ME 402/512 (CE 402) ASSIGNMENT 1. Due September 1, 1998

Some of these exercises draw on prerequisite knowledge not covered in lecture.

1. Print and read the homework grading policy at
`http://home.sprintmail.com/~zz/Teach/Continuum/HWpolicy.html`

2. Give brief *engineering* definitions for *any 12* of the following 16 terms:

elastic material	rigid body	Mohr's circle	strength
homogeneous	scalar	Shear modulus	velocity
anisotropic	vector	Hooke's law of Linear Elasticity	speed
continuous function	Poisson's ratio	stiffness	plane strain

3. Let $[A] = \begin{bmatrix} 5 & 2 \\ -12 & -5 \end{bmatrix}$ and $[B] = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. Show that [A] is a "square root" of the 2×2 identity matrix and [B] is a "square root" of the 2×2 *zero* matrix. Find the eigenvalues and eigenvectors of [A] and [B].

4. Let $[A] = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix}$ and $[B] = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 4 & 5 \end{bmatrix}$. Find $[A][B]$ and $[B][A]$.

5. Let $[A] = \begin{bmatrix} 2 & 1 & 1 \\ 7 & 1 & 2 \\ 21 & 0 & 4 \end{bmatrix}$, $\{b\} = \begin{Bmatrix} 1 \\ 0 \\ 2 \end{Bmatrix}$, $\{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$

   (a) Find the cofactor matrix $[A]^C$, and verify that $[A][A]^{CT} = \det[A]\, I$.

   (b) Use the adjugate formulation to obtain $[A]^{-1}$.

   (c) Find the solution {x} to the equation [A]{x}={b}.

6U. For the matrices in problem 4, verify that $([A][B])^T = [B]^T[A]^T$.

6G. Analytically determine the exact intersection points of the unit sphere, the *y*-*z* plane, and the paraboloid $z = x^2 + y^2$. Taking $\{x^o, y^o, z^o\} = \{1, -1, 1\}$ as the starting guess for an iterative numerical solution of this problem, find the next guess using the Newton-Raphson method

7. Consider traditional polar coordinates: $x = r\cos\theta$ and $y = r\sin\theta$.

   Find the following derivatives: $\left(\frac{\partial x}{\partial r}\right)_\theta$, $\left(\frac{\partial x}{\partial \theta}\right)_r$, $\left(\frac{\partial x}{\partial r}\right)_y$, $\left(\frac{\partial x}{\partial \theta}\right)_y$, $\left(\frac{\partial x}{\partial y}\right)_\theta$, $\left(\frac{\partial x}{\partial y}\right)_r$,

   $\left(\frac{\partial \theta}{\partial r}\right)_x$, $\left(\frac{\partial \theta}{\partial x}\right)_r$, $\left(\frac{\partial \theta}{\partial r}\right)_y$, $\left(\frac{\partial \theta}{\partial x}\right)_y$, $\left(\frac{\partial \theta}{\partial y}\right)_x$, $\left(\frac{\partial \theta}{\partial y}\right)_r$. If $z = r - k\theta$, where k is a constant, find $\left(\frac{\partial y}{\partial r}\right)_z$.

ME 402/512 (CE 402) ASSIGNMENT 2.

Due Sept. 8, 1998

1. Do problem 2A1 in text.

2. Do problem 2A2 in text.

3. Do problem 2A5 in text.

4. Do problem 2A7 in text.

5. Do problem 2A8 in text.

6. Do problem 2A11 in text.

7G. Do problem 2A4 in text.

ME 402/512 (CE 402) ASSIGNMENT 3.

Due Sept. 15, 1998

1. Let $\underset{\sim}{\mathbf{x}} = x_i \underset{\sim}{\mathbf{e}}_i$ and $\underset{\sim}{\mathbf{v}} = -\underset{\sim}{\mathbf{e}}_1 + 4\underset{\sim}{\mathbf{e}}_2 - 2\underset{\sim}{\mathbf{e}}_3$.

   Find the most general form for $\underset{\sim}{\mathbf{x}}$ such that $\underset{\sim}{\mathbf{x}}$ is perpendicular to $\underset{\sim}{\mathbf{v}}$.

2. Given $\{\mathbf{u}\} = \begin{Bmatrix} 4 \\ 3 \\ 2 \end{Bmatrix}, \{\mathbf{v}\} = \begin{Bmatrix} -2 \\ 4 \\ 3 \end{Bmatrix}, [\mathbf{T}] = \begin{bmatrix} -1 & 2 & 3 \\ 3 & 2 & 1 \\ 4 & 3 & 4 \end{bmatrix}$.

   Find the components of the following scalars, vectors, or tensors:

   $\underset{\approx}{\mathbf{T}} \bullet \underset{\sim}{\mathbf{u}}$, $\underset{\sim}{\mathbf{u}} \bullet \underset{\approx}{\mathbf{T}}^T$, $\underset{\sim}{\mathbf{v}} \bullet \underset{\approx}{\mathbf{T}} \bullet \underset{\sim}{\mathbf{u}}$, $\underset{\sim}{\mathbf{u}} \otimes \underset{\sim}{\mathbf{v}}$.

3. Show that the relation $\underset{\sim}{\mathbf{v}} = (\underset{\sim}{\mathbf{v}} \bullet \underset{\sim}{\mathbf{n}})\underset{\sim}{\mathbf{n}} + \underset{\sim}{\mathbf{n}} \times \underset{\sim}{\mathbf{v}} \times \underset{\sim}{\mathbf{n}}$ holds for all unit vectors $\underset{\sim}{\mathbf{n}}$, and that this represents an alternative resolution of $\underset{\sim}{\mathbf{v}}$ into vectors parallel and perpendicular to $\underset{\sim}{\mathbf{n}}$.
   Hint: utilize an identity proved in homework #2 and our class discussion of the projection theorem.

4. Do problem 2B2 in the textbook. *Tips: recall from class that* $T_{ij} = \underset{\sim}{\mathbf{e}}_i \bullet \underset{\approx}{\mathbf{T}} \bullet \underset{\sim}{\mathbf{e}}_j$ *and* $\underset{\sim}{\mathbf{u}} \times \underset{\sim}{\mathbf{v}} = \varepsilon_{ijk} u_j v_k \underset{\sim}{\mathbf{e}}_i$. *In your work, use our notation* $\underset{\approx}{\mathbf{T}} \bullet \underset{\sim}{\mathbf{a}}$, *not the book's notation* $\mathbf{Ta}$. *For all handwritten work, indicate vectors with a single underline and tensors with a double underline (for word processor typesetting, use bold). The array associated with a vector* $\mathbf{v}$ *is denoted* $\{\mathbf{v}\}$. *The matrix associated with a tensor is denoted* $[\underset{\approx}{\mathbf{T}}]$.

