make-up
FINAL EXAM

Test Date . . . . Any time

Begin time . . . . 7:30 AM
End time . . . . 8:45 AM

Instructions to test takers:

Instructor will NOT be available during the broadcast. In fairness to all test takers, no questions may be asked during the exam. If you believe a question is ill-posed, you should state *why* you think it is ill-posed, and you should then answer the question that you believe was meant to be asked.

- Write your name on every sheet.
- 1 hour and 15 minutes plus 20 extra minutes to read the optional problems.
- Closed book.
- No notes or handouts.
- No calculator.
- No student-supplied paper. All work must be done on the exam itself. Extra sheets are provided at the back of the exam. Do not unstaple.
- If you run out of time, be sure to describe in words what you would have done if you had more time.
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NOTE: The following problems are worth a total of 260 points, of which you may choose any 95 points as bonus points. In other words,

your score = points earned / 165

50 points

1. Discuss the role played by kinematics, balance laws, and constitutive equations in continuum mechanics. On what foundations do each of these rest? Your answer should contain the following elements:

   - A definition of the word "continuum."
   - *Detailed* outline of the derivation of Cauchy's equations of motion by giving
     (a) Euler's first and second laws.
     (b) Stress Principle of Euler and Cauchy
     (c) Cauchy's fundamental theorem -- discuss the basic ideas of the proof.
     (d) The final steps that lead to Cauchy's equations.
   - A summary of the fundamental equations of mechanics used in boundary value problems, including a description of the basic form of initial and boundary conditions, with examples.
   - A definition of the principle of material frame indifference (PMFI), including a motivational example of a constitutive law that does *not* satisfy PMFI.

40 points

2. Consider the equilibrium twist of a cylindrical solid rubber rod of radius R. Take the constitutive equation as

$$\underset{\approx}{\sigma} = -p\underset{\approx}{\mathbf{I}} + 2C\underset{\approx}{\mathbf{F}} \bullet \underset{\approx}{\mathbf{F}}^{T}, \tag{2.1}$$

where $\underset{\approx}{\sigma}$ is the Cauchy stress tensor, p is a scalar field, C is a material constant, and $\underset{\approx}{\mathbf{F}}$ is the deformation gradient. Assume one end of the rod is fixed and that the lateral surface is stress free.

(a) Taking the axis of the rod to be aligned with the 3-direction, explain why the following mapping function sensibly describes twist of the rod:

$$x_1 = X_1\cos(kX_3) - X_2\sin(kX_3) \tag{2.2}$$
$$x_2 = X_1\sin(kX_3) - X_2\cos(kX_3) \tag{2.3}$$
$$x_3 = X_3 \tag{2.4}$$

where k is a constant (called the "twist per unit length").

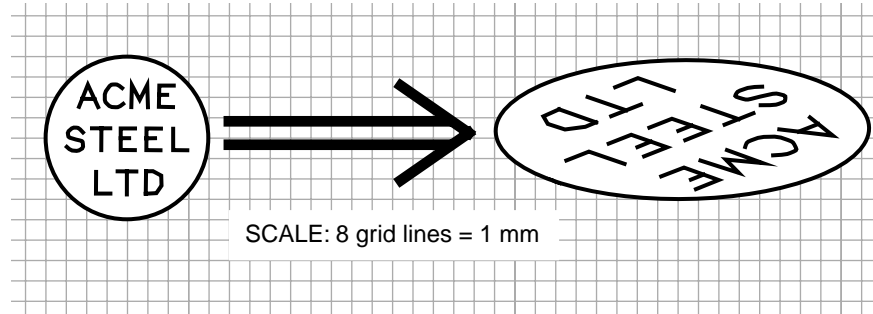*(b)* Prove that the deformation gradient associated with the above mapping is

$$[\underset{\approx}{\mathbf{F}}] = \begin{bmatrix} c & -s & -kx_2 \\ s & c & kx_1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{where} \qquad c \equiv \cos(kx_3) \quad \text{and} \quad s \equiv \sin(kx_3) \tag{2.5}$$

(c) Use Cauchy's first law in equilibrium form (*i.e., no body forces or accelerations*) to prove that the field p in Eq. (2.1) is $\quad p = 2C + Ck^2(R^2 - r^2)$, where $\quad r^2 = x_1^2 + x_2^2$.

