\left[\frac{D_{cr}\mathbf{D}}{DT}\right] = [\mathbf{0}\,] + [\mathbf{DW} - \mathbf{WD}] = \begin{bmatrix} -\frac{k^2}{2} & 0 & 0 \\ 0 & \frac{k^2}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{ii}$$

Substituting the above two equations and **D** from the previous example into Eq. (8.19.4), we obtain

$$\tau_{11} - k\,\lambda_1\,\tau_{12} = -\mu\,\lambda_2 k^2 \tag{iii}$$

$$\tau_{12} + \frac{k\lambda_1}{2}(\tau_{11} - \tau_{22}) = \mu k \tag{iv}$$

$$\tau_{13} - \frac{k\lambda_1}{2}\tau_{23} = 0 \tag{v}$$

$$\tau_{22} + k\lambda_1\tau_{12} = \mu\lambda_2 k^2 \tag{vi}$$

$$\tau_{23} + \frac{k\lambda_1}{2}\tau_{13} = 0 \tag{vii}$$

$$\tau_{33} = 0 \tag{viii}$$

Proceeding as in Example 8.19.1, we obtain the apparent viscosity η and the normal stress functions as:

$$\eta(k) \equiv \frac{\tau_{12}}{k} = \mu\,\frac{1 + \lambda_1\lambda_2 k^2}{1 + \lambda_1^2 k^2} \tag{ix}$$

$$\sigma_1 \equiv T_{11} - T_{22} = \frac{2\mu(\lambda_1 - \lambda_2)k^2}{1 + \lambda_1^2 k^2} \tag{x}$$

$$\sigma_2 \equiv T_{22} - T_{33} = -\frac{\mu(\lambda_1 - \lambda_2)k^2}{1 + \lambda_1^2 k^2} \tag{xi}$$

(c) *The Oldroyd 3-constant fluid*

The Oldroyd 3-constant model (also known as the Oldroyd fluid A) is defined by the following constitutive equation:

$$\mathbf{T} = -p\,\mathbf{I} + \boldsymbol{\tau} \tag{8.19.5a}$$

$$\boldsymbol{\tau} + \lambda_1\frac{D_{up}\boldsymbol{\tau}}{Dt} = 2\mu\left(\mathbf{D} + \lambda_2\frac{D_{up}\mathbf{D}}{Dt}\right) \tag{8.19.5b}$$

where D_{up}/Dt denote the Oldroyd upper convected derivative defined in Section 8.18. That is

$$\frac{D_{up}\boldsymbol{\tau}}{Dt} = \frac{D_{cr}\boldsymbol{\tau}}{Dt} - (\boldsymbol{\tau}\mathbf{D} + \mathbf{D}\boldsymbol{\tau}) \tag{i}$$

and

$$\frac{D_{up}\mathbf{D}}{Dt} = \frac{D_{cr}\mathbf{D}}{Dt} - (\mathbf{D}^2 + \mathbf{D}^2) \tag{ii}$$

where again D_{cr}/Dt denote the corotational derivative. By considering the simple shearing flow as was done in the previous two models, we can obtain that the viscosity of this fluid is a constant independent of the shear rate k, i.e.,

$$\eta(k) = \mu \tag{iii}$$

The normal stress functions are:

$$\sigma_1 \equiv T_{11} - T_{22} = 2\mu\,(\lambda_1 - \lambda_2)\,k^2 \tag{iv}$$

$$\sigma_2 \equiv T_{22} - T_{33} = -2\mu\,(\lambda_1 - \lambda_2)\,k^2 \tag{v}$$

(d)*The Oldroyd 4-constant fluid*

the Oldroyd 4-constant fluid is defined by the following constitutive equation:

$$\mathbf{T} = -p\,\mathbf{I} + \boldsymbol{\tau} \tag{8.19.6a}$$

$$\boldsymbol{\tau} + \lambda_1 \frac{D_{up}\boldsymbol{\tau}}{Dt} + \mu_o\,(\mathrm{tr}\,\boldsymbol{\tau}\,)\mathbf{D} = 2\mu\left(\mathbf{D} + \lambda_2 \frac{D_{up}\mathbf{D}}{Dt}\right) \tag{8.19.6b}$$

We note that in this model an additional term $\mu_o(\mathrm{tr}\,\boldsymbol{\tau}\,)\mathbf{D}$ is added to the left hand side. This term is obviously an objective term since both $\boldsymbol{\tau}$ and $\mathbf{D}$ are objective. The inclusion of this term will make the viscosity of the fluid dependent on the rate of deformation.

By considering the simple shearing flow as was done in the previous models, we can obtain the apparent viscosity to be

$$\eta(k) \equiv \frac{\tau_{12}}{k} = \mu \frac{1 + \lambda_2 \mu_o k^2}{1 + \lambda_1 \mu_o k^2} \tag{i}$$

The normal stress functions are:

$$\sigma_1 = T_{11} - T_{22} = 2\mu\,\lambda_1 \left(\frac{\eta(k)}{\mu} - \frac{\lambda_2}{\lambda_1}\right) \tag{ii}$$

$$\sigma_2 \equiv T_{22} - T_{33} = 0 \tag{iii}$$

Part C Viscometric Flows of an Incompressible Simple Fluid

8.20 Viscometric Flow

Viscometric flows may be defined to be the class of flows which satisfies the following conditions:

(i) At all time and at every material point, the history of the relative right Cauchy-Green deformation tensor can be expressed as

$$\mathbf{C}_t(\tau) = \mathbf{I} + (\tau - t)\mathbf{A}_1 + \frac{(\tau - t)^2}{2}\mathbf{A}_2. \tag{8.20.1}$$

(ii) There exists an orthogonal basis $(\mathbf{n}_i)$, with respect to which, the only nonzero Rivlin-Erickson tensors are given by

$$[\mathbf{A}_1] = \begin{bmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{\mathbf{n}_i} \qquad [\mathbf{A}_2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{\mathbf{n}_i} \tag{8.20.2}$$

The orthogonal basis $\{\mathbf{n}_i\}$ in general depends on the position of the material element.