5. Do problem 2B3 in the textbook.
   Tip: recall that linearity means $\underset{\approx}{\mathbf{T}} \bullet (\alpha_1 \underset{\sim}{\mathbf{u}}_1 + \alpha_2 \underset{\sim}{\mathbf{u}}_2) = \alpha_1 \underset{\approx}{\mathbf{T}} \bullet \underset{\sim}{\mathbf{u}}_1 + \alpha_2 \underset{\approx}{\mathbf{T}} \bullet \underset{\sim}{\mathbf{u}}_2$.

6. Do problem 2B4 in the textbook. *This problem should take you less than a minute to complete if you recall that the columns of* $[\underset{\approx}{\mathbf{T}}]$ *are* $\{\underset{\sim}{\mathbf{g}}_i\} = \{\underset{\approx}{\mathbf{T}} \bullet \underset{\sim}{\mathbf{e}}_i\}$.

7U. Do problem 2B5 in the textbook.

7G. Show that each term in problem #3 (above) is not linear with respect to $\mathbf{n}$, but each term *is* linear with respect to $\mathbf{v}$. Using *exclusively* direct notation, show that $(\underset{\sim}{\mathbf{v}} \bullet \underset{\sim}{\mathbf{n}})\underset{\sim}{\mathbf{n}} = \underset{\approx}{\mathbf{P}} \bullet \underset{\sim}{\mathbf{v}}$, where $\underset{\approx}{\mathbf{P}} \equiv \mathbf{nn}$. Using *exclusively* direct notation, show that $\underset{\sim}{\mathbf{n}} \times \underset{\sim}{\mathbf{v}} \times \underset{\sim}{\mathbf{n}} = \underset{\sim}{\mathbf{Q}} \bullet \underset{\sim}{\mathbf{v}}$, where $\underset{\sim}{\mathbf{Q}} = \underset{\approx}{\mathbf{I}} - \underset{\approx}{\mathbf{P}}$. *Hint: in order to demonstrate this result without having to resort to indicial notation, you will need to make use of the direct notation identity given in the book's problem 2A8 on page 69. Keep in mind:* $\underset{\sim}{\mathbf{n}} \bullet \underset{\sim}{\mathbf{n}} = 1$ *and* $\underset{\sim}{\mathbf{v}} = \underset{\approx}{\mathbf{I}} \bullet \underset{\sim}{\mathbf{v}}$.

ME 402/512 (CE 402) ASSIGNMENT 4. Due Sept. 22, 1998

1. Find the *direct notation* expressions for the tensors $\underset{\approx}{\mathbf{T}}$, $\underset{\approx}{\mathbf{S}}$, and $\underset{\approx}{\mathbf{Y}}$ such that

   (i) $\underset{\sim}{\mathbf{a}} \times (\underset{\sim}{\mathbf{b}} \times \underset{\sim}{\mathbf{c}}) = \underset{\approx}{\mathbf{T}} \bullet \underset{\sim}{\mathbf{a}}$

   (ii) $\underset{\sim}{\mathbf{a}} \times (\underset{\sim}{\mathbf{b}} \times \underset{\sim}{\mathbf{c}}) = \underset{\approx}{\mathbf{S}} \bullet \underset{\sim}{\mathbf{b}}$

   (iii) $\underset{\sim}{\mathbf{a}} \times (\underset{\sim}{\mathbf{b}} \times \underset{\sim}{\mathbf{c}}) = \underset{\approx}{\mathbf{Y}} \bullet \underset{\sim}{\mathbf{c}}$

   Hint: for all cases, simply rearrange the identity of problem 2A8 in the book. You should not have to resort to indicial or matrix notation to do this problem.

2. Determine what the question marks stand for in the following identity:

   $$(\underset{\sim}{\mathbf{a}} \times \underset{\sim}{\mathbf{b}}) \times (\underset{\sim}{\mathbf{c}} \times \underset{\sim}{\mathbf{d}}) = [\mathbf{?}, \mathbf{?}, \mathbf{?}]\mathbf{?} - [\mathbf{?}, \mathbf{?}, \mathbf{?}]\mathbf{?}$$

   Hint: recall the triple scalar product is defined $[\underset{\sim}{\mathbf{u}}, \underset{\sim}{\mathbf{v}}, \underset{\sim}{\mathbf{w}}] \equiv (\underset{\sim}{\mathbf{u}} \times \underset{\sim}{\mathbf{v}}) \bullet \underset{\sim}{\mathbf{w}} = \varepsilon_{ijk} u_i v_j w_k$.

3. Using *only* direct notation, show that $(\underset{\sim}{\mathbf{u}}\underset{\sim}{\mathbf{v}})^k = (\underset{\sim}{\mathbf{u}} \bullet \underset{\sim}{\mathbf{v}})^{k-1}\underset{\sim}{\mathbf{u}}\underset{\sim}{\mathbf{v}}$, where k is a positive integer exponent. *[Bonus points for anyone who proves this using <u>proper</u> proof by induction.]*

4. *(For the following problem, leave your answers in terms of $\sqrt{2}$.)* Suppose that the angles between respective base vectors in two systems are as listed in the table.

	$\underset{\sim}{\mathbf{e}}_1^*$	$\underset{\sim}{\mathbf{e}}_2^*$	$\underset{\sim}{\mathbf{e}}_3^*$
$\underset{\sim}{\mathbf{e}}_1$	90°	45°	135°
$\underset{\sim}{\mathbf{e}}_2$	45°	60°	60°
$\underset{\sim}{\mathbf{e}}_3$	45°	120°	120°

   (a) Express each $\underset{\sim}{\mathbf{e}}_i^*$ as a linear combination of the $\underset{\sim}{\mathbf{e}}_j$ vectors; Express each $\underset{\sim}{\mathbf{e}}_j$ as a linear combination of the $\underset{\sim}{\mathbf{e}}_i^*$ vectors.

   (b) Obtain the components of the transformation matrix and verify that the transformation preserves handedness.

