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ANALYSIS OF ELASTOPLASTIC REISSNER'S PLATES WITH MULTILAYERED APPROACH BY THE BOUNDARY ELEMENT METHOD

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Abstract. *The boundary element method applied to Reissner's theory for elastoplastic layered plate bending analysis is considered. As the plate is layered, the plasticity progress can be evaluated along the plate thickness. Considering the integral equations of the problem and a local coordinate system, the stresses and strains at the mid surface of each layer are obtained. Performing integrations along all layers, the bending moments and shear forces are obtained for points situated at the mid surface of the plate. It is assumed that the plastic strains are only due to bending. In each layer, the value of the yield stress for the mid surface is considered for all points located along its thickness. The integral equation is discretized using the boundary element method. An incremental-iterative algorithm for solving the system of nonlinear equations is used. Increments in stresses and strains are calculated in each layer. Some applications were solved and compared with other numerical and analytical results.*

1 INTRODUCTION

Recently, due to a large research effort, great advances have been obtained in the development of the boundary element method (BEM) to allow the analysis of several kinds of problems, such as bending of Reissner´s plates[1, 2] and elastoplastic analysis[3-6].

The aim is to get a formulation with Reissner´s theory for the analysis of multilayered elastoplastic plate, as an extension of previous works[5, 6]. In layered plates, the plasticity progress can be evaluated along the plate thickness. The layers are assumed to have isotropic and homogeneous material, to be perfectly joined and to have constant thickness. The yield stress is considered independently in each layer. The usual procedures employed in the boundary element method[7] are used in the numerical implementation and an incremental iterative algorithm is presented to solve the elastoplastic problem.

Reissner´s plate theory[8] allows for transverse shear effects and therefore, it holds for thin and thick plates.

The Cartesian tensor notation is used, in which Greek indices are denotated by subscripts 1,2 and Latin ones by 1,2,3. The thickness h of the plate is constant and the distributed load q per unit area is known. Cartesian plane (x_1, x_2) is defined in the plate undeformed mid surface and x_3 is the global coordinate along the plate thickness.

2 BASIC DEFINITIONS

The bending and transverse shear strains $\chi_{\alpha\beta}$ and φ_α, respectively, defined for plate mid surface, are

$$\chi_{\alpha\beta} = \frac{1}{2}(\phi_{\alpha,\beta} + \phi_{\beta,\alpha}) \tag{1}$$

$$\varphi_\alpha = (\omega_{,\alpha} + \phi_\alpha) \tag{2}$$

where ϕ_α and ω are the rotations and displacement, respectively, both defined for plate mid surface. The bending and shear strains defined for points situated through the thickness are:

$$\varepsilon_{\alpha\beta} = x_3 \chi_{\alpha\beta} \tag{3}$$

$$\gamma_\alpha = \varphi_\alpha \tag{4}$$

For simplicity of notation, ϕ_α and ω will be written simply as u_i. The load conditions of the plate are considered as $\sigma_{33} = \pm q/2$ and $\sigma_{\alpha 3} = 0$ to $x_3 = \pm h/2$, the normal stress σ_{33} are considered negligible.

Tractions are defined by:

$$p_\alpha = M_{\alpha\beta} n_\beta \tag{5}$$

$$p_3 = Q_\beta n_\beta \tag{6}$$

in which, n_α are direction cosines of the outward normal to the plate boundary.

The following boundary conditions are considered:

$$u_i = \overline{u}_i \quad \text{at} \quad \Gamma_u \tag{7}$$

$$p_i = \overline{p}_i \quad \text{at} \quad \Gamma_p \tag{8}$$

so that $\Gamma = \Gamma_u + \Gamma_p$, where Γ is the total boundary.

One can define:

$$\chi_{\alpha\beta} = \chi^e_{\alpha\beta} + \chi^p_{\alpha\beta} \tag{9}$$

$$\varphi_\alpha = \varphi^e_\alpha \tag{10}$$

as bending and shear strains, respectively, where $(.^e)$ denote the elastic part and $(.^p)$ the plastic one.