(d) Prove that the torque applied to the non-fixed end of the rod is $\pi CR^4 k$.

(e) Prove that the axial force is $-\frac{1}{2}\pi CR^4 k^2$.

30 points

3. The ACME steel company stretches and bends a *thin* sheet of steel to form an airplane part. To estimate the strains caused by the forming process, the company paints *tiny* circular logos on the surface of the steel before deformation. The deformation of one of the logos occurs in the 1-2 plane and is accurately drawn here to scale, where the laboratory grid is provided for reference and is fixed in space (it is not painted on the steel).

SCALE: 8 grid lines = 1 mm

(a) Assuming that the out-of-plane *shear* strains e_{13} and e_{23} are negligible and that the material is *incompressible*, use the provided grid to estimate the full 3×3 (not 2×2) matrix of the deformation gradient tensor with respect the laboratory basis. *Accuracy to within 1/2 a grid-width gets full credit.*

(b) For the illustrated deformation, estimate the polar decomposition rotation angle in *degrees* by drawing an *appropriate* line of particles in both the initial and deformed configurations and explaining why the polar rotation angle must be the angle between those two material lines. *Accuracy to within ±15° gets full credit.* LABLE THE ROTATION ANGLE IN THE DRAWING, SHOWING WHETHER CLOCKWISE OR COUNTER CLOCKWISE.

20 points

4. Circle TRUE or FALSE (A statement if false if *any part* of it is false.)

    For "false" statements, use the space below to provide a counterexample or compelling argument that proves the statement is false:

    (a) True/False Symmetry of the stress tensor is not valid if the body has an angular acceleration.

    (b) True/False If all of the particles in a body are moving along circular paths centered about the origin, then the velocity must be given by $\underset{\sim}{\mathbf{w}} \times \underset{\sim}{\mathbf{x}}$, where $\underset{\sim}{\mathbf{x}}$ is the position vector and $\underset{\sim}{\mathbf{w}}$ is the vorticity vector.

    (c) True/False On the plane of maximum normal stress, the shearing stress is always zero.

    (d) True/False On the plane of maximum shearing stress, the normal stress is zero.

    (e) True/False The Jacobian, $J \equiv \det \underset{\approx}{\mathbf{F}}$ is zero if the volume does not change.

    (f) True/False Conservation of angular momentum implies $\underset{\sim}{\mathbf{x}} \times (\underset{\approx}{\sigma} \bullet \nabla) = \underset{\sim}{\mathbf{x}} \times (\rho \underset{\sim}{\mathbf{a}} - \rho \underset{\sim}{\mathbf{b}})$

    (g) True/False A homogeneously deforming body has constant volume if $\underset{\approx}{\mathbf{F}} = \underset{\approx}{\mathbf{I}}$.

    (h) True/False If a homogeneously deforming body has constant volume, then $\underset{\approx}{\mathbf{F}} = \underset{\approx}{\mathbf{I}}$.

    (i) True/False Presuming the term "dyad" is defined, a tensor can be defined as any sum of dyads.

    (j) True/False A tensor can be defined as a set of numbers T_{ij} referenced to a basis such that the matrix of numbers becomes $T_{ij}{}^{*} = Q_{mi}Q_{nj}T_{mn}$ upon a change a change of basis, where Q_{ij} are the direction cosines defined such that $\underset{\sim}{\mathbf{e}}_i{}^{*} = Q_{ki}\underset{\sim}{\mathbf{e}}_k$.

    (k) True/False A tensor can be defined as a linear transformation taking vectors to vectors.

    (l) True/False A deformation is a proper rotation if $\det \underset{\approx}{\mathbf{F}} = 1$.