   (c) Find $[\underset{\approx}{\mathbf{T}}]_{\mathbf{e}}$ if $[\underset{\approx}{\mathbf{T}}]_{\mathbf{e}^*} = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 6 & 0 \\ 0 & 0 & 4 \end{bmatrix}_{\mathbf{e}^*}$

BONUS QUESTIONS:

5U. Use the ε-delta identity to determine what the question marks stand for in the following identity: $(\underset{\sim}{\mathbf{a}} \times \underset{\sim}{\mathbf{b}}) \times \underset{\sim}{\mathbf{c}} = (\underset{\sim}{\mathbf{?}} \bullet \underset{\sim}{\mathbf{?}})\underset{\sim}{\mathbf{?}} - (\underset{\sim}{\mathbf{?}} \bullet \underset{\sim}{\mathbf{?}})\underset{\sim}{\mathbf{?}}$. Comparing this formula with the result in problem 2A8 in the book, show that $\underset{\sim}{\mathbf{a}} \times \underset{\sim}{\mathbf{b}} \times \underset{\sim}{\mathbf{c}}$ is poor notation because it is ambiguous without parentheses. Explain why $\underset{\sim}{\mathbf{n}} \times \underset{\sim}{\mathbf{u}} \times \underset{\sim}{\mathbf{n}}$ is acceptable because it is *not* ambiguous.

5G. *(i)* Prove that $\underset{\sim}{\mathbf{a}} \otimes (\underset{\sim}{\mathbf{b}} + \underset{\sim}{\mathbf{c}}) = \underset{\sim}{\mathbf{a}} \otimes \underset{\sim}{\mathbf{b}} + \underset{\sim}{\mathbf{a}} \otimes \underset{\sim}{\mathbf{c}}$.

(ii) Prove that any sum of four dyads can always be reduced to the sum of three or fewer dyads. *Hint: for our three dimensional space, any set of four vectors is always linearly dependent and therefore one of those four vectors can always expressed as a linear combination of the other three vectors; use this fact in combination with the result from part (i).*

ME 402/512 (CE 402) ASSIGNMENT 5.

Due Sept. 29, 1998

In preparation for the exam, solve any six of the following problems from the textbook:

2B6
2B7
2B8
2B9
2B10
2B11
2B14
2B15
2B17
2B19
2B26
2B27
2B28
2B30
2B34

ME 402/512 (CE 402) ASSIGNMENT 6.

Due Sept. 29, 1998

Solve the following problems from the textbook:

2B20
2B22
2B23
2B35

ME 402/512 (CE 402) ASSIGNMENT 7. Due Oct. 13, 1998

1. Let $[\underset{\approx}{\mathbf{T}}] = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 0 \\ 4 & 5 & 0 \end{bmatrix}$. (a) Compute the characteristic equation by directly expanding $\det[\underset{\approx}{\mathbf{T}} - \lambda\underset{\approx}{\mathbf{I}}] = 0$. (b) Compute the characteristic invariants $\{I_1, I_2, I_3\}$ for the tensor $\underset{\approx}{\mathbf{T}}$.

   (c) Verify that the formulas $I_2 = tr\underset{\approx}{\mathbf{T}}^C$ and $I_2 = \frac{1}{2}[(\mathrm{tr}\underset{\approx}{\mathbf{T}})^2 - \mathrm{tr}(\underset{\approx}{\mathbf{T}}^2)]$ give the same result. (d) Verify that the characteristic equation is indeed $\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$.

2. Using the tensor $\underset{\approx}{\mathbf{T}}$ from question #1, evaluate (a) $(\underset{\approx}{\mathbf{T}} \bullet \underset{\sim}{\mathbf{e}}_2) \times \underset{\sim}{\mathbf{e}}_1$, (b) $\underset{\approx}{\mathbf{T}}\mathbf{:}(\underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_1)$, (c) $(\underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_2)\mathbf{:}\underset{\approx}{\mathbf{T}}$, (d) $\underset{\approx}{\mathbf{T}}\mathbf{:}(\underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_2)$, (e) $\underset{\approx}{\mathbf{T}}\mathbf{:}\underset{\approx}{\mathbf{T}}$, (f) $\underset{\approx}{\mathbf{T}}\mathbf{:}\underset{\approx}{\mathbf{T}}^T$.

3. Let **a** be a vector that is fixed in a rigid body; i.e., $\underset{\sim}{\mathbf{a}} = \underset{\approx}{\mathbf{Q}} \bullet \underset{\sim}{\mathbf{a}}_o$, where $\underset{\sim}{\mathbf{a}}_o$ is the unrotated (initial) orientation of the vector. Suppose the rotation $\underset{\approx}{\mathbf{Q}}$ varies with time. Noting that the *initial* orientation $\underset{\sim}{\mathbf{a}}_o$ is forever constant, prove that $\dot{\mathbf{a}} = \underset{\approx}{\Omega} \bullet \underset{\sim}{\mathbf{a}} = \omega \times \mathbf{a}$, where $\underset{\approx}{\Omega} = \dot{\underset{\approx}{\mathbf{Q}}} \bullet \underset{\approx}{\mathbf{Q}}^T$, and the angular velocity ω is the dual vector of $\underset{\approx}{\Omega}$. (Hint: see lecture).

4. [Modified from **Dynamics of Polymeric Liquids**, R.B. Bird *et. al*]
   Consider a rigid structure (such as the frame of a carnival ride) that may be approximated as point particles joined by massless rods. The particles are numbered 1, 2, 3, ... N, and the particle masses are m_p (p = 1,2, ..., N). The locations of the particles with respect to the center of mass are $\underset{\approx}{\mathbf{R}}_p$. The entire structure rotates on an axis passing through the center of mass with an angular velocity $\underset{\sim}{\mathbf{w}}$. Using the result from the previous problem, show that the angular momentum with respect to the center of mass is

$$\underset{\sim}{\mathbf{H}} = \sum_{p=1}^{N} m_p[\underset{\approx}{\mathbf{R}}_p \times (\underset{\sim}{\mathbf{w}} \times \underset{\approx}{\mathbf{R}}_p)] \tag{1}$$

   Then show that this expression may be rewritten as

$$\underset{\sim}{\mathbf{H}} = \underset{\approx}{\Phi} \bullet \underset{\sim}{\mathbf{w}} \tag{2a}$$

   where $\underset{\approx}{\Phi} = \sum_{p=1}^{N} m_p[(\underset{\sim}{\mathbf{R}}_p \bullet \underset{\sim}{\mathbf{R}}_p)\underset{\approx}{\mathbf{I}} - \underset{\sim}{\mathbf{R}}_p\underset{\sim}{\mathbf{R}}_p]$ is the moment of inertia tensor. (2b)

   Describe why Eqs. (2) might be more convenient than Eq. (1). For N=1, m_1=1, and $\underset{\sim}{\mathbf{R}}_1 = x\underset{\sim}{\mathbf{e}}_1 + y\underset{\sim}{\mathbf{e}}_2 + z\underset{\sim}{\mathbf{e}}_3$, find the matrix of $\underset{\approx}{\Phi}$. Defining the kinetic energy of rotation as $K = \sum_{p=1}^{N} \frac{1}{2}m_p(\dot{\underset{\sim}{\mathbf{R}}}_p \bullet \dot{\underset{\sim}{\mathbf{R}}}_p)$, prove that $K = \frac{1}{2}\underset{\approx}{\Phi}\mathbf{:}\underset{\sim}{\mathbf{w}}\underset{\sim}{\mathbf{w}}$.