The equilibrium equations are:

$$M_{\alpha\beta,\beta} - Q_\alpha = 0 \tag{11}$$

$$Q_{\alpha,\alpha} + q = 0 \tag{12}$$

The moments and shear forces are given by:

$$M_{\alpha\beta} = \frac{D(1-\nu)}{2}\left[2\chi_{\alpha\beta} + \frac{2\nu}{1-\nu}\chi_{\gamma\gamma}\delta_{\alpha\beta}\right] + \frac{\nu q}{(1-\nu)\lambda^2}\delta_{\alpha\beta} - M^p_{\alpha\beta} \tag{13}$$

$$Q_\alpha = \frac{D(1-\nu)\lambda^2}{2}\varphi_\alpha \tag{14}$$

in which $M^p_{\alpha\beta}$ denote the plastic moment components, so that:

$$M^p_{\alpha\beta} = \frac{D(1-\nu)}{2}\left[2\chi^p_{\alpha\beta} + \frac{2\nu}{1-\nu}\chi^p_{\gamma\gamma}\delta_{\alpha\beta}\right] \tag{15}$$

In equations (13) to (15), $\delta_{\alpha\beta}$ is the Kronecker delta, ν is Poisson's ratio, λ is a constant defined as $\frac{\sqrt{10}}{h}$, D is the bending rigidity defined as $\frac{Eh^3}{12(1-\nu)^2}$, E is Young's modulus.

3 MULTILAYERED PLATE ASSUMPTIONS

The co-ordinate system, notation and geometry for layered plates are shown in Figure 1. The symbol k denotes the number of a layer, which assumes integer values and starts from the plate bottom, and N denotes the number of layers. The plate mid surface defines the domain Ω and Γ represents the boundary of that region. The global co-ordinate along the thickness h is represented by x_3 and x_{3k} denotes the local co-ordinate along the k-layer thickness h_k, defined by $x_{3k} = x_3 - x^0_{3k}$, in which x^0_{3k} denotes the distance of the mid surface of the k-layer from the (x_1, x_2)

plane. Then, the moments and shear forces are obtained as sums of integrals calculated for each layer:

$$M_{\alpha\beta} = \sum_{k=1}^{N} \int_{-h_k/2}^{h_k/2} \sigma_{\alpha\beta} x_3 dx_3 \tag{16}$$

$$Q_{\alpha} = \sum_{k=1}^{N} \int_{-h_k/2}^{h_k/2} \sigma_{\alpha 3} dx_3 \tag{17}$$

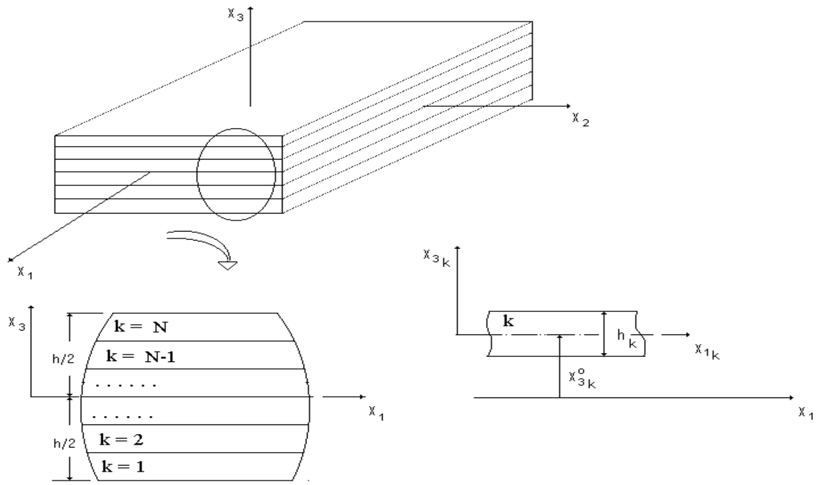

Figure 1: Geometry and notation for multilayered plates.