30 points

Undergrads: choose either one of the following problems

Grads: Solve 5G. If you're stuck, solve 5U for a 5 point penalty.

5U. State the law of conservation of mass and derive the continuity equation in two ways:

(a) Using an Eulerian control volume

(b) Using a Lagrangian control volume

Show that the result from part (a) is identical to that of part (b).

5G. For thermodynamics, the mechanical power (force times velocity) imparted on a body B is defined

$$P^M = \int_{\partial B^*} \underset{\sim}{t} \bullet \underset{\sim}{v} \; dS + \int_{B^*} \underset{\sim}{b} \bullet \underset{\sim}{v} \; \rho \; dV \tag{5.1}$$

Prove that $P^M = \dot{K} + P^S$, where $K = \dfrac{1}{2} \int_{B^*} \underset{\sim}{v} \bullet \underset{\sim}{v} \; \rho \; dV$ is the bulk kinetic energy of the

body and $P^S = \int_{B^*} \underset{\sim}{\sigma} : \underset{\sim}{D} \; dV$. What is the physical interpretation of P^S?

25 points

6. A second-order tensor in 3D space is "isotropic" if and only if it can be written in the form $\alpha \underset{\approx}{\mathbf{I}}$ for some scalar α. A tensor is "deviatoric" if and only if its trace is zero.

   (a) Prove that any tensor $\underset{\approx}{\mathbf{A}}$ can be decomposed uniquely into an isotropic part $\underset{\approx}{\mathbf{A}}^{\text{iso}}$ plus a deviatoric part $\underset{\approx}{\mathbf{A}}^{\text{dev}}$, where $\underset{\approx}{\mathbf{A}}^{\text{iso}} = \frac{1}{3}(\text{tr}\underset{\approx}{\mathbf{A}})\underset{\approx}{\mathbf{I}}$ and $\underset{\approx}{\mathbf{A}}^{\text{dev}} = \underset{\approx}{\mathbf{A}} - \frac{1}{3}(\text{tr}\underset{\approx}{\mathbf{A}})\underset{\approx}{\mathbf{I}}$.

   (b) If $\underset{\approx}{\mathbf{A}}$ is isotropic and $\underset{\approx}{\mathbf{B}}$ is deviatoric, prove that $\underset{\approx}{\mathbf{A}}:\underset{\approx}{\mathbf{B}} = 0$.

   (c) Use the above result to prove that $\underset{\approx}{\mathbf{T}}:\underset{\approx}{\mathbf{S}} = \underset{\approx}{\mathbf{T}}^{\text{iso}}:\underset{\approx}{\mathbf{S}}^{\text{iso}} + \underset{\approx}{\mathbf{T}}^{\text{dev}}:\underset{\approx}{\mathbf{S}}^{\text{dev}}$ for any tensors $\underset{\approx}{\mathbf{T}}$ and $\underset{\approx}{\mathbf{S}}$.

   (d) Prove that, for any tensor $\underset{\approx}{\mathbf{T}}$, the eigenvectors of $\underset{\approx}{\mathbf{T}}^{\text{dev}}$ are identical to those of $\underset{\approx}{\mathbf{T}}$. (hint: use the fundamental definition of the eigenproblem).

7. Consider a body B with boundary ∂B having an outward unit normal $\underset{\sim}{\mathbf{n}}$

    (a) Prove that, no matter how the body is shaped,

    $$\iint_{\partial B} \underset{\sim}{\mathbf{n}}\, dA \;=\; \underset{\sim}{\mathbf{0}} \tag{7.1}$$

    (b) Let B_o be the reference configuration associated with B. Let $\mathbf{N}$ denote the outward normal to the boundary ∂B_o. Convert the above integral to an integral in the reference configuration and then provide appropriate arguments to prove that

    $$(J\underset{\sim}{\mathbf{F}}^{-T}) \bullet \overset{\leftarrow}{\nabla}_o \;=\; \underset{\sim}{\mathbf{0}} \tag{7.2}$$

