APPENDIX: MATRICES

Matrix

A matrix is an aggregate of elements arranged in rectangular array.

The following is a matrix of m rows and n columns

$$[\mathbf{T}] = \begin{bmatrix} T_{11} & T_{12} & \dots & T_{1n} \\ T_{21} & T_{22} & \dots & T_{2n} \\ \dots & \dots & \dots & \dots \\ T_{m1} & T_{m2} & \dots & T_{mn} \end{bmatrix} \tag{1}$$

There are $m \times n$ elements in the above matrix. The first subscript of each element refers to the row in which the element is located and the second subscript refers to the column. Thus T_{32} is located in the third row and second column of the matrix. In general, T_{ij} is the element in the ith row and j^{th} column. The matrix in Eq.(1) may also be denoted by

$$[\mathbf{T}] = [T_{ij}] \tag{2}$$

where $i = 1,2,...m$ and $j = 1,2,...n$

Whenever necessary, we shall use the notation $[\mathbf{T}]^{\mathbf{m}\times\mathbf{n}}$ to indicate there are m rows and n columns in the matrix [T].

Transpose of a Matrix

Let $[S]^{m\times n} = [S_{ij}]$ be a matrix of m rows and n columns. If $S_{ij} = T_{ji}$, then [S] is called the transpose of [T], and will be denoted by $[\mathbf{S}]^T$.

For example, if

$$[\mathbf{T}]^{2\times 3} = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

then

$$[\mathbf{S}]^{3\times 2} = \begin{bmatrix} 1 & 4 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Square Matrix

A matrix with equal number of rows and columns is called a square matrix. For example,

$$[\mathbf{T}] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

is a square matrix of the third order. The elements T_{11}, T_{22}, T_{33} are called the diagonal elements. Example:

$$[\mathbf{T}] = \begin{bmatrix} T_{11} & 4 & -1 \\ 4 & T_{22} & 2 \\ -1 & 2 & T_{33} \end{bmatrix}$$

Note that for a **symmetric matrix**

$$[\mathbf{T}] = [\mathbf{T}]^T$$

Diagonal Matrix

A square matrix with the property that all nondiagonal elements are zero is called a diagonal matrix. Example:

$$[\mathbf{T}] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Note that a diagonal matrix is a symmetric matrix.

Scalar Matrix

A diagonal matrix with the property that $T_{11} = T_{22} = T_{33} = \alpha$ is called a scalar matrix. Example:

$$[\mathbf{T}] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Identity Matrix

A scalar matrix with the property that the diagonal elements equal unity is called the identity matrix. We shall denote the identity matrix by [**I**]. Thus

$$[\mathbf{I}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Row Matrix

A matrix with only one row is called a row matrix. Only one index is necessary to locate the position of an element. Thus

$$[\mathbf{a}] = [a_1, a_2, a_3] = [a_i]_r$$

where the subscript r indicates that [**a**] is a row matrix.

Column Matrix

A matrix with only one column is called a column matrix. Thus

$$[\mathbf{a}] = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = [a_i]_c$$

is a column matrix. Note that

$$[\mathbf{a}]_c^T \text{ is the row matrix given by } [a_1, a_2, a_3]$$

Matrix Operation

1. If $[\mathbf{T}]^{m\times n} = [T_{ij}]$ and $[\mathbf{S}]^{m\times n} = [S_{ij}]$, then $[\mathbf{T}] = [\mathbf{S}]$, if and only if $T_{ij} = S_{ij}$

2. If α is a scalar, then

$$\alpha[\mathbf{T}] = [\mathbf{T}]\alpha = [\alpha T_{ij}]$$

Example:

$$3\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$$

3. $[\mathbf{T}]^{m\times n} + [\mathbf{S}]^{m\times n} \equiv [T_{ij} + S_{ij}]$

Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 3 & 6 \end{bmatrix}$$

The following rules follows from the operation rules of scalars

$$[\mathbf{T}] + [\mathbf{S}] = [\mathbf{S}] + [\mathbf{T}]$$
$$[\mathbf{T}] + ([\mathbf{S}] + [\mathbf{R}]) = ([\mathbf{T}] + [\mathbf{S}]) + [\mathbf{R}]$$
$$(\alpha + \beta)[\mathbf{T}] = \alpha[\mathbf{T}] + \beta[\mathbf{T}]$$
$$\alpha[\mathbf{T} + \mathbf{S}] = \alpha[\mathbf{T}] + \alpha[\mathbf{S}]$$

4. If $[T_{ij}]^{m\times n}$ is a matrix of m rows and n columns and $[S_{ij}]^{n\times p}$ is a matrix of n rows and p columns, then

$$[\mathbf{T}]^{m\times n}\,[\mathbf{S}]^{n\times p} \equiv [\mathbf{R}]^{m\times p}$$

where the elements of $[\mathbf{R}]$ are given by

$$R_{ij} = \sum_{k=1}^{n} T_{ik}\,S_{kj} \qquad i=1,2,\ldots m,\ \ j=1,2,\ldots p$$

Example:

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \\ S_{31} & S_{32} \end{bmatrix}$$

$$= \begin{bmatrix} T_{11}S_{11} + T_{12}S_{21} + T_{13}S_{31} & T_{11}S_{12} + T_{12}S_{22} + T_{13}S_{32} \\ T_{21}S_{11} + T_{22}S_{21} + T_{23}S_{31} & T_{21}S_{12} + T_{22}S_{22} + T_{23}S_{32} \end{bmatrix},$$

$$[a_1, a_2, a_3] \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

It is important to note that only when the number of columns of the first matrix is the same as the number of rows of the second matrix is multiplication defined. Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

But

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

is not defined.