4 INTEGRAL EQUATIONS

The integral equation for displacements in elastoplastic analysis of plate bending, where initial plastic moments are considered, is expressed by[5, 6]:

$$C_{ij}(\xi)u_j(\xi) + \int_\Gamma P^*_{ij}(\xi,x)u_j(x)d\Gamma(x) = \int_\Gamma U^*_{ij}(\xi,x)p_j(x)d\Gamma(x) +$$

$$+ \int_\Omega \left[U^*_{i3}(\xi,x) - \frac{\nu}{(1-\nu)\lambda^2} U^*_{i\alpha,\alpha}(\xi,x) \right] q(x)d\Omega(x) + \int_\Omega \chi^*_{\alpha\beta i}(\xi,x) M^p_{\alpha\beta}(x)d\Omega(x) \tag{18}$$

In this expression, the second term of the left hand side is understood as a Cauchy principal value integral; ξ and x are source point and field point, respectively; $C_{ij}(\xi)$ is equal to $\frac{1}{2}\delta_{ij}$ for points ξ in a smooth boundary and equal to δ_{ij} for internal points; and the plastic moment $M^{p}_{\alpha\beta}$ is written as:

$$M^{p}_{\alpha\beta} = \sum_{k=1}^{N} \int_{-h_k/2}^{h_k/2} [\sigma_{\alpha\beta}]_k \, x_3 dx_3 \tag{19}$$

The expressions for U^*_{ij} and P^*_{ij} were presented in Refs. [1,2] and for $\chi^*_{\alpha\beta i}$ in Refs. [5,6].

For a constant load *q,* the transformation of the domain integral in expression (18) into a boundary integral yields:

$$C_{ij}(\xi)u_j(\xi) + \int_\Gamma P^*_{ij}(\xi,x)u_j(x)d\Gamma(x) = \int_\Gamma U^*_{ij}(\xi,x)p_j(x)d\Gamma(x) +$$

$$+ q\int_\Gamma \left[V^*_{i,\alpha}(\xi,x) - \frac{\nu}{(1-\nu)\lambda^2} U^*_{i\alpha,\alpha}(\xi,x) \right] n_\alpha(x)d\Gamma(x) + \int_\Omega \chi^*_{\alpha\beta i}(\xi,x) M^{p}_{\alpha\beta}(x)d\Omega(x) \tag{20}$$

where V^*_i are solutions of the equation $\nabla^2 V^*_i = U^*_{i3}$ and the expressions for $V^*_{i,\alpha}$ were presented in Refs. [1,2]. The expression (20) represents three integral equations, two of them for rotations (i = α = 1, 2) and the third for deflection (i = 3).

The expressions for moments and shear forces at an internal point ξ are written as[5, 6]:

$$M_{\alpha\beta}(\xi) = \int_\Gamma U^*_{\alpha\beta k}(\xi,x)p_k(x)d\Gamma(x) - \int_\Gamma P^*_{\alpha\beta k}(\xi,x)u_k(x)d\Gamma(x) +$$

$$+ q\int_\Gamma W^*_{\alpha\beta}(\xi,x)d\Gamma(x) + \frac{\nu}{(1-\nu)\lambda^2} q\delta_{\alpha\beta} +$$

$$+ \int_\Omega \chi^*_{\alpha\beta\gamma\theta}(\xi,x) M^{p}_{\gamma\theta}(x)d\Omega(x) - \frac{1}{2}\left[2(1+\nu)M^{p}_{\alpha\beta}(x) + (1-3\nu)\delta_{\alpha\beta} M^{p}_{\theta\theta}(x)\right] \tag{21}$$

and

$$Q_\beta(\xi) = \int_\Gamma U^*_{3\beta k}(\xi,x)p_k(x)d\Gamma(x) - \int_\Gamma P^*_{3\beta k}(\xi,x)u_k(x)d\Gamma(x) +$$

$$+ q\int_\Gamma W^*_{3\beta}(\xi,x)d\Gamma(x) + \int_\Omega \chi^*_{3\beta\gamma\theta}(\xi,x) M^{p}_{\gamma\theta}(x)d\Omega(x) \tag{22}$$

The expressions of the $U^*_{ijk}, P^*_{ijk}, W^*_{i\beta}$ were presented in Refs. [1,2] and $\chi^*_{i\beta\gamma\theta}$ in Refs. [5,6].

5 CONSTITUTIVE EQUATIONS

The following yield stress function is considered for elastoplastic material:

$$\Phi(\sigma_{\alpha\beta}, k) = f(\sigma_{\alpha\beta}) - \Psi(k) = \sigma_e - \sigma_0 = 0 \tag{23}$$

where the parameter k is a constant, σ_0 is initial yield stress under uniaxial tension and σ_e is the effective stress, calculated from von Mises and Tresca yield criteria.