ME 402/512 (CE 402) ASSIGNMENT 8. .

Due Oct. 27, 1998

1. For a continuous *rigid* body, the inertia tensor is defined $\underset{\approx}{\Phi} = \iiint[(\underset{\sim}{\mathbf{X}} \bullet \underset{\sim}{\mathbf{X}})\underset{\approx}{\mathbf{I}} - \underset{\sim}{\mathbf{X}}\underset{\sim}{\mathbf{X}}]dm$, where $\underset{\sim}{\mathbf{X}}$ is the position vector from the center of mass to the mass element dm. Suppose the body is subjected to a rigid rotation $\underset{\approx}{\mathbf{Q}}$ such that each $\mathbf{X}$ becomes $\underset{\sim}{\mathbf{x}} = \underset{\approx}{\mathbf{Q}} \bullet \underset{\sim}{\mathbf{X}}$.

   (a) Noting that $\underset{\approx}{\mathbf{Q}}$ is independent of position, prove the new inertia tensor (defined by $\underset{\approx}{\phi} = \iiint[(\underset{\sim}{\mathbf{x}} \bullet \underset{\sim}{\mathbf{x}})\underset{\approx}{\mathbf{I}} - \underset{\sim}{\mathbf{x}}\underset{\sim}{\mathbf{x}}]dm$) is related to the original inertia tensor by $\underset{\approx}{\phi} = \underset{\approx}{\mathbf{Q}} \bullet \underset{\approx}{\Phi} \bullet \underset{\approx}{\mathbf{Q}}^T$.

   (b) Recalling $\dot{\underset{\approx}{\mathbf{Q}}} = \underset{\approx}{\Omega} \bullet \underset{\approx}{\mathbf{Q}}$, where $\underset{\approx}{\Omega}$ is the skew-symmetric angular velocity tensor, prove that $\dot{\underset{\approx}{\phi}} = \underset{\approx}{\Omega} \bullet \underset{\approx}{\phi} - \underset{\approx}{\phi} \bullet \underset{\approx}{\Omega}$. (This type of expression plays a key role later in the course.)

   (c) Letting $\underset{\sim}{\mathbf{H}} = \underset{\approx}{\phi} \bullet \underset{\sim}{\omega}$, where $\underset{\sim}{\omega}$ is the dual vector of $\underset{\approx}{\Omega}$, prove that $\dot{\underset{\sim}{\mathbf{H}}} = \underset{\approx}{\Omega} \bullet \underset{\sim}{\mathbf{H}} + \underset{\approx}{\phi} \bullet \dot{\underset{\sim}{\omega}}$.

2. Let $\vec{\nabla} = \underset{\sim}{\mathbf{e}}_k \dfrac{\partial}{\partial x_k}$, $\underset{\sim}{\mathbf{x}} = x_i \underset{\sim}{\mathbf{e}}_i$, and $r = \sqrt{\underset{\sim}{\mathbf{x}} \bullet \underset{\sim}{\mathbf{x}}}$.

   Prove that (a) $\vec{\nabla}\underset{\sim}{\mathbf{x}} = \underset{\approx}{\mathbf{I}}$ (b) $\vec{\nabla} \bullet \underset{\sim}{\mathbf{x}} = 3$ (c) $\vec{\nabla} \times \underset{\sim}{\mathbf{x}} = \underset{\sim}{\mathbf{0}}$ (d) $\vec{\nabla} r = \dfrac{\underset{\sim}{\mathbf{x}}}{r}$ (e) $\vec{\nabla}\left(\dfrac{1}{r}\right) = -\dfrac{\underset{\sim}{\mathbf{x}}}{r^3}$

   (f) ***Grads only***: $\vec{\nabla}(\vec{\nabla} r^n) = n r^{n-2}[(n-2)r^{-2}\underset{\sim}{\mathbf{x}}\underset{\sim}{\mathbf{x}} + \underset{\approx}{\mathbf{I}}]$

3. Prove any three of the following identities; ***Grads prove five***.

$\vec{\nabla}(rs) = r\vec{\nabla}s + s\vec{\nabla}r$	$\vec{\nabla} \times (s\underset{\sim}{\mathbf{v}}) = (\vec{\nabla}s) \times \underset{\sim}{\mathbf{v}} + s(\vec{\nabla} \times \underset{\sim}{\mathbf{v}})$
$\vec{\nabla} \bullet (s\underset{\sim}{\mathbf{v}}) = (\vec{\nabla}s) \bullet \underset{\sim}{\mathbf{v}} + s(\vec{\nabla} \bullet \underset{\sim}{\mathbf{v}})$	$\underset{\sim}{\mathbf{v}} \bullet \vec{\nabla}\underset{\sim}{\mathbf{v}} = \frac{1}{2}\vec{\nabla}(\underset{\sim}{\mathbf{v}} \bullet \underset{\sim}{\mathbf{v}}) - \underset{\sim}{\mathbf{v}} \times (\vec{\nabla} \times \underset{\sim}{\mathbf{v}})$
$\vec{\nabla} \bullet (\underset{\sim}{\mathbf{v}}\underset{\sim}{\mathbf{w}}) = (\vec{\nabla} \bullet \underset{\sim}{\mathbf{v}})\underset{\sim}{\mathbf{w}} + \underset{\sim}{\mathbf{v}} \bullet (\vec{\nabla}\underset{\sim}{\mathbf{w}})$	$s\underset{\approx}{\mathbf{I}}\mathbf{:}(\vec{\nabla}\underset{\sim}{\mathbf{v}}) = s(\vec{\nabla} \bullet \underset{\sim}{\mathbf{v}})$
$\vec{\nabla} \bullet (s\underset{\approx}{\mathbf{I}}) = \vec{\nabla}s$	$\vec{\nabla} \bullet (s\underset{\approx}{\mathbf{T}}) = (\vec{\nabla}s) \bullet \underset{\approx}{\mathbf{T}} + s(\vec{\nabla} \bullet \underset{\approx}{\mathbf{T}})$
$\nabla^2(\vec{\nabla} \bullet \underset{\sim}{\mathbf{v}}) = \vec{\nabla} \bullet \nabla^2\underset{\sim}{\mathbf{v}}$	$\vec{\nabla}(\underset{\sim}{\mathbf{v}} \bullet \underset{\sim}{\mathbf{w}}) = (\vec{\nabla}\underset{\sim}{\mathbf{v}}) \bullet \underset{\sim}{\mathbf{w}} + (\vec{\nabla}\underset{\sim}{\mathbf{w}}) \bullet \underset{\sim}{\mathbf{v}}$
$\nabla^2\underset{\sim}{\mathbf{v}} = \vec{\nabla}(\vec{\nabla} \bullet \underset{\sim}{\mathbf{v}}) - \vec{\nabla} \times (\vec{\nabla} \times \underset{\sim}{\mathbf{v}})$	$\vec{\nabla} \bullet (\underset{\sim}{\mathbf{v}} \times \underset{\sim}{\mathbf{w}}) = \underset{\sim}{\mathbf{w}} \bullet (\vec{\nabla} \times \underset{\sim}{\mathbf{v}}) - \underset{\sim}{\mathbf{v}} \bullet (\vec{\nabla} \times \underset{\sim}{\mathbf{w}})$

