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PLANE ANISOTROPIC BEAMS WITH SHEAR DEFORMATION VIA A GENERALIZED SOLUTION

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Abstract. *The natural frequencies of plane anisotropic beams with shear deformation are obtained, using a variational methodology developed by Filipich and Rosales named WEM (Whole Element Method). The method consists in extremizing an appropriate functional after using certain sequences that accomplish essential boundary conditions. Numerical examples are carried out for anisotropic cantilever beams as well as simply supported beams with narrow rectangular cross section, taking into account different fiber orientations. Comparison is made with results from Murakami-Yamakawa beam theory and Cortinez-Machado two-dimensional analysis.*

1 INTRODUCTION

Fiber-reinforced composites have been increasingly used over the past few decades in a variety of structures that require high strength and stiffness, and low weight. Many structural components made of composites have the form of beams. Accordingly, refined theories were developed in order to achieve reliable theoretical model to predict the structural behaviour, such as the important role of the shear deformability and the effects of anisotropy.

A theory of shear deformable orthotropic beams was developed by Nowinsky[1]. On the other hand, Dharmarajan and McCutchen[2] have discussed a method for obtaining shear correction factors for these types of beams. An orthotropic beam theory including normal deformability along with the shear effect was presented by Soldatos and Elishakoff [3].

In this article, the governing equations correspond to a Timoshenko-type beam model developed by Murakami-Yamakawa[4]. They obtained an anisotropic beam theory with anisotropic coupling by using a Hamilton-type principle that incorporates Reissner's semicomplementary energy function. This theory allowed them to study the effect of coupling of transverse shearing and axial stretching due to the difference between the directions of the orthotropic material axes and axis of the beam.

The methodology used herein was developed and used for different situations by Filipich and Rosales[5,6]. In the WEM solution an appropriate functional is extremized after proposing suitable sequences. These extremizing sequences are extended trigonometric series. Functions of a complete set in L_2 (convergent in the mean to square integrable functions) are linearly combined with the addition of support functions, which guarantee the uniform convergence to the continuos functions. The sequences are required to satisfy only the essential conditions. In case they are not identically verified, the functional is extended through a very convenient methodology of considering these restrictions by means of Lagrange multipliers. Natural frequencies of anisotropic cantilever beams as well as simply supported beams with narrow rectangular cross section are computed, taking into account different fiber orientations. Comparison is made with results from Murakami-Yamakawa[4] beam theory and Cortinez-Machado[7] two-dimensional analysis with FEM.

2 PROBLEM FORMULATION

An anisotropic beam with a narrow rectangular cross section is considered (see Figure 1). According to the Timoshenko-type beam model developed by Murakami-Yamakawa, the problem is governed by means of the following equations of motion

$$\begin{aligned} & N' - 2c\rho \overline{\overline{U}} = 0 \\ & M' - Q - \frac{2}{3}c^3 \rho \overline{\overline{\phi}} = 0 \\ & Q' - 2c\rho \overline{\overline{V}} = 0 \end{aligned} \tag{1}$$

where, U and V are the longitudinal and transverse beam displacement, ϕ is the rotational beam displacement, N is the longitudinal force, M is the bending force, Q is the shear force and ρ is the mass density.

$$(\cdot)' \equiv \frac{\partial(\cdot)}{\partial X}; \qquad \overline{(\cdot)} \equiv \frac{\partial(\cdot)}{\partial t} \tag{2}$$

besides, in (1)

$$N = C_{11}U' + C_{13}(\phi + V'), \quad M = C_{22}\phi', \quad Q = C_{13}U' + C_{33}(\phi + V') \tag{3}$$

where C_{11}, C_{13}, C_{22}, C_{33} are coefficients and its definitions may be found in reference[4].

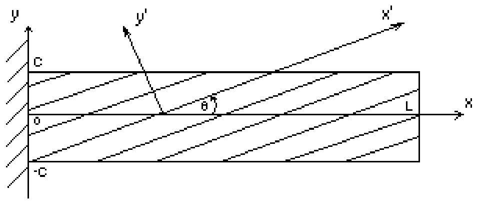

Figure 1: Analyzed Anisotropic Beam and Reference System.

