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CALCULATION OF THE STABILIZATION PARAMETERS IN SUPG AND PSPG FORMULATIONS

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Abstract. *We describe how we determine the stabilization parameters used in the stabilized finite element formulations in fluid mechanics. These formulations include the interface-tracking and interface-capturing techniques we developed for computation of flow problems with moving boundaries and interfaces. The stabilized formulations we focus on are the streamline-upwind/Petrov-Galerkin (SUPG) and pressure-stabilizing/Petrov-Galerkin (PSPG) methods. The stabilization parameters described here are designed for the semi-discrete and space-time formulations of the advection-diffusion equation and the Navier-Stokes equations of incompressible flows.*

1 INTRODUCTION

Most finite element techniques and computations reported in recent literature for computational fluid mechanics are based on stabilized formulations. The interface-tracking and interface-capturing techniques we developed in recent years (see[1–7]) for flows with moving boundaries and interfaces are also based on stabilized formulations. An interface-tracking technique, such as the Deforming-Spatial-Domain/Stabilized Space-Time (DSD/SST) formulation,[1] requires meshes that "track" the interfaces. The mesh needs to be updated as the flow evolves. In interface-capturing techniques, such as one designed for two-fluid flows, the computations are based on spatial domains that are typically not moving or deforming. An interface function, marking the location of the interface, needs to be computed to "capture" the interface over the non-moving mesh.

In the interface-tracking and interface-capturing techniques we developed, we use the streamline-upwind/Petrov-Galerkin (SUPG),[8] Galerkin/least-squares (GLS),[9] and pressure-stabilizing/Petrov-Galerkin (PSPG)[1] formulations. In the interface-capturing techniques, stabilized semi-discrete formulations are used for both the Navier-Stokes equations of incompressible flows and the advection equation governing the time-evolution of an interface function marking the interface location. These stabilization techniques prevent numerical oscillations and other instabilities in solving problems with advection-dominated flows and when using equal-order interpolation functions for velocity and pressure. In these stabilized formulations, judicious selection of the stabilization parameter, which is almost always known as "τ", plays an important role in determining the accuracy of the formulation. This stabilization parameter involves a measure of the local length scale (also known as "element length") and other parameters such as the local Reynolds and Courant numbers. Various element lengths and τs were proposed starting with those in[8] and,[10] followed by the one introduced in,[11] and those proposed in the subsequently reported SUPG, GLS and PSPG methods. A number of τs, dependent upon spatial and temporal discretizations, were introduced and tested in.[12] More recently, τs which are applicable to higher-order elements were proposed in.[13]

Ways to calculate τs from the element-level matrices and vectors were first introduced in.[14] These new definitions are expressed in terms of the ratios of the norms of the relevant matrices or vectors. They take into account the local length scales, advection field and the element-level Reynolds number. Based on these definitions, a τ can be calculated for each element, or even for each element node or degree of freedom or element equation. Certain variations and complements of these new τs were introduced in.[4,15–17] In this paper, we describe the element-matrix-based and element-vector-based τs designed for the semi-discrete and space-time formulations of the advection-diffusion equation and the Navier-Stokes equations of incompressible flows. We also describe approximate versions of these τs that are based on the local length scales for the advection- and diffusion-dominated limits.

2 GOVERNING EQUATIONS

Let $\Omega_t \subset I\!R^{n_{sd}}$ be the spatial fluid mechanics domain with boundary Γ_t at time $t \in (0, T)$, where the subscript t indicates the time-dependence of the spatial domain. The Navier-Stokes equations of incompressible flows can be written on Ω_t and $\forall t \in (0, T)$ as

$$\rho(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{u} - \mathbf{f}) - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = 0, \tag{1}$$

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0, \tag{2}$$

where ρ, $\mathbf{u}$ and $\mathbf{f}$ are the density, velocity and the external force, and $\boldsymbol{\sigma}$ is the stress tensor:

$$\boldsymbol{\sigma}(p, \mathbf{u}) = -p\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(\mathbf{u}), \quad \boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}((\boldsymbol{\nabla}\mathbf{u}) + (\boldsymbol{\nabla}\mathbf{u})^T). \tag{3}$$