4. The second law of thermodynamics often implies that $\underset{\sim}{\mathbf{q}} \bullet \vec{\nabla}T < 0$, where $\underset{\sim}{\mathbf{q}}$ is the heat flux vector and T is temperature. Interpret this inequality physically. Prove that $\underset{\sim}{\mathbf{q}} = -\underset{\approx}{\mathbf{K}} \bullet \vec{\nabla}T$ is a generalization of Fourier's law ($\underset{\sim}{\mathbf{q}} = -\kappa\vec{\nabla}T$) and prove that the second law demands that the conductivity tensor $\underset{\approx}{\mathbf{K}}$ must be positive definite.

5. Construct a counter-example matrix [T] demonstrating that having all positive invariants does not necessarily make a tensor positive definite. *Hint: use a 2x2 matrix.*

6. Do problem 2C2 in the text *except* use the point (0,1,-1) in lieu of the origin (0,0,0).

ME 402/512 (CE 402) ASSIGNMENT 9. Due Nov. 3, 1998

1. A force of magnitude F acts in a direction radially away from the origin at a point $\left(\frac{a}{3}, \frac{2b}{3}, \frac{2c}{3}\right)$ on the surface of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b} + \frac{z^2}{c} = 1$. Determine the component of the force in the direction of the normal to the surface, taking a=1, b=2, c=–1, and F=4.

2. (a) If $\underset{\sim}{\mathbf{x}}$ is the position vector, use the divergence theorem to express $\int_{\partial B} \underset{\sim}{\mathbf{x}} \bullet \underset{\sim}{\mathbf{n}}\, dS$ in terms of the volume of the region B. (b) Actually perform the surface integral for a unit cube with one corner at the origin of an orthonormal laboratory coordinate system.

3. Verify that Stokes theorem holds for the plane area bounded by the square with corners (0,0), (β,0), (β,β), (0,β) in the x_1-x_2 plane if $v_1 = Ax_2$, $v_2 = Ax_3 + Bx_2$, $v_3 = Cx_1$, where A,B, and C are constants.

4. For the same vector field as in problem #4 above, verify that the divergence theorem holds for a cube with one vertex at the origin and the opposite vertex at the point (β,β,β).

5. Use indicial notation to prove that $\varphi\overleftarrow{\nabla} \times \overleftarrow{\nabla} = \underset{\sim}{\mathbf{0}}$ and that $(\underset{\sim}{\mathbf{v}} \times \overleftarrow{\nabla}) \bullet \overleftarrow{\nabla} = 0$.

6U. (a) A tensor $\mathbf{D}$ is called "negative definite" if and only if $-\mathbf{D}$ is positive definite. Explain why $\mathbf{D}$ is therefore negative definite only if its invariants satisfy $I_1 < 0$, $I_2 > 0$, and $I_3 < 0$.

(b) Find all the invariants of the following tensor $\mathbf{B}$. Is $\mathbf{B}$ positive / negative definite?

$$[\mathbf{B}] = \begin{bmatrix} 8 & 3 & 4 \\ 2 & 0 & -5 \\ -1 & 6 & 1 \end{bmatrix} \tag{1}$$

6G. Carefully read the first three pages of the document at

`http://me.unm.edu/~rmbrann/curvilinear.pdf`

Then solve Study Question (2.1) on page 7 of that document.

7. BONUS PROBLEM: Let $\mathbf{r}$ be a vector. Find the components of a fourth-order tensor $\underset{\approx}{\mathbb{U}}$ such that $\underset{\sim}{\mathbf{r}}\underset{\sim}{\mathbf{r}} - (\underset{\sim}{\mathbf{r}} \bullet \underset{\sim}{\mathbf{r}})\underset{\approx}{\mathbf{I}} = \underset{\sim}{\mathbf{r}} \bullet \underset{\approx}{\mathbb{U}} \bullet \underset{\sim}{\mathbf{r}}$, *Hint: write this expression in indicial notation and then differentiate both sides with respect to* r_α *and again with respect to* r_β, *keeping in mind:* $\partial r_i / \partial r_j = \delta_{ij}$. *You may assume without loss that* $\underset{\approx}{\mathbb{U}}$ *is symmetric in its first and last indices.*

ME 402/512 (CE 402) ASSIGNMENT 10.