The statement given in (ii) is equivalent to the following: There exists an orthogonal basis $(\mathbf{n}_i)$ with respect to which

$$\mathbf{A}_1 = k(\mathbf{N} + \mathbf{N}^T) \tag{8.20.3}$$

$$\mathbf{A}_2 = 2k^2\mathbf{N}^T\mathbf{N} \tag{8.20.4}$$

where the matrix of $\mathbf{N}$ with respect to $(\mathbf{n}_i)$ is given by

$$[\mathbf{N}] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{\mathbf{n}_i} \tag{8.20.5}$$

In the following examples, we demonstrate that simple shearing flow, plane Poiseuille flow, Poiseuille flow and Couette flow are all viscometric flows.

Example 8.20.1

Consider the uni-directional flow with a velocity field given in Cartesian coordinates as:

$$v_1 = v(x_2), \quad v_2 = v_3 = 0 \tag{i}$$

Show that it is a viscometric flow. We note that the uni-directional flow includes the simple shearing flow (where $v(x_2)=kx_2$) and the plane Poiseuille flow.

Solution. In Example 8.9.1, we obtained that for this flow, the history of $\mathbf{C}_t(\tau)$ is given by Eq. (8.20.1) and the matrix of the two non-zero Rivlin-Ericksen tensors $\mathbf{A}_1$ and $\mathbf{A}_2$, with respect to the rectangular Cartesian basis, are given by Eqs. (8.20.2) where $k = k(x_2)$. Thus, the given uni-directional flows are viscometric flows and the basis $\{\mathbf{n}_i\}$ with respect to which, $\mathbf{A}_1$ and $\mathbf{A}_2$ have the forms given in Eq. (8.20.2), is clearly given by

$$\mathbf{n}_1 = \mathbf{e}_1, \quad \mathbf{n}_2 = \mathbf{e}_2, \quad \mathbf{n}_3 = \mathbf{e}_3 \tag{ii}$$

Example 8.20.2

Consider the axisymmetirc flow with a velocity field given in cylindrical coordinates as:

$$v_r = 0, \quad v_\theta = 0, \quad v_z = v(r) \tag{i}$$

Show that this is a viscometric flow. Find the basis $\{\mathbf{n}_i\}$with respect to which, $\mathbf{A}_1$ and $\mathbf{A}_2$ have the forms given in Eq. (8.20.2).

Solution. In Example 8.9.2, we obtained that for this flow, the history of the right Cauchy-Green deformation tensor $\mathbf{C}_t(\tau)$ is given by an equation of the same form as Eq. (8.20.1)

where the two non-zero Rivlin-Ericksen tensors are given by

$$[\mathbf{A}_1]_{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z} = \begin{bmatrix} 0 & 0 & k(r) \\ 0 & 0 & 0 \\ k(r) & 0 & 0 \end{bmatrix}_{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z} \tag{ii}$$

$$[\mathbf{A}_2]_{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z} = \begin{bmatrix} 2k^2(r) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z} \tag{iii}$$

Let

$$\mathbf{n}_1 = \mathbf{e}_z, \quad \mathbf{n}_2 = \mathbf{e}_r, \quad \mathbf{n}_3 = \mathbf{e}_\theta \tag{iv}$$

then

$$(\mathbf{A}_1)_{11} = (\mathbf{A}_1)_{zz}, \quad (\mathbf{A}_1)_{12} = (\mathbf{A}_1)_{zr}, \quad (\mathbf{A}_1)_{13} = (\mathbf{A}_1)_{z\theta} \quad \text{etc.}$$

Then with respect to the basis $\{\mathbf{n}_i\}$.

$$[\mathbf{A}_1]_{\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3} = \begin{bmatrix} 0 & k(r) & 0 \\ k(r) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad [\mathbf{A}_2]_{\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2k^2(r) & 0 \\ 0 & 0 & 0 \end{bmatrix}_{\mathbf{n}_i} \tag{v}$$

Thus, this is a viscometric flow for which the basis $\{\mathbf{n}_i\}$ is related to the cylindrical basis $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ by Eq. (iv) [see figure 8.4].

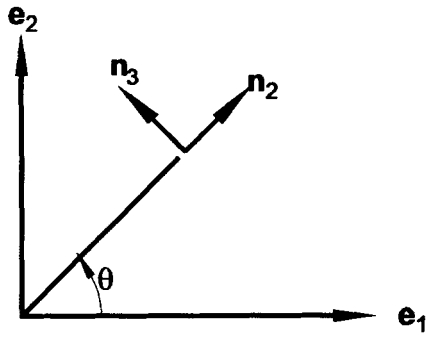

Fig. 8.4

Example 8.20.3

Consider the Couette flow with a velocity field given in cylindrical coordinates as

$$v_r = 0, \quad v_\theta = v(r) = r\,\omega(r), \quad v_z = 0 \tag{i}$$

Show that this is a viscometric flow and find the basis $\{\mathbf{n}_i\}$ with respect to which, $\mathbf{A}_1$ and $\mathbf{A}_2$ have the forms given in Eq. (8.20.2).