Here p is pressure, $\mathbf{I}$ is the identity tensor, $\mu = \rho\nu$ is viscosity, ν is the kinematic viscosity, and $\boldsymbol{\varepsilon}(\mathbf{u})$ is the strain-rate tensor. The essential and natural boundary conditions for Eq. (1) are represented as

$$\mathbf{u} = \mathbf{g} \text{ on } (\Gamma_\mathrm{t})_\mathrm{g}, \qquad \mathbf{n} \cdot \boldsymbol{\sigma} = \mathbf{h} \text{ on } (\Gamma_\mathrm{t})_\mathrm{h}, \tag{4}$$

where $(\Gamma_t)_g$ and $(\Gamma_t)_h$ are complementary subsets of the boundary Γ_t, $\mathbf{n}$ is the unit normal vector, and $\mathbf{g}$ and $\mathbf{h}$ are given functions. A divergence-free velocity field $\mathbf{u}_0(\mathbf{x})$ is specified as the initial condition.

If the problem does not involve any moving boundaries or interfaces, the spatial domain does not need to change with respect to time, and the subscript t can be dropped from Ω_t and Γ_t. This might be the case even for flows with moving boundaries and interfaces, if in the formulation used the spatial domain is not defined to be the part of the space occupied by the fluid(s). For example, we can have a fixed spatial domain, and model the fluid-fluid interfaces by assuming that the domain is occupied by two immiscible fluids, A and B, with densities ρ_A and ρ_B and viscosities μ_A and μ_B. In modeling a free-surface problem where Fluid B is irrelevant, we assign a sufficiently low density to Fluid B. An interface function ϕ serves as a marker identifying Fluid A and B with the definition $\phi = \{1 \text{ for Fluid A and } 0 \text{ for Fluid B}\}$. The interface between the two fluids is approximated to be at $\phi = 0.5$. In this context, ρ and μ are defined as

$$\rho = \phi\rho_A + (1 - \phi)\rho_B, \quad \mu = \phi\mu_A + (1 - \phi)\mu_B. \tag{5}$$

The evolution of the interface function ϕ, and therefore the motion of the interface, is governed by a time-dependent advection equation, written on Ω and $\forall t \in (0, T)$ as

$$\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}\phi = 0. \tag{6}$$

As a generalization of Eq. (6), let us consider over a domain Ω with boundary Γ the following time-dependent advection-diffusion equation, written on Ω and $\forall t \in (0, T)$ as

$$\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}\phi - \boldsymbol{\nabla} \cdot (\nu\boldsymbol{\nabla}\phi) = 0, \tag{7}$$

where ϕ represents the quantity being transported (e.g., temperature, concentration), and ν is the diffusivity. The essential and natural boundary conditions associated with Eq. (7) are represented as

$$\phi = \mathrm{g} \text{ on } \Gamma_{\mathrm{g}}, \qquad \mathbf{n} \cdot \nu \boldsymbol{\nabla} \phi = \mathrm{h} \text{ on } \Gamma_{\mathrm{h}}. \tag{8}$$

A function $\phi_0(\mathbf{x})$ is specified as the initial condition.

3 STABILIZED FORMULATION FOR ADVECTION-DIFFUSION EQUATION

Let us assume that we have constructed some suitably-defined finite-dimensional trial solution and test function spaces $\mathcal{S}_\phi^h$ and $\mathcal{V}_\phi^h$. The stabilized finite element formulation of Eq. (7) can then be written as follows: find $\phi^h \in \mathcal{S}_\phi^h$ such that $\forall w^h \in \mathcal{V}_\phi^h$:

$$\begin{aligned} &\int_\Omega w^h \left(\frac{\partial \phi^h}{\partial t} + \mathbf{u}^h \cdot \boldsymbol{\nabla} \phi^h \right) d\Omega + \int_\Omega \boldsymbol{\nabla} w^h \cdot \nu \boldsymbol{\nabla} \phi^h d\Omega - \int_{\Gamma_{\mathrm{h}}} w^h \mathrm{h}^h d\Gamma \\ + &\sum_{e=1}^{n_{el}} \int_{\Omega^e} \tau_{\mathrm{SUPG}} \mathbf{u}^h \cdot \boldsymbol{\nabla} w^h \left(\frac{\partial \phi^h}{\partial t} + \mathbf{u}^h \cdot \boldsymbol{\nabla} \phi^h - \boldsymbol{\nabla} \cdot \left(\nu \boldsymbol{\nabla} \phi^h \right) \right) d\Omega = 0. \end{aligned} \tag{9}$$

Here n_{el} is the number of elements, Ω^e is the element domain, and τ_{SUPG} is the SUPG stabilization parameter.