Due Nov. 10, 1998

1. Letting $k = \frac{1}{2}$, do problem 3.3 in the book and add the following additional tasks:

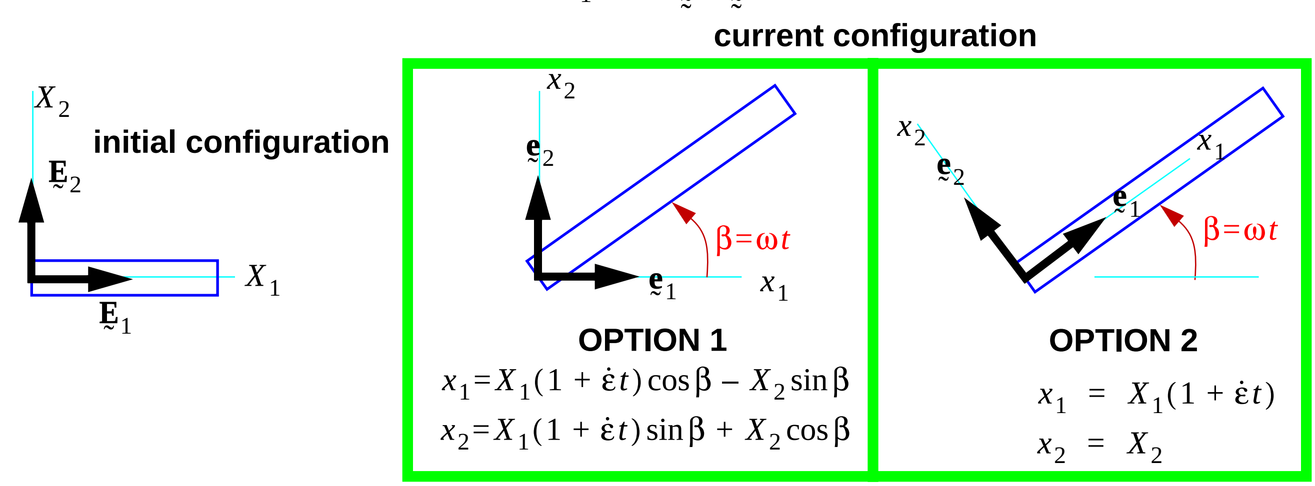
   (d) Find the matrix of the deformation gradient tensor $\underset{\approx}{\mathbf{F}}$.

   (e) Is the deformation homogeneous? Why or why not?

   (f) For $t=2$, evaluate and the deformation gradient at the material particles originally at (0,0,0) and (1,1,1) and sketch those two deformation gradients as though they were homogeneous. Are your sketches consistent with the deformed square from part (a)?

2. Solve problem 3.16 in the book. (An isotherm is a surface of constant temperature.)

3. Consider the deformation sketched below in which a bar is undergoing uniform uniaxial extension while simultaneously translating and rotating according to the given equations (in which ω and $\dot{\varepsilon}$ are constants). The translation is such that the vector connecting the initial origin to the current origin is $kt\underset{\sim}{\mathbf{E}}_1$, where k is a constant. The figure shows two possible ways that you might elect to set up a spatial basis. For each of the two options,
   (a) Determine the components of the displacement $\underset{\sim}{\mathbf{u}}$ with respect to the spatial $\underset{\sim}{\mathbf{e}}_i$ basis.
   (b) Find the components of the velocity with respect to the spatial $\underset{\sim}{\mathbf{e}}_i$ basis.
   (c) Find the mixed two-point ($\underset{\sim}{\mathbf{e}}_i\underset{\sim}{\mathbf{E}}_j$) components of the deformation gradient $\underset{\approx}{\mathbf{F}}$.
   (d) Show that your answers to the above questions give sensible answers for various limiting cases (When $\dot{\varepsilon}=0$, you expect the deformation to be rotational, so you expect $\underset{\approx}{\mathbf{F}}$ to reduce to an orthogonal rotation tensor. When both $\omega=0$ and $\dot{\varepsilon}=0$, the deformation is pure rigid translation, so you expect $\underset{\sim}{\mathbf{v}}=k\underset{\sim}{\mathbf{E}}_1$ and $\underset{\approx}{\mathbf{F}}=\underset{\approx}{\mathbf{I}}$.)



4U. Do problem 3.14 in the book.

4G. Solve Study Questions (2.2), (2.3), and (2.4) in `http://me.unm.edu/~rmbrann/curvilinear.pdf`.

ME 402/512 (CE 402) ASSIGNMENT 11. Due Nov. 17, 1998

1. Prove that $\dfrac{D(\underset{\approx}{\mathbf{F}}^{-1})}{Dt} = -\underset{\approx}{\mathbf{F}}^{-1} \bullet \underset{\approx}{\mathbf{L}}$.

2. $[\underset{\approx}{\mathbf{F}}] = \begin{bmatrix} 1 & -1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Sketch this deformation gradient tensor, including the deformation of an inscribed circle. Find the polar decomposition tensors $\underset{\approx}{\mathbf{R}}$ and $\underset{\approx}{\mathbf{V}}$. Do it two ways: (a) using the same steps as in the book's example 3.23.1 on page 130. (b) using the shortcut given in class.

3. Do problem 3.37 in the book, noting that we say "vorticity" where the book says "spin."

4. Do problem 3.34 in the book.

5U. Do problem 3.65 in the book.

5G. Solve Questions (2.5) and (3.1) in **`http://me.unm.edu/~rmbrann/curvilinear.pdf`**. For Question (2.5), violations of the summation convention will be graded severely.

ME 402/512 (CE 402) ASSIGNMENT 12.

Due Nov. 24, 1998

In preparation for the exam, solve any six of the following problems from the textbook:

3.1,
3.2, 3.4, 3.5, 3.6, 3.8, 3.9, 3.10, 3.11, 3.12 <--- pick no more than one from this line
3.17
3.39
3.40
3.41
3.61, 3.62, 3.63, 3.64 <--- pick no more than one from this line
3.66

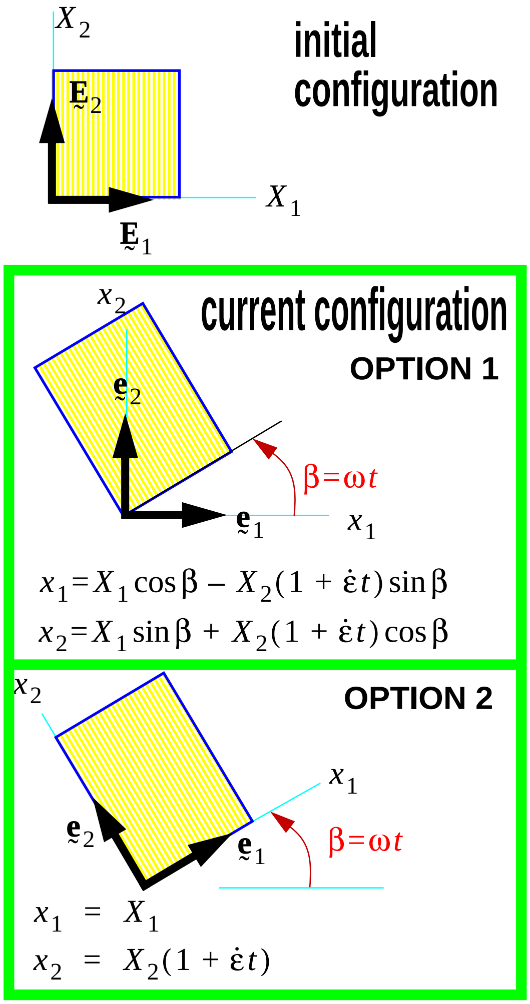
ME 402/512 (CE 402) ASSIGNMENT 13.