Let us consider normal modes of vibration

$$U_{(x,t)} = u_{(x)}\cos(\omega t); \qquad \phi_{(x,t)} = \varphi_{(x)}\cos(\omega t); \qquad V_{(x,t)} = v_{(x)}\cos(\omega t) \tag{4}$$

where ω are the sought frequencies, and $u_{(x)}$, $\varphi_{(x)}$, $v_{(x)}$ denote the amplitudes of the displacements. Likewise we define some relationships

$$x = \frac{X}{L} \qquad 0 \le x \le 1; \qquad c = 1; \qquad \lambda^* = 2\rho\omega^2 L \tag{5}$$

The substitution of (4) and (5) into (1) yields the following system

$$\begin{aligned} &N' + \lambda^* u = 0 \\ &M' - Q.L + \frac{\lambda^*}{3}\varphi = 0 \\ &Q' + \lambda^* v = 0 \end{aligned} \tag{6}$$

Now, we apply the Virtual Work theorem in (6) with an arbitrary set of infinitesimal virtual displacements δu, $\delta\varphi$, δv. Then through integration by parts we may transform (6) into:

$$\left|N\delta u\right|_0^1 - (N,\delta u') + \lambda^*(u,\delta u) = 0$$
$$\left|M\delta\varphi\right|_0^1 - (M,\delta\varphi') - (Q.L - \frac{\lambda^*}{3}\varphi,\delta\varphi) = 0 \qquad (7)$$
$$\left|Q\delta v\right|_0^1 - (Q,\delta v') + \lambda^*(v,\delta v) = 0$$

At this stage, according to the WEM method, we define the extremizing sequences:

$$u = \sum_{i=1}^{n} A_i sin(\beta_i x) + A_0 x + A^*, \qquad u' = \sum_{i=1}^{n} A_i \beta_i \cos(\beta_i x) + A_0$$
$$\varphi = \sum_{i=1}^{n} B_i \cos(\beta_i x) + B_0, \qquad \varphi' = -\sum_{i=1}^{n} B_i \beta_i sin(\beta_i x) \qquad (8)$$
$$v = \sum_{i=1}^{n} D_i sin(\beta_i x) + D_0 x + D^*, \qquad v' = \sum_{i=1}^{n} D_i \beta_i \cos(\beta_i x) + D_0$$

where $\beta_i = i\,\pi$, with i = 1, 2, 3, .., n.

For the sake of brevity the theorems and corollaries that support the theory are not include (see for instance Filipich et al[8]). Before insert (8) in (7), the geometric (essential) Boundary Conditions (BC) of the beam under study are imposed to (8). In case that one or more BC are not satisfied, (7) is extended through Lagrange multipliers to consider these restrictions. Finally, after some algebra manipulation, the characteristic equation is obtained for each beam case, as will be shown in the next section.

3 SOLUTION OF THE PROBLEM THROUGH WEM

3.1 Cantilever Beam

The Boundary Conditions in this case are:

$$U_{(0,t)} = \phi_{(0,t)} = V_{(0,t)} = N_{(L,t)} = M_{(L,t)} = Q_{(L,t)} = 0 \qquad (9)$$

After applying this BC in our extremizing sequences (8), we note that A* = D* = 0, but the essential condition φ(0) = 0 is not satisfied identically. So the Lagrange multiplier Z is added to the second equation of (7), that becomes into:

$$-(N,\delta u') + \lambda^*(u,\delta u) = 0$$
$$-(M,\delta\varphi') - (Q.L - \frac{\lambda^*}{3}\varphi,\delta\varphi) - Z(\sum \delta B_i + \delta B_0) = 0 \qquad (10)$$
$$-(Q,\delta v') + \lambda^*(v,\delta v) = 0$$