4 ELEMENT-MATRIX-BASED STABILIZATION PARAMETERS FOR ADVECTION-DIFFUSION EQUATION

Let us use the notation $\mathbf{b} : \int_{\Omega^e} (\ldots) d\Omega : \mathbf{b}_{\mathrm{V}}$ to denote the element-level matrix $\mathbf{b}$ and element-level vector $\mathbf{b}_{\mathrm{V}}$ corresponding to the element-level integration term $\int_{\Omega^e} (\ldots) d\Omega$. We define the following element-level matrices and vectors:

$$\mathbf{m} : \quad \int_{\Omega^e} w^h \frac{\partial \phi^h}{\partial t} d\Omega \quad : \mathbf{m}_{\mathrm{V}}, \tag{10}$$

$$\mathbf{c} : \quad \int_{\Omega^e} w^h \mathbf{u}^h \cdot \boldsymbol{\nabla} \phi^h d\Omega \quad : \mathbf{c}_{\mathrm{V}}, \tag{11}$$

$$\mathbf{k} : \quad \int_{\Omega^e} \boldsymbol{\nabla} w^h \cdot \nu \boldsymbol{\nabla} \phi^h d\Omega \quad : \mathbf{k}_{\mathrm{V}}, \tag{12}$$

$$\tilde{\mathbf{k}} : \quad \int_{\Omega^e} \mathbf{u}^h \cdot \boldsymbol{\nabla} w^h \; \mathbf{u}^h \cdot \boldsymbol{\nabla} \phi^h d\Omega \quad : \tilde{\mathbf{k}}_{\mathrm{V}}, \tag{13}$$

$$\tilde{\mathbf{c}} : \quad \int_{\Omega^e} \mathbf{u}^h \cdot \boldsymbol{\nabla} w^h \frac{\partial \phi^h}{\partial t} d\Omega \quad : \tilde{\mathbf{c}}_{\mathrm{V}}. \tag{14}$$

We define the element-level Reynolds and Courant numbers as follows:

$$
\begin{aligned}
Re &= \frac{\|\mathbf{u}^h\|^2}{\nu}\frac{\|\mathbf{c}\|}{\|\tilde{\mathbf{k}}\|}, && (15)\\
Cr_u &= \frac{\Delta t}{2}\frac{\|\mathbf{c}\|}{\|\mathbf{m}\|}, && (16)\\
Cr_\nu &= \frac{\Delta t}{2}\frac{\|\mathbf{k}\|}{\|\mathbf{m}\|}, && (17)\\
Cr_{\tilde{\nu}} &= \frac{\Delta t}{2}\tau_{\mathrm{SUPG}}\frac{\|\tilde{\mathbf{k}}\|}{\|\mathbf{m}\|}, && (18)
\end{aligned}
$$

where $\|\mathbf{b}\|$ is the norm of matrix $\mathbf{b}$.

The components of element-matrix-based τ_{SUPG} are defined as follows:

$$
\begin{aligned}
\tau_{\mathrm{S1}} &= \frac{\|\mathbf{c}\|}{\|\tilde{\mathbf{k}}\|}, && (19)\\
\tau_{\mathrm{S2}} &= \frac{\Delta t}{2}\frac{\|\mathbf{c}\|}{\|\tilde{\mathbf{c}}\|}, && (20)\\
\tau_{\mathrm{S3}} &= \tau_{\mathrm{S1}} Re = \left(\frac{\|\mathbf{c}\|}{\|\tilde{\mathbf{k}}\|}\right) Re. && (21)
\end{aligned}
$$

To construct τ_{SUPG} from its components we proposed in[14] the form

$$
\tau_{\mathrm{SUPG}} = \left(\frac{1}{\tau_{\mathrm{S1}}^r} + \frac{1}{\tau_{\mathrm{S2}}^r} + \frac{1}{\tau_{\mathrm{S3}}^r}\right)^{-\frac{1}{r}}, \qquad (22)
$$

which is based on the inverse of τ_{SUPG} being defined as the r-norm of the vector with components $\frac{1}{\tau_{\mathrm{S1}}}$, $\frac{1}{\tau_{\mathrm{S2}}}$ and $\frac{1}{\tau_{\mathrm{S3}}}$. We note that the higher the integer r is, the sharper the switching between τ_{S1}, τ_{S2} and τ_{S3} becomes.