The stress increment and the elastoplastic tensor are written, respectively, as

$$d\sigma_{\alpha\beta} = C^{ep}_{\alpha\beta\gamma\theta} d\varepsilon_{\gamma\theta} + \frac{\nu x_3 dq}{2(1-\nu)h}\left[3 - \left(\frac{2x_3}{h}\right)^2\right]\left[\delta_{\alpha\beta} - \frac{1}{\gamma'} C_{\alpha\beta\mu\rho} a_{\mu\rho} a_{\eta\varsigma} \delta_{\eta\varsigma}\right] \tag{24}$$

where $C_{\alpha\beta\gamma\theta}$ is the isotropic tensor of elastic constants.

$$C^{ep}_{\alpha\beta\gamma\theta} = C_{\alpha\beta\gamma\theta} - \frac{1}{\gamma'} C_{\alpha\beta\mu\rho} a_{\mu\rho} a_{\eta\varsigma} C_{\eta\varsigma\gamma\theta} \tag{25}$$

By following an initial stress procedure, it is first considered an imaginary elastic problem. In this case,

$$d\sigma^e_{\alpha\beta} = C_{\alpha\beta\gamma\theta} d\varepsilon_{\gamma\theta} + \frac{\nu x_3}{2(1-\nu)h}\left[3 - \left(\frac{2x_3}{h}\right)^2\right] dq \delta_{\alpha\beta} \tag{26}$$

and the initial stress increment will be computed as

$$d\sigma_{\alpha\beta} = d\sigma^e_{\alpha\beta} - \frac{1}{\gamma'} C_{\alpha\beta\mu\rho} a_{\mu\rho} a_{\eta\varsigma} d\sigma^e_{\eta\varsigma} \tag{27}$$

so that

$$d\sigma^p_{\alpha\beta} = d\sigma^e_{\alpha\beta} - d\sigma_{\alpha\beta} \tag{28}$$

and

$$d\sigma^p_{\alpha\beta} = \frac{1}{\gamma'} C_{\alpha\beta\mu\rho} a_{\mu\rho} a_{\eta\varsigma} d\sigma^e_{\eta\varsigma} \tag{30}$$

The entire transverse section of a layer is considered to become plastic when the effective stress reaches the yield stress at the point of mid surface of this layer.

6 BOUNDARY ELEMENTS

Quadratic boundary elements and constant triangular internal cells will be employed. This cells are necessary only in regions of the domain in which plastic strains are expected. Then, equation (20) is applied to all boundary nodes in discretized form and equations (21) and (22) are written for all moment internal points, also in discretized form. In matrix form, they become, respectively,

$$\mathbf{HU} = \mathbf{GP} + \mathbf{B} + \mathbf{TM}^p \quad (31)$$

$$\mathbf{M} = \mathbf{G'P} - \mathbf{H'U} + (\mathbf{W'} + \mathbf{V'}) + (\mathbf{T'} + \mathbf{E'})\mathbf{M}^p \quad (32)$$

$$\mathbf{Q} = \mathbf{G''P} - \mathbf{H''U} + \mathbf{W''} + \mathbf{T''M}^p \quad (33)$$

where the matrices **G'**, **G''**, **H'** and **H''** and the vectors **W'** and **W''** contain the boundary integrals related to the fundamental solution; **V'** contains the free part related to the transverse load; **T'** and **T''** contain the domain integrals that multiply the plastic moments.
Now, applying a procedure similar to that employed in Refs. [3-6], the following expressions are obtained:

$$\mathbf{y} = \mathbf{RM}^p + \mathbf{m} \quad (34)$$

$$\mathbf{M}^e = \mathbf{S'M}^p + \mathbf{n'} \quad (35)$$

$$\mathbf{Q} = \mathbf{S''M}^p + \mathbf{n''} \quad (36)$$

where:

$$\mathbf{R} = \mathbf{A}^{-1}\mathbf{T} \quad (37)$$

$$\mathbf{m} = \mathbf{A}^{-1}\mathbf{f} \quad (38)$$

$$\mathbf{S'} = \mathbf{T}^* - \mathbf{A'R} \quad (39)$$

and **f**, **f'** e **f''** are vectors that contain prescribed values.

7 SOLUTION PROCEDURE

The algorithm used to solve the elastoplastic problem in layered approach is described in what follows[9].