Solution. For the given velocity field, we obtained in Example 8.9.3

$$[\mathbf{C}_t(\tau)] = [\mathbf{I}] + (\tau - t)\begin{bmatrix} 0 & k(r) & 0 \\ k(r) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{(\tau-t)^2}{2}\begin{bmatrix} 2k^2(r) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{ii}$$

where

$$k(r) = \left(\frac{dv}{dr} - \frac{v}{r}\right) = r\frac{d\omega}{dr} \tag{iii}$$

The nonzero Rivlin-Ericksen tensors are

$$[\mathbf{A}_1]_{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z} = \begin{bmatrix} 0 & k(r) & 0 \\ k(r) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad [\mathbf{A}_2]_{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z} = \begin{bmatrix} 2k^2(r) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{iv}$$

$$[\mathbf{A}_n]_{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z} = 0 \quad \text{for } n = 3 \tag{v}$$

Comparing Eqs. (iv)(v) and (vi) with Eqs. (8.20.2), we see that the Couette flow is a viscometric flow. However, the basis $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ with respect to which, $\mathbf{A}_1$ and $\mathbf{A}_2$ have the forms given in Eq. (8.20.2), is

$$\mathbf{n}_1 = \mathbf{e}_\theta, \quad \mathbf{n}_2 = \mathbf{e}_r, \quad \mathbf{n}_3 = \mathbf{e}_z \tag{vi}$$

8.21 Stresses in Viscometric Flow of an Incompressible Simple Fluid

When a simple fluid is in viscometric flow, its history of deformation tensor $\mathbf{C}_t(t-\tau)$ is completely characterized by the two non-zero Rivlin-Ericksen tensor $\mathbf{A}_1$ and $\mathbf{A}_2$. Thus, the functional in Eq. (8.14.2) becomes simply a function of $\mathbf{A}_1$ and $\mathbf{A}_2$. That is

$$\mathbf{T} = -p\,\mathbf{I} + \mathbf{f}(\mathbf{A}_1,\mathbf{A}_2) \tag{8.21.1}$$

where the Rivlin -Ericksen tensors $\mathbf{A}_1$ and $\mathbf{A}_2$ are expressible as

$$\mathbf{A}_1 = k(\mathbf{N} + \mathbf{N}^T) \tag{8.21.2}$$

$$\mathbf{A}_2 = 2k^2\mathbf{N}^T\mathbf{N} \tag{8.21.3}$$

where the matrix of $\mathbf{N}$ relative to some choice of basis $\mathbf{n}_i$ is

$$[\mathbf{N}] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8.21.4}$$

Furthermore, the objectivity condition, Eq. (8.14.5) demands that for all orthogonal tensors $\mathbf{Q}$

$$\mathbf{Q}\mathbf{f}(\mathbf{A}_1, \mathbf{A}_2)\mathbf{Q}^T = \mathbf{f}(\mathbf{Q}\mathbf{A}_1\mathbf{Q}^T,\ \mathbf{Q}\mathbf{A}_2\mathbf{Q}^T) \tag{8.21.5}$$

In the following, we shall show that for a simple fluid in viscometric flow, with respect the basis $\mathbf{n}_i$,

$$T_{13} = T_{23} = 0$$

and that the normal stresses are all different from one another.

Let us choose a orthogonal tensor $\mathbf{Q}$ such that

$$[\mathbf{Q}]_{\mathbf{n}_i} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{i}$$

Then,

$$[\mathbf{Q}][\mathbf{N}][\mathbf{Q}^T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = [\mathbf{N}] \tag{ii}$$

Also

$$[\mathbf{Q}][\mathbf{N}^T\mathbf{N}][\mathbf{Q}^T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = [\mathbf{N}^T\mathbf{N}] \tag{iii}$$

i.e., for this choice of $\mathbf{Q}$,

$$\mathbf{Q}\mathbf{N}\mathbf{Q}^T = \mathbf{N} \tag{iv}$$

and

$$\mathbf{Q}(\mathbf{N}^T\mathbf{N})\mathbf{Q}^T = \mathbf{N}^T\mathbf{N} \tag{v}$$

Thus,

$$\mathbf{Q}\mathbf{A}_1\mathbf{Q}^T = k\mathbf{Q}(\mathbf{N} + \mathbf{N}^T)\mathbf{Q}^T = k(\mathbf{N} + \mathbf{N}^T) = \mathbf{A}_1 \tag{vi}$$

and

$$\mathbf{Q}\mathbf{A}_2\mathbf{Q}^T = 2k^2\mathbf{Q}(\mathbf{N}^T\mathbf{N})\mathbf{Q}^T = 2k^2\mathbf{N}^T\mathbf{N} = \mathbf{A}_2 \tag{vii}$$

Now, from Eq. (8.21.1), we have, for this particular choice of $\mathbf{Q}$,

$$\mathbf{Q}\mathbf{T}\mathbf{Q}^T = -p\,\mathbf{I} + \mathbf{f}(\mathbf{Q}\mathbf{A}_1\mathbf{Q}^T,\ \mathbf{Q}\mathbf{A}_2\mathbf{Q}^T) = -p\,\mathbf{I} + \mathbf{f}(\mathbf{A}_1,\ \mathbf{A}_2) \tag{viii}$$

i.e., for this $\mathbf{Q}$

$$\mathbf{Q}\mathbf{T}\mathbf{Q}^T = \mathbf{T} \tag{ix}$$

Thus,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \tag{x}$$

Carrying out the matrix multiplications, one obtains

$$\begin{bmatrix} T_{11} & T_{12} & -T_{13} \\ T_{21} & T_{22} & -T_{23} \\ -T_{31} & -T_{32} & T_{33} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \tag{xi}$$

The above equation states that

$$T_{13} = T_{23} = 0 \tag{8.21.6}$$

Since $\mathbf{A}_1$ and $\mathbf{A}_2$ depend only on k, therefore, the nonzero stress components with respect to the basis $\mathbf{n}_i$ are:

$$T_{12} = \tau(k) \tag{xii}$$

$$T_{11} = -p + \alpha(k) \tag{xiii}$$

$$T_{22} = -p + \beta(k) \tag{xiv}$$

$$T_{33} = -p + \gamma(k) \tag{xv}$$

where α, β, and γ are functions of k. Defining the normal stress functions by the equations

$$\sigma_1 \equiv T_{11} - T_{22} \tag{8.21.7}$$

$$\sigma_2 \equiv T_{22} - T_{33} \tag{8.21.8}$$

We can write the stress components of a simple fluid in viscometric flows as follows

$$T_{12} = \tau(k) \tag{8.21.9}$$

$$T_{11} = T_{22} + \sigma_1(k) \tag{8.21.9b}$$

$$T_{22} = T_{33} + \sigma_2(k) \tag{8.21.9c}$$

and

$$T_{13} = T_{23} = 0. \tag{8.21.9d}$$

As mentioned earlier in Section B, the function $\tau(k)$ is called the **shear stress function** and the function $\sigma_1(k)$, and $\sigma_2(k)$ are called the **normal stress functions.** [we recall that other definitions of the normal stress functions such as those given in Eq. (8.15.9) have also been used]. These three functions are known as the **viscometric functions.** These functions, when determined from the experiment on *one* viscometric flow of a fluid, determine completely the properties of the fluid in any other viscometric flow.