The following examples show even if both [**T**] [**S**] and [**S**] [**T**] are defined, they are in general not equal. Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 11 & 15 \end{bmatrix}$$

But

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 11 & 16 \end{bmatrix}$$

However, if [**T**] is a scalar matrix, i.e.,

$$[\mathbf{T}] = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}$$

then,

$$[\mathbf{T}]\,[\mathbf{S}] = [\mathbf{S}]\,[\mathbf{T}]$$

provided both [**T**] [**S**] and [**S**] [**T**] are defined (that is, provided [**S**] is also a square matrix and of the same order as that of the [**T**]). In particular, if [**S**] is a square matrix of the same order as that of the unit matrix [**I**], then

$$[\mathbf{I}]\,[\mathbf{S}] = [\mathbf{S}]\,[\mathbf{I}] = [\mathbf{S}]$$

It can be shown that the matrix product has the following properties:

$$[\mathbf{T}]([\mathbf{S}]\,[\mathbf{R}]) = ([\mathbf{T}]\,[\mathbf{S}])[\mathbf{R}]$$

$$\alpha[\mathbf{TS}] = (\alpha[\mathbf{T}])\,[\mathbf{S}]) = [\mathbf{T}]\,(\alpha[\mathbf{S}])$$

$$[\mathbf{T} + \mathbf{S}][\mathbf{R}] = [\mathbf{TR}] + [\mathbf{SR}]$$

$$[\mathbf{R}]\,[\mathbf{T} + \mathbf{S}] = [\mathbf{R}]\,[\mathbf{T}] + [\mathbf{R}]\,[\mathbf{S}]$$

The Reversed Rule for a Transposed Product

In the following, we shall show that the transpose of a product of matrices is equal to the product of their transpose in reverse order, i.e.,

$$[\mathbf{TS}]^T = [\mathbf{S}]^T\,[\mathbf{T}]^T$$

Proof: Let

$$[\mathbf{T}] = [T_{ij}]^{m\times n}\ , \qquad [\mathbf{S}] = [S_{ij}]^{n\times p}$$

$$[\mathbf{T}]^T \equiv [\mathbf{A}] = [A_{ij}]^{n\times m} \qquad [\mathbf{S}]^T \equiv [\mathbf{B}] = [B_{ij}]^{p\times n}$$

$$[\mathbf{TS}] \equiv [\mathbf{C}] = [C_{ij}]^{m\times p} \qquad [\mathbf{TS}]^T \equiv [\mathbf{D}] = [D_{ij}]^{p\times m}$$

and

$$[\mathbf{S}]^T\,[\mathbf{T}]^T = [\mathbf{B}]\,[\mathbf{A}] \equiv [\mathbf{E}] = [E_{ij}]^{p\times m}$$

then,

$$C_{ij} = \sum_{k=1}^{n} T_{ik}S_{kj} \quad i = 1,2,...m,\ j = 1,2,..p$$

so that

$$D_{ij} = C_{ji} = \sum_{k=1}^{n} T_{jk}S_{ki} \quad j = 1,2,...m,\ i = 1,2,..p$$

On the other hand

$$E_{ij} = \sum_{k=1}^{n} B_{ik}A_{kj} \quad i = 1,2,..p,\ j = 1,2,...m$$

$$= \sum_{k=1}^{n} S_{ki}T_{jk} \quad i = 1,2,..p,\ j = 1,2,...m$$

Thus,

$$E_{ij} = D_{ij}$$

That is

$$[\mathbf{TS}]^T = [\mathbf{S}]^T\,[\mathbf{T}]^T$$

Inverse of a Matrix

A square matrix [S] is called the inverse of the square matrix [T] if

$$[\mathbf{T}]\,[\mathbf{S}] = [\mathbf{I}]$$

where [I] is the unit matrix. We shall denote [S] by $[\mathbf{T}]^{-1}$. Thus,

$$[\mathbf{T}]\,[\mathbf{T}]^{-1} = [\mathbf{I}]$$

It can be proved that if the determinant of [T], ie., $|T_{ij}|$ is not equal to zero, then the inverse of [T] exists and that

$$[\mathbf{T}]\,[\mathbf{T}]^{-1} = [\mathbf{T}]^{-1}\,[\mathbf{T}] = [\mathbf{I}]$$

where $[\mathbf{T}]^{-1}$ is unique. The proof will not be given here.

The Reverse Rule of the Inverse of a Product of Matrices.

In the following we shall show that the inverse of a product of matrices is equal to the product of their inverses in reversed order, i.e.,

$$[\mathbf{TS}]^{-1} = [\mathbf{S}]^{-1}\,[\mathbf{T}]^{-1}$$

Proof.

$$[\mathbf{TS}]\,[\mathbf{S}]^{-1}\,[\mathbf{T}]^{-1} = [\mathbf{T}]\,[\mathbf{S}]\,[\mathbf{S}]^{-1}\,[\mathbf{T}]^{-1} = [\mathbf{T}]\,[\mathbf{T}]^{-1} = [\mathbf{I}].$$

Thus, $[\mathbf{S}]^{-1}\,[\mathbf{T}]^{-1}$ is the unique inverse of [TS].

Differentiation of a Matrix

If T_{ij} are functions of x, y, z, t, then

$$\frac{\partial}{\partial x}[\mathbf{T}] = \left[\frac{\partial T_{ij}}{\partial x}\right]$$

$$\frac{\partial}{\partial y}[\mathbf{T}] = \left[\frac{\partial T_{ij}}{\partial y}\right]$$

etc.