Due Dec 1, 1998

1. Let $\underset{\sim}{\mathbf{x}}$ denote the position vector. (a) Recalling that $\underset{\sim}{\mathbf{x}}\overleftarrow{\nabla} = \underset{\approx}{\mathbf{I}}$, use indicial notation to prove that $(\underset{\sim}{\mathbf{x}} \times \underset{\approx}{\sigma}) \bullet \overleftarrow{\nabla} = \underset{\sim}{\mathbf{x}} \times (\underset{\approx}{\sigma} \bullet \overleftarrow{\nabla}) - \underset{\approx}{\varepsilon}{:}\underset{\approx}{\sigma}$. (b) Prove that $\underset{\approx}{\varepsilon}{:}\underset{\approx}{\sigma} = \underset{\sim}{0}$ if and only if $\mathrm{skw}(\underset{\approx}{\sigma}) = \underset{\approx}{0}$.

2. Let $\underset{\sim}{\mathbf{N}}\,dA_o$ and $\underset{\sim}{\mathbf{n}}\,dA$ denote an area elements in the reference and spatial configurations, respectively. Suppose that two tensors, $\underset{\approx}{\sigma}$ and $\underset{\approx}{\hat{\tau}}$, are desired to satisfy a relationship $\underset{\approx}{\sigma} \bullet \underset{\sim}{\mathbf{n}}\,dA = \underset{\approx}{\hat{\tau}} \bullet \underset{\sim}{\mathbf{N}}\,dA_o$. Use Nanson's relation to express $\underset{\approx}{\hat{\tau}}$ as a function of $\underset{\approx}{\sigma}$ and $\underset{\approx}{\mathbf{F}}$.

3. Consider the deformation sketched at right in which a bar is undergoing uniform uniaxial extension while simultaneously translating and rotating according to the given equations (in which ω and $\dot{\varepsilon}$ are constants). The translation is such that the vector connecting the initial origin to the current origin is $kt\underset{\sim}{\mathbf{E}}_2$, where k is a constant. The figure shows two possible ways that you might elect to set up a spatial basis. For each of the two options,
(a) Determine the components of the displacement $\underset{\sim}{\mathbf{u}}$ with respect to the spatial $\underset{\sim}{\mathbf{e}}_i$ basis.
(b) Find the components of the velocity with respect to the spatial $\underset{\sim}{\mathbf{e}}_i$ basis.
(c) Find the mixed two-point ($\underset{\sim}{\mathbf{e}}_i\underset{\sim}{\mathbf{E}}_j$) components of the deformation gradient $\underset{\approx}{\mathbf{F}}$.
(d) Find the rate of deformation and vorticity.
(e) Find the Lagrange and Euler strain tensors.
(f) Show that your answers to the above questions give sensible answers for various limiting cases for which you know what the answer should be.



4U. Show that $\underset{\approx}{\mathbf{I}} - \underset{\approx}{\mathbf{V}}^{-1}$ is a strain measure of the Seth-Hill form with $k=-1$. In other words, show that it generalizes the uniaxial strain measure $(L-L_o)/L$.

4G. Solve problems (3.2) and (3.4) in **`http://me.unm.edu/~rmbrann/curvilinear.pdf`**.

It is not necessary to show your work for Study Question (3.2). Your grade is based solely on whether you get the entries right or wrong. Problem (3.3) has been skipped because both (3.3) and (3.4) deal with situation of different tensors having identical components with respect to different bases. Understanding these problems is essential for you to communicate effectively with researchers who prefer to work exclusively in general curvilinear coordinates.

ME 402/512 (CE 402) ASSIGNMENT 14.

Due Dec. 8, 1998

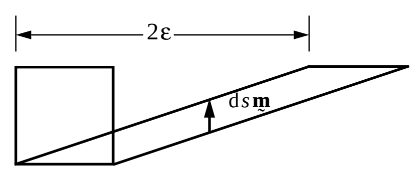
1. A function $f(\underset{\sim}{\mathbf{x}})$ is called homogeneous of degree y if

$$f(\alpha\underset{\sim}{\mathbf{x}}) = \alpha^y f(\underset{\sim}{\mathbf{x}}) \qquad \forall\alpha > 0 \tag{2}$$

   Let $\mathbf{f}'(\underset{\sim}{\mathbf{x}})$ denote the derivative of the function $f(\underset{\sim}{\mathbf{x}})$ with respect to $\underset{\sim}{\mathbf{x}}$.
   (a) Prove that $\mathbf{f}'(\underset{\sim}{\mathbf{x}})$ is homogeneous of degree y-1.
   (b) Differentiate both sides of (2) with respect to α and use (a) to prove that $\mathbf{f}'(\underset{\sim}{\mathbf{x}}) \bullet \underset{\sim}{\mathbf{x}} = yf(\underset{\sim}{\mathbf{x}})$.

2. Prove that the rate of Lagrange strain is $\dot{\underset{\approx}{\mathbf{E}}} = \underset{\approx}{\mathbf{F}}^T \bullet \underset{\approx}{\mathbf{D}} \bullet \underset{\approx}{\mathbf{F}}$.

3. (a) Prove that the Euler strain $\underset{\approx}{\mathbf{e}}$ (denoted $\underset{\approx}{\mathbf{e}}*$ in the book) is related to the spatial displacement gradient $\underset{\approx}{\mathbf{h}}$ according to the formula $\underset{\approx}{\mathbf{e}} = \frac{1}{2}(\underset{\approx}{\mathbf{h}} + \underset{\approx}{\mathbf{h}}^T - \underset{\approx}{\mathbf{h}}^T \bullet \underset{\approx}{\mathbf{h}})$.
   (b) Consider a material element $dS\underset{\sim}{\mathbf{M}}$ in the reference configuration that deforms to $ds\underset{\sim}{\mathbf{m}} = \underset{\approx}{\mathbf{F}} \bullet dS\underset{\sim}{\mathbf{M}}$ in the current configuration, prove that $\underset{\sim}{\mathbf{m}} \bullet \underset{\approx}{\mathbf{e}} \bullet \underset{\sim}{\mathbf{m}} = \frac{1}{2}\left(1 - \frac{(dS)^2}{(ds)^2}\right)$.
   (c) Find the mistakes(s) on page 142 in the book.
   (d) Consider simple shear (with $\tan\gamma = 2\varepsilon$). Find the matrix for $\underset{\approx}{\mathbf{e}}$ in the limits that $\varepsilon \ll 1$ and $\varepsilon \gg 1$, and compare with the similar results derived in class for Lagrange strain.