In order to obtain our characteristic equation, we need to replace (8) in (3)

$$N = \sum_{i=1}^{n} P_i \cos(\beta_i x) + P_0 , \qquad M = \sum_{i=1}^{n} T_i sin(\beta_i x), \qquad Q = \sum_{i=1}^{n} R_i \cos(\beta_i x) + R_0 \tag{11}$$

where

$$\begin{aligned} P_i &= \frac{C_{11}}{L}\beta_i A_i + \frac{C_{13}}{L}(B_i L + \beta_i D_i) \quad , \quad P_0 = \frac{C_{11}}{L} A_0 + \frac{C_{13}}{L}(B_0 L + D_0) \\ T_i &= -\frac{C_{22}}{L}\beta_i B_i \\ R_i &= \frac{C_{13}}{L}\beta_i A_i + \frac{C_{33}}{L}(B_i L + \beta_i D_i) \quad , \quad R_0 = \frac{C_{13}}{L} A_0 + \frac{C_{33}}{L}(B_0 L + D_0) \end{aligned} \tag{12}$$

The next step consist on substituting each term of (10) for these last relationships, which yield:

$$\begin{cases} (N, \delta u') = \frac{1}{2}\sum_{i=1}^{n} \delta A_i \beta_i P_i + \delta A_0 P_0 \\ (u, \delta u) = \sum_{i=1}^{n} \delta A_i [\frac{A_i}{2} + A_0 I_i^1] + \delta A_0 [\sum_{i=1}^{n} A_i I_i^1 + \frac{A_0}{3}] \end{cases} \tag{13}$$

$$\begin{cases} (M, \delta\varphi') = -\frac{1}{2}\sum_{i=1}^{n} \delta B_i \beta_i T_i \\ (Q.L - \frac{\lambda^*}{3}\varphi, \delta\varphi) = \frac{1}{2}\sum_{i=1}^{n} \delta B_i \gamma_i + \delta B_0 \gamma_0 \end{cases} \tag{14}$$

$$\begin{cases} (Q, \delta v') = \frac{1}{2}\sum_{i=1}^{n} \delta D_i \beta_i R_i + \delta D_0 R_0 \\ (v, \delta v) = \sum_{i=1}^{n} \delta D_i [\frac{D_i}{2} + D_0 I_i^1] + \delta D_0 [\sum_{i=1}^{n} D_i I_i^1 + \frac{D_0}{3}] \end{cases} \tag{15}$$

where

$$\begin{aligned} & Q.L - \frac{\lambda^*}{3}\varphi = \sum_{i=1}^{n} \gamma_i Cos(\beta_i x) + \gamma_0 \\ & \gamma_i = (R_i L - \frac{\lambda^*}{3} B_i) \quad , \quad \gamma_0 = (R_0 L - \frac{\lambda^*}{3} B_0) \end{aligned} \tag{16}$$

$$I_i^1 = \int_0^1 x.Sen(\beta_i x) dx \tag{17}$$

Finally introducing (13), (14) and (15) into (10) and grouping the variations of the unknowns, we have the following two system of equations:

$$\begin{cases} \sum_{i=1}^{n} \delta A_i [-\frac{\beta_i}{2} P_i + \lambda^* (\frac{A_i}{2} + A_0 I_i^1)] = 0 \\ \sum_{i=1}^{n} \delta B_i [-Z + \frac{\beta_i}{2} T_i - \frac{\gamma_i}{2}] = 0 \\ \sum_{i=1}^{n} \delta D_i [-\frac{\beta_i}{2} R_i + \lambda^* (\frac{D_i}{2} + D_0 I_i^1)] = 0 \end{cases} \tag{18}$$

$$\begin{cases} \delta A_0 [-P_0 + \lambda^* (\sum_{i=1}^{n} A_i I_i^1 + \frac{A_0}{3})] = 0 \\ \delta B_0 [-Z - \gamma_0] = 0 \\ \delta D_0 [-R_0 + \lambda^* (\sum_{i=1}^{n} D_i I_i^1 + \frac{D_0}{3})] = 0 \end{cases} \tag{19}$$

Once the terms between brackets are equated to zero, the characteristic equation to solve the natural frequencies is found.

3.2 Simply Supported Beam

For this case we have the following Boundary Conditions:

$$U_{(0,t)} = M_{(0,t)} = V_{(0,t)} = N_{(L,t)} = M_{(L,t)} = V_{(L,t)} = 0 \tag{20}$$

Afterward of replacing this BC in the extremizing sequences (8), we notice that all the essential conditions are satisfied identically; hence the Lagrange multiplier is not needed, as is the case in the cantilever beam. Consequently we will use the equation (7) to solve the problem. Besides we note that $A^* = D_0 = D^* = 0$.
The procedure is analogous to developed in the above section.