The components of the element-vector-based τ_{SUPG} are defined as follows:

$$
\begin{aligned}
\tau_{\mathrm{SV1}} &= \frac{\|\mathbf{c}_{\mathrm{V}}\|}{\|\tilde{\mathbf{k}}_{\mathrm{V}}\|}, && (23)\\
\tau_{\mathrm{SV2}} &= \frac{\|\mathbf{c}_{\mathrm{V}}\|}{\|\tilde{\mathbf{c}}_{\mathrm{V}}\|}, && (24)\\
\tau_{\mathrm{SV3}} &= \tau_{\mathrm{SV1}} Re = \left(\frac{\|\mathbf{c}_{\mathrm{V}}\|}{\|\tilde{\mathbf{k}}_{\mathrm{V}}\|}\right) Re. && (25)
\end{aligned}
$$

With these three components,

$$
(\tau_{\mathrm{SUPG}})_{\mathrm{V}} = \left(\frac{1}{\tau_{\mathrm{SV1}}^r} + \frac{1}{\tau_{\mathrm{SV2}}^r} + \frac{1}{\tau_{\mathrm{SV3}}^r}\right)^{-\frac{1}{r}}. \qquad (26)
$$

Remark 1 *The definition of τ_{SUPG} given by Eqs. (23)-(26) can be seen as a nonlinear definition because it depends on the solution. However, in marching from time level n to $n+1$ the element vectors can be evaluated at level n. This might be preferable in some cases, as it spares us from ending up with a nonlinear semi-discrete equation system.*

5 STABILIZED FORMULATION FOR NAVIER-STOKES EQUATIONS

Let us assume that we have some suitably-defined finite-dimensional trial solution and test function spaces for velocity and pressure: $\mathcal{S}_{\mathbf{u}}^h$, $\mathcal{V}_{\mathbf{u}}^h$, $\mathcal{S}_p^h$ and $\mathcal{V}_p^h = \mathcal{S}_p^h$. The stabilized finite element formulation of Eqs. (1)-(2) can then be written as follows: find $\mathbf{u}^h \in \mathcal{S}_{\mathbf{u}}^h$ and $p^h \in \mathcal{S}_p^h$ such that $\forall \mathbf{w}^h \in \mathcal{V}_{\mathbf{u}}^h$ and $q^h \in \mathcal{V}_p^h$:

$$
\begin{aligned}
& \int_\Omega \mathbf{w}^h \cdot \rho \left(\frac{\partial \mathbf{u}^h}{\partial t} + \mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{u}^h - \mathbf{f} \right) d\Omega + \int_\Omega \boldsymbol{\varepsilon}(\mathbf{w}^h) : \boldsymbol{\sigma}(p^h, \mathbf{u}^h) d\Omega - \int_{\Gamma_\mathrm{h}} \mathbf{w}^h \cdot \mathbf{h}^h d\Gamma \\
+ \; & \int_\Omega q^h \boldsymbol{\nabla} \cdot \mathbf{u}^h d\Omega + \sum_{e=1}^{n_{el}} \int_{\Omega^e} \frac{1}{\rho} [\tau_{\mathrm{SUPG}} \rho \mathbf{u}^h \cdot \nabla \mathbf{w}^h + \tau_{\mathrm{PSPG}} \nabla q^h] \cdot \\
& \qquad \left[\rho \left(\frac{\partial \mathbf{u}^h}{\partial t} + \mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{u}^h \right) - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(p^h, \mathbf{u}^h) - \rho \mathbf{f} \right] d\Omega \\
+ \; & \sum_{e=1}^{n_{el}} \int_{\Omega^e} \tau_{\mathrm{LSIC}} \boldsymbol{\nabla} \cdot \mathbf{w}^h \rho \boldsymbol{\nabla} \cdot \mathbf{u}^h d\Omega \; = \; 0. \qquad (27)
\end{aligned}
$$

Here τ_{PSPG} and τ_{LSIC} are the PSPG and LSIC (least-squares on incompressibility constraint) stabilization parameters.