    (c) Now consider the first Piola-Kirchhoff stress, defined $\underset{\sim}{\hat{\tau}} \;=\; \underset{\sim}{\sigma} \bullet \underset{\sim}{\mathbf{F}}^C$ where $\underset{\sim}{\sigma}$ is the Cauchy stress and $\underset{\sim}{\mathbf{F}}^C$ is the cofactor of the deformation gradient tensor. Use the above result to prove that $\dfrac{1}{\rho_o}(\underset{\sim}{\hat{\tau}} \bullet \overset{\leftarrow}{\nabla}_o) \;=\; \dfrac{1}{\rho}(\underset{\sim}{\sigma} \bullet \overset{\leftarrow}{\nabla})$, where ρ and ρ_o are the current and reference densitities, respectively.

40 points

8. For large deformation plasticity, the following constitutive equation has been proposed when plastic flow occurs:

$$\underset{\approx}{\mathbf{D}} \;=\; \frac{9}{4}\, \frac{\underset{\approx}{\mathbf{S}}{:}\dot{\underset{\approx}{\sigma}}}{H\bar{\sigma}^2}\; \underset{\approx}{\mathbf{S}} \tag{8.1}$$

where $\underset{\approx}{\mathbf{D}}$ is the "rate" of deformation,

$\underset{\approx}{\sigma}$ is the Cauchy stress,

$\underset{\approx}{\mathbf{S}}$ is the stress deviator, i.e., $\underset{\approx}{\mathbf{S}} \equiv \underset{\approx}{\sigma} - \frac{1}{3}(\mathrm{tr}\underset{\approx}{\sigma})\; \underset{\approx}{\mathbf{I}}$,

$\bar{\sigma} \equiv \sqrt{\frac{3}{2}\mathrm{tr}(\underset{\approx}{\mathbf{S}}^2)}$ is the "effective" stress,

and H is a scalar-valued function of $\bar{D}$, where $\bar{D} \;=\; \sqrt{\frac{2}{3}\mathrm{tr}(\underset{\approx}{\mathbf{S}}^2)}$.
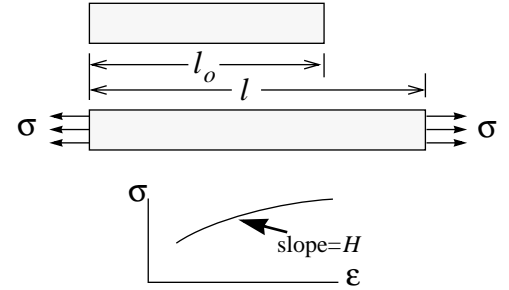
Suppose a bar is subjected to a quasistatic interval of uniaxial tensile stress during which $\sigma_{11}=\sigma$ and all other $\sigma_{ij}=0$.

(a) Prove that $\bar{\sigma} \;=\; \sigma$.

(b) Prove that $\underset{\approx}{\mathbf{S}}{:}\dot{\underset{\approx}{\sigma}} \;=\; \frac{2}{3}\sigma\dot{\sigma}$

(c) Prove that H must be the slope of the uniaxial stress-strain curve σ vs. logarithmic strain $\varepsilon=\ln(l/l_o)$.

(d) Is the constituive law of Eq. (8.1) frame invariant?

Scratch Sheet #1

Scratch Sheet #2

Scratch Sheet #3

Scratch Sheet #3

9. An incompressible rigid-plastic material behaves according to the constitutive equation

$$\underset{\approx}{\sigma} = -p\underset{\approx}{\mathbf{I}} + \frac{\sqrt{2}k}{\sqrt{\underset{\approx}{\mathbf{D}}:\underset{\approx}{\mathbf{D}}}}\underset{\approx}{\mathbf{D}} \tag{9.1}$$

where k is constant, p is a hydrostatic pressure, and $\underset{\approx}{\mathbf{D}}$ is the rate of deformation. A thin circular sheet undergoes an axisymmetrical deformation due to the boundary being displaced radially. Let the outer radius at time t be $R = R(t)$.