It can be shown that

$$\tau(-k) = -\tau(k) \tag{8.21.10}$$

$$\sigma_1(-k) = \sigma_1(k) \tag{8.21.11}$$

$$\sigma_2(-k) = \sigma_2(k) \tag{8.21.12}$$

That is, τ is an odd function of k, while σ_1 and σ_2 are even functions of k.

For the fluid in simple shearing flow, k is a constant so that all stress components are independent of spatial positions. Being accelerationless, it is clear that all momentum equations are satisfied so long as k remains constant. For a Newtonian fluid, such as water, the simple shearing flow gives

$$\tau(k) = \mu k, \quad \sigma_1(k) = 0, \quad \sigma_2 = 0 \tag{8.21.13}$$

For a non-Newtonian fluid, such as a polymeric solution, for small k, the viscometric functions can be approximated by a few terms of their Taylor series expansion. Noting the τ is an odd function of k, we have

$$\tau(k) = \mu k + \mu_1 k^3 + \ldots \tag{8.21.14}$$

and noting that σ_1 and σ_2 are even function of k, we have

$$\sigma_1 = s_1 k^2 + \ldots \tag{8.21.15}$$

$$\sigma_2 = s_2 k^2 + \ldots \tag{8.21.16}$$

Since the deviation from Newtonian behavior is of the order of k^2 for σ_1 and σ_2 and of k^3 for τ, therefore, it is expected that the deviation of the normal stresses will manifest themselves within the range of k in which the response of the shear stress remains essentially the same as that of a Newtonian fluid.

8.22 Channel Flow

We now consider the steady shearing flow between two infinite parallel fixed plates. That is,

$$v_1 = v(x_2), \; v_2 = v_3 = 0 \tag{8.22.1}$$

with

$$v(\pm\frac{h}{2}) = 0 \tag{8.22.2}$$

We saw in Section 8.20 that the basis $\mathbf{n}_i$ for which the stress components are given by Eqs. (8.21.9), is the Cartesian basis. $\mathbf{e}_i$. That is, with $k(x_2) \equiv dv/dx_2$

$$\tau_{12} = \tau(k), \; \tau_{13} = \tau_{23} = 0, \quad \tau_{11} = \tau_{22} + \sigma_1(k), \; \tau_{22} = \tau_{33} + \sigma_2(k) \tag{8.22.3}$$

Substituting the above equation in the equations of motion, we get, in the absence of body forces, [noting that k depends only on x_2]

$$-\frac{\partial p}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} = 0, \qquad \frac{\partial \tau_{22}}{\partial x_2} - \frac{\partial p}{\partial x_2}, \qquad \frac{\partial p}{\partial x_3} = 0 \tag{i}$$

Differentiating the above three equations with respect to x_1, we get

$$\frac{\partial}{\partial x_1}\frac{\partial p}{\partial x_1} = \frac{\partial}{\partial x_2}\frac{\partial p}{\partial x_1} = \frac{\partial}{\partial x_3}\frac{\partial p}{\partial x_1} = 0 \tag{ii}$$

Thus, $-\dfrac{\partial p}{\partial x_1}$ = a constant. Let this constant be denoted by f, which is the driving force for the flow, we have,

$$-\frac{\partial p}{\partial x_1} = -f \tag{8.22.4}$$

Now, the first equation in Eq. (i) gives

$$\frac{\partial \tau(k)}{\partial x_2} - \frac{\partial p}{\partial x_1} = 0 \tag{iii}$$

so that

$$\tau(k) = -f x_2 \tag{8.22.5}$$

where the integration constant is taken to be zero because the flow is symmetric with respect to the plane $x_2 = 0$. Inverting Eq. (8.22.5), we have,

$$k = \tau^{-1}(-f x_2) \equiv \gamma(-f x_2) = -\gamma(f x_2) \tag{8.22.6}$$

where $\gamma(S)$, the inverse function of $\tau(k)$, is an odd function because $\tau(k)$ is an odd function. Since $k(x_2) \equiv dv/dx_2$, therefore, the above equation gives

$$\frac{dv}{dx_2} = -\gamma(f x_2) \tag{8.22.7}$$

Integrating, we get

$$v(x_2) = -\int_{-\frac{h}{2}}^{x_2} \gamma(f x_2) d x_2 \tag{8.22.8}$$

For a given simple fluid with a known shear stress function $\tau(k)$, $\gamma(S)$ is also known, the above equation can be integrated to give the velocity distribution in the channel. The volume flux per unit width Q is given by

$$Q = \int_{-\frac{h}{2}}^{\frac{h}{2}} v(x_2) d x_2 \tag{8.22.9}$$

Equation (8.22.9) can be written in a form suitable for determining the function $\gamma(S)$ from an experimentally measured relationship between Q and f. Indeed, integration by part gives

$$Q = x_2 v(x_2)\Big]_{-h/2}^{h/2} - \int_{-h/2}^{h/2} x_2 dv = -\int_{-h/2}^{h/2} x_2 \frac{dv}{dx_2} d x_2 \tag{iv}$$

Using Eq. (8.22.7), we obtain

$$Q = \int_{-h/2}^{h/2} x_2 \gamma(f x_2) d x_2 = \frac{1}{f^2} \int_{-fh/2}^{fh/2} S \gamma(S) d S \tag{v}$$

or,

$$Q = = \frac{2}{f^2} \int_{o}^{fh/2} S \gamma(S) d S \tag{8.22.10}$$

Thus,

$$\frac{\partial(f^2 Q)}{\partial f} = 2\left(\frac{fh}{2}\right) \gamma\left(\frac{fh}{2}\right) \left(\frac{h}{2}\right) \tag{vi}$$

so that

$$\gamma(\frac{fh}{2}) = \frac{2}{fh^2} \frac{\partial}{\partial f}(f^2 Q) \tag{8.22.11}$$

Now, if the variation of Q with the driving force f (the pressure gradient), is measured experimentally, then the right hand side of the above equation is known so that the inverse shear stress function $\gamma(S)$ is obtained from the above equation.