4 NUMERICAL RESULTS

The natural frequencies obtained with WEM are compared with results from Murakami-Yamakawa[4] (M-Y) beam theory and Cortinez-Machado[7] (FEM) two-dimensional analysis with FEM. In Cortinez-Machado, the beam was modelled by means of a plane state of stress corresponding to an anisotropic elastic body. Natural frequencies were determined by means of the finite element system Flex-Pde[9].

Table 1,2,3 and 4 show the non-dimensional natural frequencies of vibration given by

$$\overline{\omega} = \frac{c.\omega}{(c/L)^2 \sqrt{E_{11}/3\rho}} \tag{21}$$

Two different material properties were considered,
Material 1: E_{22}/E_{11}= 0.1, G_{12}/E_{11}= 0.0333, ν_{12}= 0.3
Material 2: E_{22}/E_{11}= 0.333, G_{12}/E_{11}= 0.1667, ν_{12}= 0.25

Table 1 and 2 show the natural frequencies of vibration for cantilever beams, with the slenderness ratios *c/L* = 1/10 and 1/30, as a function of the θ angle between the strong fiber direction of the material and the longitudinal x-axis (see Figure 1). It may be noted that the M-Y and FEM results are practically coincident with the present ones for all the cases analyzed. However, in the short beam case (*c/L* = 1/10) with material 1 (more anisotropic), the third frequency obtained with the present work and with the FEM model, was not detected by the M-Y calculation. For that reason it was necessary to calculate the fourth frequency in order to verify that this corresponds at the third of M-Y.

c/L=1/10	0º			30º		
mode	M-Y	FEM	WEM	M-Y	FEM	WEM
1	2,80	2,81	2,80	1,36	1,47	1,32
2	10,15	10,32	10,13	7,10	7,56	7,07
3	27.20	21,28	20,34	11.20	11.65	11,02
4		27.20	27,20		18,60	17,20

c/L=1/10	60º			90º		
mode	M-Y	FEM	WEM	M-Y	FEM	WEM
1	1,05	1,05	1,03	1,10	1,07	1,07
2	6,00	5,99	5,79	6,00	5,75	5,71
3	8,15	8.19	8,13	8.50	8.60	8,60
4		14,97	14,30		13,66	13,44

c/L=1/30	0º			30º		
mode	M-Y	FEM	WEM	M-Y	FEM	WEM
1	3,40	3,42	3,41	1,35	1,39	1,35
2	18,30	18,87	18,33	8,30	8,60	8,33
3	43,50	45,45	43,33	23,50	23,55	22,66

c/L=1/30	60º			90º		
mode	M-Y	FEM	WEM	M-Y	FEM	WEM
1	1,05	1,06	1,05	1,10	1,10	1,10
2	7,00	6,67	6,48	6,90	6,85	6,78
3	17,90	18,66	17,76	18,40	18,75	18,36

Table 1: Non-dimensional natural frequencies ϖ for cantilever beams,
material 1 with *c/L* = 1/10 and *c/L* = 1/30.

c/L=1/10	0º			30º		
mode	M-Y	FEM	WEM	M-Y	FEM	WEM
1	3,30	3,32	3,31	2,55	2,61	2,56
2	16,10	16,29	16,08	13,70	13,96	13,64
3	27.20	27.22	27,20	20,70	20.82	20,75

c/L=1/10	60º			90º		
mode	M-Y	FEM	WEM	M-Y	FEM	WEM
1	2,04	2,07	2,06	2,00	1,98	1,98
2	11,40	11,56	11,42	11,00	10,97	10,90
3	16.10	16.37	16,33	15.90	15.71	15,71

c/L=1/30	0º			30º		
mode	M-Y	FEM	WEM	M-Y	FEM	WEM
1	3,50	3,50	3,49	2,61	2,64	2,63
2	21,00	21,28	21,03	16,30	16,30	16,15
3	55,80	57,02	55,60	43,70	44,48	43,79

c/L=1/30	60º			90º		
mode	M-Y	FEM	WEM	M-Y	FEM	WEM
1	2,13	2,11	2,10	2,00	2,02	2,02
2	13,20	13,14	12,96	12,50	12,55	12,47
3	35,60	36,37	35,43	34,00	34,74	34,02

Table 2: Non-dimensional natural frequencies ϖ for cantilever beams, material 2 with *c/L* = 1/10 and *c/L* = 1/30.