(a) Using the fact that the material is incompressible, use the constitutive law to prove that

$p = -\frac{1}{3}\mathrm{tr}\underset{\approx}{\sigma}$.

(b) Consider an infinitesimal fiber originally located at radius r_o and oriented in the θ-direction. Under symmetrical loading, this fiber deforms to a new radius r, and its orientation remains in the θ-direction. Use geometrical arguments to explain why the stretch of this fiber (*i.e.*, the ratio of its deformed length to its initial length) must be equal to r/r_o.

(c) Symmetry also demands that radially oriented fibers must remain radially oriented. Let λ denote the stretch of an infinitesimal radial fiber. Use incompressiblity to prove that the matrix of the deformation gradient with respect to the cylindrical basis triad $\{\underset{\sim}{\mathbf{e}}_r, \underset{\sim}{\mathbf{e}}_\theta, \underset{\sim}{\mathbf{e}}_z\}$ must be

$$\underset{\approx}{\mathbf{F}} \rightarrow \begin{bmatrix} \lambda & 0 & 0 \\ 0 & r/r_o & 0 \\ 0 & 0 & r_o/(\lambda r) \end{bmatrix} \tag{9.2}$$
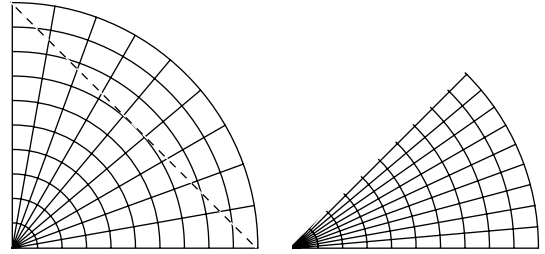
(d) Use the above result to prove that

$$\underset{\approx}{\mathbf{D}} \rightarrow \begin{bmatrix} \dot{\lambda}/\lambda & 0 & 0 \\ 0 & \dot{r}/r & 0 \\ 0 & 0 & -\frac{\dot{\lambda}}{\lambda} - \frac{\dot{r}}{r} \end{bmatrix} \quad \text{and} \quad \underset{\approx}{\mathbf{D}}:\underset{\approx}{\mathbf{D}} = 2\left[\left(\frac{\dot{\lambda}}{\lambda}\right)^2 - \frac{\dot{\lambda}}{\lambda}\frac{\dot{r}}{r} + \left(\frac{\dot{r}}{r}\right)^2\right] \tag{9.3}$$

(e) Taking the stress to be diagonal with respect to the cylindrical triad, find the equation in the book which says the radial component of the divergence of stress is given by

$\frac{1}{r}(\sigma_{rr} + r\sigma_{rr,\,r} - \sigma_{\theta\theta})$.

10. If $\{r_o, \theta_o, z_o\}$ are the initial cylindrical coordinates of a particle, then the deformed locations of that particle are $r = r_o$, $\theta = \theta_o/(1+kt)$, $z = z_o$. The figure at right shows this deformation as viewed down the z-axis at the moment when $kt = 1$.

(a) Sketch the deformed location of the dotted line.

(b) Find the deformation gradient as a function of time for the point originally located at $r_o=R$, $\theta_o=0$, and $z_o=0$. *You do not need to know cylindrical coordinates to solve this problem.*

11. Find the eigenvalues and eigenvectors of the tensor:

$$\underset{\approx}{\mathbf{T}} = a \sum_{i=1}^{3} \sum_{j=1}^{3} \underset{\sim}{\mathbf{e}}_i \underset{\sim}{\mathbf{e}}_j \qquad (11.1)$$

where the vectors $\{ \underset{\sim}{\mathbf{e}}_1, \underset{\sim}{\mathbf{e}}_2, \underset{\sim}{\mathbf{e}}_3 \}$ are an orthonormal basis.