Example 8.22.1

Use Eq. (8.22.8) to calculate the volume discharge per unit width across a cross section of the channel for a Newtonian fluid.

Solution. For a Newtonian fluid,

$$S = \tau(k) = \mu k, \text{ so that } k = \frac{S}{\mu} \equiv \gamma(S) \tag{i}$$

Thus,

$$\gamma(-fx) = \frac{-fx}{\mu} \tag{ii}$$

$$v(x_2) = \int_{-h/2}^{x_2} -\frac{fx}{\mu} dx = -\left[\frac{f}{\mu}\frac{x^2}{2}\right]_{-h/2}^{x_2} = -\frac{f}{\mu}\left(\frac{x_2^2}{2} - \frac{h^2}{8}\right) \tag{iii}$$

and

$$Q = -\int_{-d/2}^{d/2} x\left(-f\frac{x}{\mu}\right) dx = \left[\frac{fx^3}{\mu\, 3}\right]_{-d/2}^{d/2} = \frac{f d^3}{12\mu} \tag{iv}$$

8.23 Couette Flow

Couette flow is defined to be the two dimensional steady laminar flow between two concentric infinitely long cylinders which rotate with angular velocities Ω_1 and Ω_2. The velocity field is given by

$$v_r = 0, \quad v_\theta = v(r) = r\,\omega(r), \quad v_z = 0 \tag{8.23.1}$$

and in the absence of body forces, there is no pressure gradient in the θ and z directions.

In example 8.20.3, we see that the Couette flow is a viscometric flow and with

$$\mathbf{n}_1 = \mathbf{e}_\theta, \quad \mathbf{n}_2 = \mathbf{e}_r, \quad \mathbf{n}_e = \mathbf{e}_z \tag{i}$$

the nonzero Rivlin-Ericksen tensors are given by

$$[\mathbf{A}_1]_{\mathbf{e}_\theta,\mathbf{e}_r,\mathbf{e}_z} = \begin{bmatrix} 0 & k(r) & 0 \\ k(r) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{ii}$$

$$[\mathbf{A}_2]_{\mathbf{e}_\theta,\mathbf{e}_r,\mathbf{e}_z} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2k^2(r) & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{iii}$$

where

$$k(r) = r\frac{d\omega}{dr} \tag{iv}$$

Thus, the stress components with respect to the basis $\{\mathbf{n}_i\}$ are given by (see Section 8.21)

$$\tau_{\theta r} = \tau(k) \tag{8.23.2}$$

$$\tau_{\theta\theta} - \tau_{rr} = \sigma_1(k) \tag{8.23.3}$$

$$\tau_{rr} - \tau_{zz} = \sigma_2(k) \tag{8.23.4}$$

$$\tau_{rz} = \tau_{\theta z} = 0 \tag{8.23.5}$$

where $\tau(k)$, $\sigma_1(k)$, and $\sigma_2(k)$ are the shear stress function, the first normal stress function and the second normal stress function respectively. These three functions completely characterize the fluid in any viscometric flow, of which the present Couette flow is one. For a given simple fluid, these three functions are assumed to be known. On the other hand, we may use any one of the viscometric flows to measure these functions for use with the same fluid in other viscometric flows.

Let us first assume that we know these functions, then our objective is to find the velocity distribution, $v(r)$ and the stress distributions $\tau_{ij}(r)$ in this flow when the externally applied torque M per unit height in the axial direction is given.

In cylindrical coordinates, the equations of motion are

$$\frac{d\tau_{rr}}{dr} + \frac{1}{r}(\tau_{rr} - \tau_{\theta\theta}) - \frac{dp}{dr} = -\rho\, r\, \omega^2 \tag{8.23.6}$$

$$\frac{1}{r}\frac{d}{dr}(r^2 \tau_{r\theta}) = 0 \tag{8.23.7}$$

The z-equation of motion is identically satisfied, in view of Eq. (8.23.5) and the fact that τ_{zz} does not depend on z.

Equation (8.23.7) gives

$$\tau_{r\theta} = \frac{C}{r^2} \tag{v}$$

where C is the integration constant. The torque per unit height of the cylinders needed to maintain the flow is clearly given by

$$M = (2\pi\, r\, \tau_{r\theta})r \tag{8.23.8}$$

thus,

$$C = \frac{M}{2\pi} \tag{vi}$$

Now, to find the velocity distribution $v(r)$, from the known shear stress function $\tau(k)$, we first note that $\tau_{r\theta} = \tau(k)$ so that by Eq. (8.23.8), we have

$$\tau(k) = \frac{M}{2\pi\, r^2} \tag{vii}$$

Now, we wish to determine the function $k(r) \equiv r\dfrac{d\omega(r)}{dr}$ in Couette flow. To do this, we let

$$S(r) \equiv \tau(k(r)) \tag{8.23.9a}$$

and

$$k(r) = \gamma(S(r)) \tag{8.23.9b}$$

where $\gamma(S)$ is the inverse of the function$\tau(k)$, and is therefore a known function when the function $\tau(k)$is known.