The incremental process starts with the reduction of the maximum equivalent stress $\sigma_e^{\max}$ evaluated at internal cell points to the initial yield stress σ_0. An initial load factor is therefore computed as:

$$\lambda_0 = \frac{\sigma_0}{\sigma_e^{max}} \quad (40)$$

The next values of the load factor are calculated as:

$$\lambda_i = \lambda_{i-1} + \Delta\lambda_i \quad (41)$$

where $\Delta\lambda_i$ is defined for the first yielding by

$$\Delta\lambda_i = \beta_i \lambda_0 \tag{42}$$

and β_i is a given percentage.

For each λ_i, the increment of the initial stress is iteratively computed by the following iterative incremental procedure:

i- Compute the increment of elastic moment by:

$$\Delta\mathbf{M}^e = \mathbf{S}'\Delta\mathbf{M}^p + \Delta\lambda_i \mathbf{n} \tag{43}$$

for the first iteration, or by the expression:

$$\Delta\mathbf{M}^e = \mathbf{S}'\Delta\mathbf{M}^p \tag{44}$$

in the following iterations.

ii- Compute the increment of the elastic stress for mid surface points of each layer by:

$$\Delta\sigma^e_{\alpha\beta} = \frac{12\Delta M^e_{\alpha\beta}}{h^3} x_3 \tag{45}$$

iii- Compute the increment of the real stress for mid surface points of each layer by:

$$\Delta\sigma_{\alpha\beta} = \Delta\sigma^e_{\alpha\beta} - \frac{1}{\gamma'} C_{\alpha\beta\mu\rho} a_{\mu\rho} a_{\eta\varsigma} \Delta\sigma^e_{\eta\varsigma} \tag{46}$$

iv- Verify the convergence, comparing $\Delta\varepsilon^p_e$ calculated with its accumulated value obtained from the load increment, to conclude if it can be neglected.

v- Compute the increment of initial stress for mid surface points of each layer by:

$$\Delta\sigma^p_{\alpha\beta} = \Delta\sigma^e_{\alpha\beta} - \Delta\sigma_{\alpha\beta} \tag{47}$$

vi- Accumulate the values of initial and real stresses for mid surface points of each layer by:

$$\sigma^p_{\alpha\beta} = \sigma^p_{\alpha\beta} + \Delta\sigma^p_{\alpha\beta} \tag{48}$$

$$\sigma_{\alpha\beta} = \sigma_{\alpha\beta} + \Delta\sigma_{\alpha\beta} \tag{49}$$

vii- Compute the increments of residual moment by:

$$\Delta M^p_{\alpha\beta} = \sum_{k=1}^{N} \int_{-h_k/2}^{h_k/2} [\Delta\sigma^p_{\alpha\beta}] x_3 dx_3 \tag{50}$$

viii- Compute the increments of real moment by:

$$\Delta M_{\alpha\beta} = \Delta M^e_{\alpha\beta} - \Delta M^p_{\alpha\beta} \tag{51}$$

ix- Continue with the next point, restarting the process from item (ii) until all points have been considered.
x- Restarting new iteration with item (i).

8 NUMERICAL RESULTS

We solve two numerical examples and compare our results with those available in the literature.

Simply Supported Square Plate Under Uniform Loading: As shown in Figure 2, it is considered a square plate of tickness h=0.01, simply supported, under uniform loading q. Von Mises yield criterion and plastic ideally material (H'=0) are considered in the numerical analysis. The following material properties are used: ν =0.3; E=10.92 and σ_0=1,600. Due to symmetry, only a quarter of the plate is discretized.

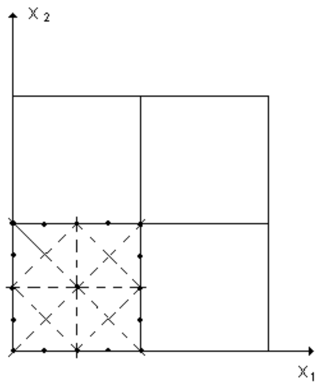

Figure 2: Square plate. Boundary elements and internal cells.

