BONUS PROBLEM WORTH 15 HOMEWORK POINTS

In homework #8, there is a formula for the gradient of the dot product between two vectors:

$$\vec{\nabla}(\underset{\sim}{\mathbf{v}} \bullet \underset{\sim}{\mathbf{w}}) = (\vec{\nabla}\underset{\sim}{\mathbf{v}}) \bullet \underset{\sim}{\mathbf{w}} + (\vec{\nabla}\underset{\sim}{\mathbf{w}}) \bullet \underset{\sim}{\mathbf{v}} \tag{1}$$

However, if you look up the gradient of a dot product in a math handbook such as the CRC, you will find the following *horrendous* formula:

$$\vec{\nabla}(\underset{\sim}{\mathbf{v}} \bullet \underset{\sim}{\mathbf{w}}) = (\underset{\sim}{\mathbf{w}} \bullet \vec{\nabla})\underset{\sim}{\mathbf{v}} + (\underset{\sim}{\mathbf{v}} \bullet \vec{\nabla})\underset{\sim}{\mathbf{w}} + \underset{\sim}{\mathbf{w}} \times (\vec{\nabla} \times \underset{\sim}{\mathbf{v}}) + \underset{\sim}{\mathbf{v}} \times (\vec{\nabla} \times \underset{\sim}{\mathbf{w}}), \tag{2}$$

In eq (2), the notation $(\underset{\sim}{\mathbf{w}} \bullet \vec{\nabla})\underset{\sim}{\mathbf{v}}$ means the same thing as $\underset{\sim}{\mathbf{w}} \bullet (\vec{\nabla}\underset{\sim}{\mathbf{v}})$. In other words, the *i*th component of the first term on the right-hand-side of eq (2) is $w_k\left(\frac{\partial}{\partial x_k}v_i\right) = w_k v_{i,k}$.

This is <u>not the same</u> as of the first term on the right-hand-side of eq (1), for which the *i*th component is $\left(\frac{\partial}{\partial x_i}v_k\right)w_k = w_k v_{k,i}$.

(a) Use indicial notation to prove that Eqs. (1) and (2) are *both* correct. Which of these equations would *you* rather use in your applications?

(b) In this class, our proofs of formulas like this have employed indicial notation with respect to a rectangular CARTESIAN basis. The last step in our indicial derivations is to express the final result in direct notation. Explain why this process yields a direct notation formula that is valid for ANY basis (even nonorthonormal curvilinear).

(c) Consider cylindrical coordinates. Then, for example, $\underset{\sim}{\mathbf{v}} = v_r\underset{\sim}{\mathbf{e}}_r + v_\theta\underset{\sim}{\mathbf{e}}_\theta + v_z\underset{\sim}{\mathbf{e}}_z$, where the components are all functions of (r, θ, z).

(i) GRADS: Explain why $\underset{\sim}{\mathbf{v}} \bullet \underset{\sim}{\mathbf{w}} = v_r w_r + v_\theta w_\theta + v_z w_z$, despite the fact that this is a curvilinear coordinate system.

(ii) EVERYONE: Because the coordinates are curvilinear, the formulas for the gradient take on a different form. The radial component of the first term on the right hand side of Eq. (1) is $(\vec{\nabla}\underset{\sim}{\mathbf{v}})_{rr}w_r + (\vec{\nabla}\underset{\sim}{\mathbf{v}})_{r\theta}w_{r\theta} + (\vec{\nabla}\underset{\sim}{\mathbf{v}})_{rz}w_z$, where the $rr, r\theta, rz$ components of $\vec{\nabla}\underset{\sim}{\mathbf{v}}$ are obtained from the formulas provided in the book for the gradient of vector in cylindrical coordinates [Beware: when the book says $\nabla\underset{\sim}{\mathbf{v}}$, it means what we call $\underset{\sim}{\mathbf{v}}\overleftarrow{\nabla} = (\vec{\nabla}\underset{\sim}{\mathbf{v}})^T$. Thus, you must use the transpose of the book's formulas if you really want $\vec{\nabla}\underset{\sim}{\mathbf{v}}$] Using the book's formulas, write out an explicit expression for the radial component of the first term on the right hand side of Eq. (1) if $v_r = r\theta$, $v_\theta = \theta z r^2$, $v_z = r^4\sin\theta$, $w_r = r$, $w_\theta = z\theta$, and $w_z = 0$.