Since $k(r) = r\dfrac{d\omega}{dr}$, therefore, from Eq. (8.23.9b), we have

$$r\frac{d\omega}{dr} = \gamma(S) \tag{viii}$$

where from Eqs. (vii) and (8.23.9a),

$$S = \frac{M}{2\pi r^2} \tag{ix}$$

Now,

$$\frac{d\omega}{dr} = \frac{d\omega}{dS}\frac{dS}{dr} = \frac{d\omega}{dS}\left(\frac{M}{2\pi}\right)\left(\frac{-2}{r^3}\right) = \frac{M}{2\pi r^2}\frac{2}{r}\frac{d\omega}{dS} = -2Sr\frac{d\omega}{dS} \tag{x}$$

Thus,

$$\gamma(S) = r\frac{d\omega}{dr} = -2S\frac{d\omega}{dS} \tag{xi}$$

from which we have

$$d\omega = -\frac{\gamma(S)}{2S}dS \tag{xii}$$

Integration of the above equation gives

$$\omega(r) = \Omega_1 - \frac{1}{2}\int_{\frac{M}{2\pi R_1^2}}^{\frac{M}{2\pi r}} \frac{\gamma(S)}{S}dS \tag{8.23.10}$$

where Ω_1 is the angular velocity of the inner cylinder. For given $\gamma(S)$, the above equation gives the desired function $\omega(r)$ from which $v_\theta = r\omega(r)$ can be obtained.

Next, we wish to obtain the normal stresses T_{rr} at the two cylindrical surfaces $r = R_1$ and $r = R_2$.

From the r-equation of motion, Eq. (8.23.6), we have

$$\frac{d\tau_{rr}}{dr} + \frac{1}{r}(\tau_{rr} - \tau_{\theta\theta}) = -r\rho\omega^2 \tag{xiii}$$

That is, using Eq. (8.23.3)

$$\frac{d\tau_{rr}}{dr} - \frac{\sigma_1}{r} = -r\rho\omega^2 \tag{xiv}$$

Integrating, we get

$$\tau_{rr}(r) - \tau_{rr}(R_1) = \int_{r=R_1}^{r} \frac{\sigma_1}{r}dr - \rho\int_{r=R_1}^{r} r\,\omega^2 dr \tag{xv}$$

In particular,

$$[-T_{rr}(R_2)]-[-T_{rr}(R_1)] = \rho\int_{R_1}^{R_2} r\,\omega^2(r)dr - \int_{R_1}^{R_2} \frac{\sigma_1}{r}dr \tag{8.23.11}$$

In the right hand side of the above equation, the first term is always positive, stating that the centrifugal force effects always make the pressure on the outer cylinder larger than that on the inner cylinder. On the other hand, for a fluid which has a positive normal stress function $\bar{\sigma}_1$, the second term in the above equation is negative, stating that the contribution to the pressure difference due to the normal stress effect is in the opposite direction to that due to the centrifugal force effect. Indeed, all known polymeric solutions have a positive $\bar{\sigma}_1$ and in many instances, this normal stress effects actually causes the pressure on the inner cylinder to be larger than that on the outer cylinder.

We now consider the reverse problem of determining the material function $\gamma(S)$ and therefore $\tau(S)$ from a measured relationship between the torque M needed to maintain the Couette flow and the angular velocity difference $\Omega_2 - \Omega_1$.

Since

$$\tau_1 = \frac{M}{2\pi R_1^2} \tag{xvi}$$

therefore,

$$\tau_2 = \frac{M}{2\pi R_2^2} = \frac{M}{2\pi R_1^2}\left(\frac{R_1}{R_2}\right)^2 \tag{xvii}$$

That is

$$\tau_2 = \beta\tau_1 \tag{xviii}$$

where

$$\beta = \left(\frac{R_1}{R_2}\right)^2 < 1 \tag{xix}$$

Now, from Eq. (8.23.10), we have

$$\Delta\Omega = \int_{\tau_2}^{\tau_1} \frac{\gamma(S)}{2S}dS \tag{8.23.12}$$

where

$$\Delta\Omega = \Omega_2 - \Omega_2 \tag{xx}$$

and

$$\tau_1 = \frac{M}{2\pi R_1^2} \text{ and } \tau_2 = \frac{M}{2\pi R_2^2} \tag{xxi}$$

Differentiating the above equation with respect to M gives

$$2M\frac{\partial \Delta\Omega}{\partial M} = \gamma(\tau_1) - \gamma(\tau_2) \tag{8.23.13}$$

Using Eq. (xix), we have

$$2M\frac{\partial \Delta\Omega}{\partial M} = \gamma(\tau_1) - \gamma(\beta\tau_1) \tag{8.23.14}$$

Defining

$$\Gamma(\tau_1) \equiv \gamma(\tau_1) - \gamma(\beta\tau_1) \tag{8.23.15}$$

we have,

$$2M\frac{\partial \Delta\Omega}{\partial M} = \Gamma(\tau_1) \tag{8.23.16}$$

i.e.,

$$2M\frac{\partial \Delta\Omega}{\partial M} = \Gamma\left(\frac{M}{2\pi R_1^2}\right) \tag{8.23.17}$$

Equation (8.23.17) allows the determination of $\Gamma(\tau_1)$from experimental results relating $\Delta\Omega$ with M. To obtain $\gamma(S)$ and therefore $\tau(k)$, from $\Gamma(\tau_1)$, we note the following

$$\Gamma(\tau_1) = \gamma(\tau_1) - \gamma(\beta\tau_1) \tag{xxii}$$

$$\Gamma(\beta\tau_1) = \gamma(\beta\tau_1) - \gamma(\beta^2\tau_1) \tag{xxiii}$$

$$\Gamma(\beta^2\tau_1) = \gamma(\beta^2\tau_1) - \gamma(\beta^3\tau_1) \tag{xxiv}$$

Thus,

$$\sum_n^N \Gamma(\beta^n\tau_1) = \gamma(\tau_1) - \gamma(\beta^{N+1}\tau_1) \tag{xxv}$$

Since $\beta < 1$, $\lim_{N\to\infty} \beta^{N+1} \to 0$, $\gamma(\beta^{N+1}\tau_1) \to \gamma(0) = 0$, therefore,

$$\gamma(\tau) = \sum_{n=0}^{\infty} \Gamma(\beta^n\tau) \tag{8.23.18}$$

Thus, from the known function $\Gamma(\tau)$, the above equation allows one to obtain the inverse shear function $\gamma(S)$ from which the shear function $\tau(S)$ can be obtained.