The results obtained with layered approach for that plate are shown in the Figure 3, in which the load displacement curves are presented, corresponding to the point situated at the center of the plate. The results presented for the same problem in Ref.[5] using the BEM with nonlayered approach and in Ref. [10] using the finite element method (FEM) are also shown in Figure 3.

Rectangular beam: In this example, it is considered a simply supported rectangular beam under uniform loading with the following characteristics: length l=3,000mm, width w=150mm and thickness h=900mm. The discretization is shown in Figure 4. The material parameters are: H'= 0; ν =0.3; E=210kN/mm^2 and σ_0=0.25kN/mm^2.

The results are presented in Figure 5, and the value 4.46mm was obtained.The analytical result obtained for the point located at the center of the mid surface using Timoshenko's theory[11], which considers the effects of transverse shear strains, is 4.48mm.

9 CONCLUSIONS

An application of the boundary element method to the analysis of multilayered elastoplastic plates is presented in this work. The formulation utilizes Reissner´s plate bending theory, which takes into account transverse shear deformation. The solution procedure considers the stresses for mid surface points of each layer. The layered approach allows the plasticity progress to be evaluated through the thickness. It was verified that results are in good agreement with other numerical and analytical ones.

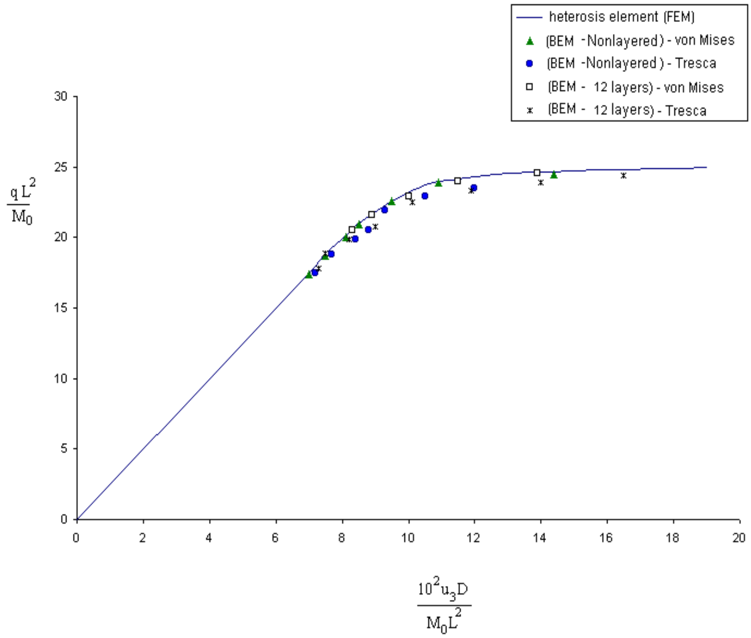

Figure 3: Square plate. Load displacement curves.

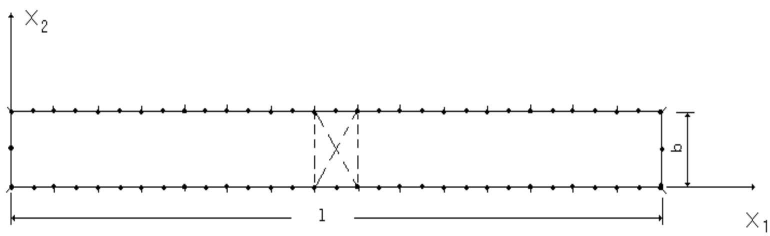

Figure 4: Rectangular beam. Boundary elements and internal cells.

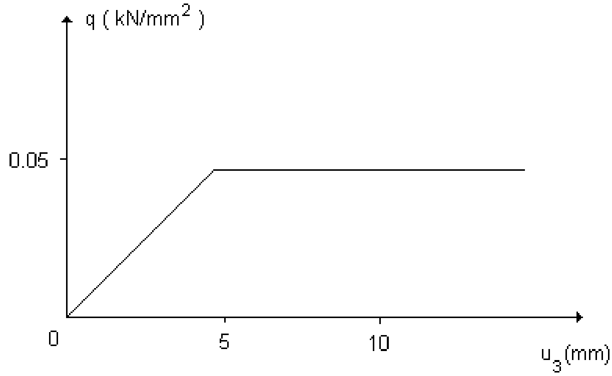

Figure 5: Rectangular beam. Load displacement curves.

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