6 ELEMENT-MATRIX-BASED STABILIZATION PARAMETERS FOR NAVIER-STOKES EQUATIONS

We define the following element-level matrices and vectors:

$$\mathbf{m}: \quad \int_{\Omega^e} \mathbf{w}^h \cdot \rho \frac{\partial \mathbf{u}^h}{\partial t} d\Omega \quad : \mathbf{m}_{\mathrm{V}}, \tag{28}$$

$$\mathbf{c}: \quad \int_{\Omega^e} \mathbf{w}^h \cdot \rho (\mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{u}^h) d\Omega \quad : \mathbf{c}_{\mathrm{V}}, \tag{29}$$

$$\mathbf{k}: \quad \int_{\Omega^e} \boldsymbol{\varepsilon}(\mathbf{w}^h) : 2\mu \boldsymbol{\varepsilon}(\mathbf{u}^h) d\Omega \quad : \mathbf{k}_{\mathrm{V}}, \tag{30}$$

$$\mathbf{g}: \quad \int_{\Omega^e} (\boldsymbol{\nabla} \cdot \mathbf{w}^h) p^h d\Omega \quad : \mathbf{g}_{\mathrm{V}}, \tag{31}$$

$$\mathbf{g}^{\mathrm{T}}: \quad \int_{\Omega^e} q^h (\boldsymbol{\nabla} \cdot \mathbf{u}^h) d\Omega \quad : \mathbf{g}_{\mathrm{V}}^{\mathrm{T}}, \tag{32}$$

$$\tilde{\mathbf{k}}: \quad \int_{\Omega^e} (\mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{w}^h) \cdot \rho (\mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{u}^h) d\Omega \quad : \tilde{\mathbf{k}}_{\mathrm{V}}, \tag{33}$$

$$\tilde{\mathbf{c}}: \quad \int_{\Omega^e} (\mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{w}^h) \cdot \rho \frac{\partial \mathbf{u}^h}{\partial t} d\Omega \quad : \tilde{\mathbf{c}}_{\mathrm{V}}, \tag{34}$$

$$\tilde{\gamma}: \quad \int_{\Omega^e} (\mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{w}^h) \cdot \boldsymbol{\nabla} p^h d\Omega \quad : \tilde{\gamma}_{\mathrm{V}}, \tag{35}$$

$$\beta: \quad \int_{\Omega^e} \boldsymbol{\nabla} q^h \cdot \frac{\partial \mathbf{u}^h}{\partial t} d\Omega \quad : \beta_{\mathrm{V}}, \tag{36}$$

$$\gamma: \quad \int_{\Omega^e} \boldsymbol{\nabla} q^h \cdot (\mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{u}^h) d\Omega \quad : \gamma_{\mathrm{V}}, \tag{37}$$

$$\theta: \quad \int_{\Omega^e} \boldsymbol{\nabla} q^h \cdot \boldsymbol{\nabla} p^h d\Omega \quad : \theta_{\mathrm{V}}, \tag{38}$$

$$\mathbf{e}: \quad \int_{\Omega^e} (\boldsymbol{\nabla} \cdot \mathbf{w}^h) \rho (\boldsymbol{\nabla} \cdot \mathbf{u}^h) d\Omega \quad : \mathbf{e}_{\mathrm{V}}. \tag{39}$$

Remark 2 *In the definition of the element-level matrices listed above, we assume that* $\mathbf{u}^h$ *appearing in the advective operator (i.e. in* $\mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{u}^h$ *and* $\mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{w}^h$*) is evaluated at time level* n *rather than* $n+1$*. The definition would essentially be the same if we, alternatively, assumed that it is evaluated at time level* $n+1$ *but nonlinear iteration level* i *rather than* $i+1$*. Except, in the first option, in the advective operator we use* $(\mathbf{u}^h)_n$*, whereas in the second option we use* $(\mathbf{u}^h)^i_{n+1}$*. The second option can be seen as a nonlinear definition. The first option might be preferable in some cases, as it spares us from another level of nonlinearity coming from the way* τ *is defined. In the definition of the element-level-vectors, we face the same choices in terms of the evaluation of* $\mathbf{u}^h$ *in the advective operator.*

The element-level Reynolds and Courant numbers are defined the same way as they were defined before, as given by Eqs. (15)-(18). The components of the element-matrix-based τ_{SUPG} are defined the same way as they were defined before, as given by Eqs. (19)-(21). τ_{SUPG} is constructed from its components the same way as it was constructed before, as give by Eq. (22). The components of the element-vector-based τ_{SUPG} are defined the same way as they were defined before, as given by Eqs. (23)-(25). The construction of $(\tau_{\mathrm{SUPG}})_{\mathrm{V}}$ is also the same as it was before, given by Eq. (26).