If the gap $R_2 - R_1$ is very small, the rate of shear k will be essential a constant independent of r given by

$$k = \frac{R_1 \Delta\Omega}{R_2 - R_1}$$

Thus, $k = \gamma(\tau_1)$ leads to

$$\gamma\left(\frac{M}{2\pi R_1}\right) = \frac{R_1 \Delta\Omega}{R_2 - R_1} \tag{8.23.19}$$

By measuring the relationship between M and $\Delta\Omega$, the above equation determines the inverse shear stress function $\gamma(S)$.

PROBLEMS

8.1. Show that for an incompressible Newtonian fluid in Couette flow, the pressure at the outer cylinder ($r = R_o$) is always larger than that at the inner cylinder. That is, obtain

$$[-T_{rr}(R_o)] - [-T_{rr}(R_i)] = \rho \int_{R_i}^{R_0} r\, \omega^2(r) dr$$

8.2. Obtain the force-displacement relationship for N-Maxwell elements connected in parallel. Neglect inertia effects.

8.3. Obtain the force-displacement relationship for the Kelvin-Voigt solid which consists of a dashpot (with damping coefficient η) and a spring (with spring constant G) connected in parallel. Also, obtain its relaxation function. Neglect inertia effects.

8.4. Obtain the force-displacement relationship for a dashpot (damping coefficient η_o) and a Kelvin-Voigt solid (damping coefficient η and spring constant G, see the previous problem) connected in series. Also, obtain the relaxation function.

8.5. A linear Maxwell fluid, defined by Eq.(8.1.1), is between two parallel plates which are one unit apart. Starting from rest, at time $t = 0$, the top plate is given a displacement $u = v_o t$ while the bottom plate remains fixed. Neglect inertia effects, obtain the shear stress history.

8.6. Obtain Eq.(8.3.1) by solving the linear, nonhomogeneous ordinary differential equation Eq.(8.1.1b).

8.7. Show that for the linear Maxwell fluid defined by Eqs. (8.1.1)

$$\int_{-\infty}^{t} \phi(t-t')J(t')dt' = t$$

where $\phi(t)$ is the relaxation function and $J(t)$ is the creep compliance function.

8.8. Obtain the storage modulus and loss modulus for the linear Maxwell fluid with a continuous relaxation spectrum defined by Eq.(8.4.1).

8.9. Show that the viscosity μ of a linear Maxwell fluid defined by Eqs. (8.1.1) is related to the relaxation function $\phi(t)$ and the memory function $f(s)$ by the relation

$$\mu = \int_{o}^{\infty} \phi(s)ds = -\int_{o}^{\infty} s f(s) ds$$

8.10. Show that the relaxation function for the Jeffrey's model, Eq.(8.2.5) with $a_2 = 0$ is given by

$$\phi(t) = \frac{b_o}{a_1}\left[\left(1-\frac{b_1}{b_o a_1}\right)e^{-t/a_1} + \frac{2b_1}{b_o}\delta(t)\right]$$

where $\delta(t)$is the Dirac function.

8.11. For the following velocity field, obtain (a) the particle pathline equation using the current time as the reference time, (b) the relative right Cauchy-Green deformation tensor and (c) the Rivlin-Ericksen tensor.

$$v_1 = 0, \quad v_2 = v(x_1), \quad v_3 = 0$$

8.12. Given the velocity field

$$v_1 = -kx_1, \quad v_2 = kx_2, \quad v_3 = 0$$

(a) Obtain the relative right Cauchy-Green deformation tensor.

(b) Using Eq.(8.9.2), obtain the Rivlin-Ericksen tensors.

(c) Obtain the Rivlin-Ericksen tensors from the recursive equation, Eq.(8.10.3).

(d) Is this velocity field a viscometric flow?

8.13. Do the previous problem for the velocity field

$$v_1 = kx_1, \quad v_2 = kx_2, \quad v_3 = -2kx_3$$

8.14. Given the velocity field

$$v_1 = kx_2, \quad v_2 = kx_1, \quad v_3 = 0$$

(a) Obtain the pathline equations using the current time as the reference time.

(b) Obtain the relative right Cauchy-Green deformation tensor.

(c) Using Eq.(8.9.2) to obtain the Rivlin-Ericksen tensor.

(d) Using Eq.(8.10.3) to obtain the Rivlin-Ericksen tensor.

8.15. Given the velocity field

$$v_1 = -kx_1, \quad v_2 = kx_2, \quad v_3 = 0$$

(a) Obtain the stress field **T** for a Newtonian fluid.

(b) Obtain the co-rotational derivative of the stress tensor **T**.

(c) Obtain the upper convected derivative of the stress tensor **T**.

(d) Obtain the lower convected derivative of the stress tensor **T**.

8.16. Do the previous problem for the following velocity field

$$v_1 = kx_1, \quad v_2 = kx_2, \quad v_3 = -2kx_3$$

8.17. Given the velocity field

$$v_1 = -kx_1, \quad v_2 = kx_2, \quad v_3 = 0$$

(a) Obtain the stress field **T** for a second-order fluid.

(b) Obtain the co-rotational derivative of the stress tensor **T**.

(c) Obtain the upper convected derivative of the stress tensor **T**.

(d) Obtain the lower convected derivative of the stress tensor **T**.

8.18. Do the previous problem for the following velocity field

$$v_1 = k x_1, \quad v_2 = k x_2, \quad v_3 = -2 k x_3$$

8.19 . Derive Eqs.(8.8.4).