Table 3 and 4 show the results corresponding to simply supported beams for material 1 and 2, respectively. The comparison with the M-Y and FEM model shows a good agreement for the long beam cases for both materials. But, there is a discrepancy with the third frequency of M-Y results, corresponding to the short beam case (*c/L* = 1/10) with material 1 (more anisotropic).

c/L=1/10	0º			30º		
mode	M-Y	FEM	WEM	M-Y	FEM	WEM
1	6,60	6,76	6,65	3,60	3,67	3,60
2	17,00	17,17	16,44	10.50	10.64	10,48
3	27.40	27,26	25,98	12,60	13,48	12,61
4		27.47	27,20		26,22	24,14

c/L=1/10	60º			90º		
mode	M-Y	FEM	WEM	M-Y	FEM	WEM
1	2,80	2,84	2,82	2,90	2,92	2,91
2	8.00	8.25	8,12	9.00	8.69	8,60
3	10,20	10,37	10,19	10,00	10,06	9,97
4		20,83	20,22		19,19	18,85

c/L=1/30	0º			30º		
mode	M-Y	FEM	WEM	M-Y	FEM	WEM
1	9,30	9,36	9,26	3,80	3,80	3,78
2	32,20	32,92	31,85	15,00	15,01	14,86
3		63,21	59,89	32.20	31.92	31,51

c/L=1/30	60º			90º		
mode	M-Y	FEM	WEM	M-Y	FEM	WEM
1	2,90	2,94	2,93	3,10	3,10	3,09
2	11,87	11,71	11,56	12,50	12,19	12,08
3	24.37	24.69	24,38	26.00	26,07	25,81

Table 3: Non-dimensional natural frequencies ϖ for simply supported beams, material 1 with *c/L* = 1/10 and *c/L* = 1/30.

It should be noted that the numerical values of [4] were read from graphics. The discrepancy among the results from[7] and the present ones may be due to the difference in the modelization (i.e., a 2D beam in[7] and a Timoshenko beam in the present work).

c/L=1/10	0º			30º		
mode	M-Y	FEM	WEM	M-Y	FEM	WEM
1	8,75	8,81	8,78	7,00	7,00	6,93
2	27.50	27.49	27,20	20.50	20.60	20,43
3	27,70	28,66	27,79	23,85	24,95	23,88

c/L=1/10	60º			90º		
mode	M-Y	FEM	WEM	M-Y	FEM	WEM
1	5,60	5,66	5,64	5,49	5,41	5,41
2	16.25	16.50	16,30	16.00	15.87	15,71
3	20,20	20,38	20,14	19,10	19,22	19,14

c/L=1/30	0º			30º		
mode	M-Y	FEM	WEM	M-Y	FEM	WEM
1	9,70	9,75	9,72	7,40	7,35	7,33
2	37,50	37,74	37,32	28,13	28,99	28,70
3		80,92	79,02		61.81	61,35

c/L=1/30	60º			90º		
mode	M-Y	FEM	WEM	M-Y	FEM	WEM
1	5,90	5,89	5,88	5,70	5,67	5,66
2	23,10	23,34	23,15	22,50	22,41	22,25
3	48.75	49.46	48,90	47.50	47.60	47,12

Table 4: Non-dimensional natural frequencies ϖ for simply supported beams, material 2 with c/L = 1/10 and c/L = 1/30.

5 CONCLUSIONS

The solution of the vibration problem of the plane anisotropic beams with shear deformation has been addressed by means of a generalized solution. The eigenvalues found via this solution are exact, as shown by Filipich et al.[10]. Satisfactory accuracy was computed increasing the number of terms to N = 3000. In the application of WEM an extremizing sequence is proposed and introduced in the governing equations and the boundary conditions are imposed to the sequence. After simple algebraic handling, it is possible to systematically arrive to a sequence which involves sums of trigonometric functions. The WEM method constitutes a suitable tool for the solution of vibration problems, among them.

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