   (e) The figure at right shows simple shear of a unit cube at the instant when ε = what?. If $ds\underset{\sim}{\mathbf{m}}$ is a material element that is instantaneously vertical as shown, *accurately* draw the material element $dS\underset{\sim}{\mathbf{M}}$. Use a ruler (yuk!) to actually measure the lengths ds and dS and verify the result from (b).

4. Construct a basis $\{\underset{\sim}{\mathbf{E}}_1, \underset{\sim}{\mathbf{E}}_2, \underset{\sim}{\mathbf{E}}_3\}$ such that $\underset{\sim}{\mathbf{E}}_3$ makes equal angles with the three principal directions of stress (choose $\underset{\sim}{\mathbf{E}}_1$ and $\underset{\sim}{\mathbf{E}}_2$ based on convenience). The plane spanned by $\underset{\sim}{\mathbf{E}}_1$ and $\underset{\sim}{\mathbf{E}}_2$ is called the octahedral plane. (a) Find an expression for the traction vector $\underset{\sim}{\mathbf{t}}$ on this plane in terms of principal values of $\underset{\approx}{\sigma}$. (b) Show that the octahedral normal stress, $|t_3| = \frac{1}{3}\mathrm{tr}\underset{\approx}{\sigma}$. (c) Express the octahedral shear stress, defined $\tau^{\mathrm{oct}} \equiv \sqrt{(t_1)^2 + (t_2)^2}$ as a function of the principal stresses.

5. Do problem 7.7 in the book. For this problem, the “cylindrical control volume” is Eulerian, not Lagrangian, and it therefore can increase in mass via flux through the boundary.

6. Do problem 7.8 in the book.

7G. Solve problems (3.5) and (4.1) in **`http://me.unm.edu/~rmbrann/curvilinear.pdf`**.

ME 402/512 (CE 402) BONUS . Due Dec 18 (or Dec. 20 if faxed to 505-844-5095), 1998

1. (a) Prove that $\underset{\sim}{\mathbf{v}}\overleftarrow{\nabla}_o = \dot{\underset{\approx}{\mathbf{F}}}$

   (b)Recall that mechanical power is defined $\mathbb{P}_M = \int_{\partial B} (\underset{\approx}{\sigma} \bullet \underset{\sim}{\mathbf{n}}) \bullet \underset{\sim}{\mathbf{v}} dA + \int_B (\underset{\sim}{\mathbf{b}} \bullet \underset{\sim}{\mathbf{v}}) \rho dV$. By a change of variables, show that $\mathbb{P}_M = \int_{\partial B_o} (\hat{\underset{\approx}{\tau}} \bullet \underset{\sim}{\mathbf{N}}) \bullet \underset{\sim}{\mathbf{v}} dA_o + \int_{B_o} (\underset{\sim}{\mathbf{b}} \bullet \underset{\sim}{\mathbf{v}})\ \rho_o dV_o$. Following the methods used in class for the spatial expression, apply the divergence theorem, and use the reference form of Cauchy's first law to prove that $\mathbb{P}_M = \dot{K} + \int_B P_s\ \rho_o dV_o$, where K is the kinetic energy and the stress power is $P_s = \frac{1}{\rho_o} \hat{\underset{\approx}{\tau}} : \dot{\underset{\approx}{\mathbf{F}}}$. Incidentally, because of this relationship, the PK1 stress is said to be "work conjugate" to the deformation gradient $\underset{\approx}{\mathbf{F}}$.

2. Consider a closed line integral $Y = \oint_C \hat{\underset{\sim}{\mathbf{t}}} \bullet \underset{\sim}{\mathbf{v}} ds$ where $\hat{\underset{\sim}{\mathbf{t}}} ds$ represents the increment around a closed curve C. If the vector $\underset{\sim}{\mathbf{v}}$ is the velocity, then Y is called the "circulation." Suppose, for this problem, that the curve C moves with the material. Then $\hat{\underset{\sim}{\mathbf{t}}} ds$ is a spatial material element and can be written $d\underset{\sim}{\mathbf{x}}$. For this case, perform an appropriate change of variables to convert the spatial integral to a reference integral (permitting you to bring the rate inside the integral — why?) and then convert back to spatial form to demonstrate that $\frac{D}{Dt}\oint_C \underset{\sim}{\mathbf{v}} \bullet d\mathbf{x} = \oint_C (\underset{\sim}{\mathbf{a}} + \underset{\sim}{\mathbf{v}} \bullet \underset{\approx}{\mathbf{L}}) \bullet d\mathbf{x}$, where $\underset{\sim}{\mathbf{a}} = \dot{\underset{\sim}{\mathbf{v}}}$ and $\underset{\approx}{\mathbf{L}}$ is the velocity gradient.

3. Page 319 in the book states that the material time derivative of an objective tensor is in general non-objective. Let $\bar{\underset{\approx}{\sigma}} \equiv \underset{\approx}{\mathbf{R}}^T \bullet \underset{\approx}{\sigma} \bullet \underset{\approx}{\mathbf{R}}$ denote the "unrotated" Cauchy stress, where $\underset{\approx}{\mathbf{R}}$ denotes the rotation tensor from the polar decomposition. Show that $\bar{\underset{\approx}{\sigma}}$ is a material tensor. Let $\dot{\underset{\approx}{\sigma}}$ denote the material time rate of the Cauchy stress and let $\underset{\approx}{\Omega} \equiv \dot{\underset{\approx}{\mathbf{R}}} \bullet \underset{\approx}{\mathbf{R}}^T$. Prove that $\dot{\bar{\underset{\approx}{\sigma}}} = \underset{\approx}{\mathbf{R}}^T \bullet \overset{\circ}{\underset{\approx}{\sigma}} \bullet \underset{\approx}{\mathbf{R}}$, where $\overset{\circ}{\underset{\approx}{\sigma}} = \dot{\underset{\approx}{\sigma}} - \underset{\approx}{\Omega} \bullet \underset{\approx}{\sigma} + \underset{\approx}{\sigma} \bullet \underset{\approx}{\Omega}$ is called the polar "rate" of stress. Explain why the polar rate can be interpreted as the part of the stress rate *not* attributable to rigid rotation. What connection (if any) does the polar "rate" have with part (b) of question #1 in homework assignment 8?

4U. Solve problems 4.12 and 4.41 in the book.

5U. Solve problems (4.2), (4.3), and (4.4) in
`http://me.unm.edu/~rmbrann/curvilinear.pdf`