8.20. Derive Eqs. (8.8.9b) and (8.8.9e).

8.21. Derive Eq. (8.10.3).

8.22. Derive Eq. (8.18.5)

8.23. Show from Eq.(8.18.15), that Oldroyd's upper convected derivative is objective.

8.24. The Reiner-Rivlin fluid is defined by the constitutive equation

$$\mathbf{T} = -p\,\mathbf{I} + \boldsymbol{\tau}$$

$$\boldsymbol{\tau} = \phi_1(I_2, I_3)\mathbf{D} + \phi_2(I_2, I_3)\mathbf{D}^2$$

where I_i are the scalar invariants of $\mathbf{D}$. Obtain the stress components for this fluid in a simple shearing flow.

8.25. The exponential of a tensor $\mathbf{A}$ is defined as

$$\exp\mathbf{A} = \mathbf{I} + \sum_{1}^{\infty} \frac{1}{n\,!}\mathbf{A}^n$$

If $\mathbf{A}$ is an objective tensor, is $\exp \mathbf{A}$ also objective?

8.26. Why is it that the following constitutive equation is not acceptable?

$$\mathbf{T} = -p\,\mathbf{I} + \boldsymbol{\tau}$$

$$\boldsymbol{\tau} = \alpha\,(\nabla\mathbf{v})$$

where $\mathbf{v}$ is the velocity vector and α is a constant.

8.27. Let $d\mathbf{a}$ and $d\mathbf{A}$ denote the differential area vector at time τ and at time t respectively. For an incompressible fluid, show that

$$\left[\frac{D^N da^2}{D\tau^N}\right]_{\tau=t} = d\mathbf{A}\cdot\left[\frac{D^N \mathbf{C}_t^{-1}}{D\tau^N}\right]_{\tau=t} d\mathbf{A} \equiv -d\mathbf{A}\cdot\mathbf{M}_N\, d\mathbf{A}$$

where da is the magnitude of $d\mathbf{A}$. and the tensor $\mathbf{M}_N$ are known as the White-Metzner tensors

8.28. (a) Verify that Oldroyd's lower convected derivatives of the identity tensor are the Rivlin-Ericksen tensors $\mathbf{A}_N$.

(b) Verify that Oldroyd's upper convected derivatives of the identity tensor are the negative White-Metzner tensors (see Prob. 8.27).

8.29. Show that the derivative given in Eq.(xviii) of Section 8.18 is objective.

8.30. Obtain Eq. (8.18.12) for Oldroyd's lower convected derivative.

8.31. Obtain Eq.(8.18.18) for Oldroyd's upper convected derivative.

8.32. Show that the lower convected derivative of the first Rivlin-Ericksen tensor $\mathbf{A}_1$ is the second Rivlin-Ericksen tensor $\mathbf{A}_2$.

8.33. Consider the following constitutive equation

$$\boldsymbol{\tau} + \lambda \frac{D_*\boldsymbol{\tau}}{Dt} = 2\mu\mathbf{D}$$

where $\dfrac{D_*\boldsymbol{\tau}}{Dt} = \dfrac{D_{cr}\boldsymbol{\tau}}{Dt} + a(\mathbf{D}\boldsymbol{\tau} + \boldsymbol{\tau}\mathbf{D})$ and a is a constant. Obtain the shear stress function and the two normal stress functions for this fluid. We note that $a = 1$ corresponds to Eq.(8.18.4) and $= -1$ corresponds to Eq.(8.18.20).

8.34. Let $\mathbf{Q}$ be a tensor whose matrix with respect to the basis $\mathbf{n}_i$ is

$$[\mathbf{Q}]_{\mathbf{n}_i} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) Verify the following relations for the tensor $\mathbf{N}$ whose matrix with respect to $\mathbf{n}_i$ is given by Eq.(8.20.5): $\mathbf{QNQ}^T = -\mathbf{N}$ and $\mathbf{QN}^T\mathbf{NQ} = \mathbf{N}^T\mathbf{N}$

(b) For $\mathbf{A}_1$ and $\mathbf{A}_2$ given by Eq.(8.20.3) and Eq.(8.20.4), verify the relations

$$\mathbf{QA}_1\mathbf{Q}^T = -\mathbf{A}_1 \quad \text{and} \quad \mathbf{QA}_2\mathbf{Q}^T = \mathbf{A}_2$$

(c) From $\mathbf{T} = -p\mathbf{I} + \mathbf{f}(\mathbf{A}_1, \mathbf{A}_2)$ where $\mathbf{A}_1$, and $\mathbf{A}_2$ are given by Eq.(8.20.3) and (8.20.4) and

$$\mathbf{Q}\,\mathbf{f}(\mathbf{A}_1,\mathbf{A}_2)\mathbf{Q}^T = \mathbf{f}(\mathbf{QA}_1\mathbf{Q}^T, \mathbf{QA}_2\mathbf{Q}^T)$$

show that

$$\mathbf{QT}(k)\mathbf{Q}^T = \mathbf{T}(-k)$$

(d) From the results of part (c), show that the viscometric functions have the properties

$$\tau(k) = -\tau(-k), \quad \sigma_1(k) = \sigma_1(-k), \quad \sigma_2(k) = \sigma_2(-k)$$

8.35. For the velocity field given in Example 8.20.2, i.e.,

$$v_r = 0, \quad v_\theta = 0, \quad v_z = v(r),$$

(a) Obtain the stress components in terms of the shear stress function $\tau(k)$ and the normal stress functions $\sigma_1(k)$ and $\sigma_2(k)$, where $k = dv/dr$. (b) Obtain the following velocity distribution for the Poiseuille flow under a pressure gradient of $-f$:

$$v(r) = \int_r^R \gamma\left(\frac{fr}{2}\right) dr$$

where γ is the inverse shear stress function

(c) Obtain the relation

$$\gamma\left(\frac{Rf}{2}\right) = \frac{1}{\pi R^3 f^2}\frac{\partial(f^3 Q)}{\partial f}$$

where Q is the volume flux.