The components of the element-matrix-based τ_{PSPG} are defined as follows:

$$\tau_{\mathrm{P1}} = \frac{\|\mathbf{g}^{\mathrm{T}}\|}{\|\gamma\|}, \tag{40}$$

$$\tau_{\mathrm{P2}} = \frac{\Delta t}{2}\frac{\|\mathbf{g}^{\mathrm{T}}\|}{\|\beta\|}, \tag{41}$$

$$\tau_{\mathrm{P3}} = \tau_{\mathrm{P1}} Re = \left(\frac{\|\mathbf{g}^{\mathrm{T}}\|}{\|\gamma\|}\right) Re. \tag{42}$$

τ_{PSPG} is constructed from its components as follows:

$$\tau_{\mathrm{PSPG}} = \left(\frac{1}{\tau_{\mathrm{P1}}^{r}} + \frac{1}{\tau_{\mathrm{P2}}^{r}} + \frac{1}{\tau_{\mathrm{P3}}^{r}}\right)^{-\frac{1}{r}}. \tag{43}$$

The components of the element-vector-based τ_{PSPG} are defined as follows:

$$\tau_{\mathrm{PV1}} = \tau_{\mathrm{P1}}, \tag{44}$$

$$\tau_{\mathrm{PV2}} = \tau_{\mathrm{PV1}}\frac{\|\gamma_{\mathrm{V}}\|}{\|\beta_{\mathrm{V}}\|}, \tag{45}$$

$$\tau_{\mathrm{PV3}} = \tau_{\mathrm{PV1}} Re. \tag{46}$$

With these components,

$$(\tau_{\mathrm{PSPG}})_{\mathrm{V}} = \left(\frac{1}{\tau_{\mathrm{PV1}}^{r}} + \frac{1}{\tau_{\mathrm{PV2}}^{r}} + \frac{1}{\tau_{\mathrm{PV3}}^{r}}\right)^{-\frac{1}{r}}. \tag{47}$$

The element-matrix-based τ_{LSIC} is defined as follows:

$$\tau_{\mathrm{LSIC}} = \frac{\|\mathbf{c}\|}{\|\mathbf{e}\|}. \tag{48}$$

We define the element-vector-based τ_{LSIC} as:

$$(\tau_{\mathrm{LSIC}})_{\mathrm{V}} = \tau_{\mathrm{LSIC}}. \tag{49}$$

Remark 3 *We can also calculate a separate τ for each element node, or degree of freedom, or element equation. In that case, each component of τ would be calculated separately for each element node, or degree of freedom, or element equation. For this, we first represent an element matrix $\mathbf{b}$ in terms of its row matrices: $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{n_{ex}}$ and an element vector $\mathbf{b}_V$ in terms of its subvectors: $(\mathbf{b}_V)_1, (\mathbf{b}_V)_2, \ldots, (\mathbf{b}_V)_{n_{ex}}$. If we want a separate τ for each element node, then $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{n_{ex}}$ and $(\mathbf{b}_V)_1, (\mathbf{b}_V)_2, \ldots, (\mathbf{b}_V)_{n_{ex}}$ would be the row matrices and subvectors corresponding to each element node, with $n_{ex} = n_{en}$, where n_{en} is the number of element nodes. If we want a separate τ for each degree of freedom, then $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{n_{ex}}$ and $(\mathbf{b}_V)_1, (\mathbf{b}_V)_2, \ldots, (\mathbf{b}_V)_{n_{ex}}$ would be the row matrices and subvectors corresponding to each degree of freedom, with $n_{ex} = n_{dof}$, where n_{dof} is the number of degrees of freedom. If we want a separate τ for each element equation, then $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{n_{ex}}$ and $(\mathbf{b}_V)_1, (\mathbf{b}_V)_2, \ldots, (\mathbf{b}_V)_{n_{ex}}$ would be the row matrices and subvectors corresponding to each element equation, with $n_{ex} = n_{ee}$, where n_{ee} is the number of element equations. Based on this, the components of τ would be calculated using the norms of these row matrices or subvectors, instead of the element matrices or vectors. For example, a separate τ_{S1} or τ_{SV1} for each element node would be calculated by using the expression*

$$(\tau_{S1})_a = \frac{\|\mathbf{c}_a\|}{\|\tilde{\mathbf{k}}_a\|}, \quad a = 1, 2, \ldots, n_{en} \tag{50}$$

or

$$(\tau_{SV1})_a = \frac{\|(\mathbf{c}_V)_a\|}{\|(\tilde{\mathbf{k}}_V)_a\|}, \quad a = 1, 2, \ldots, n_{en}. \tag{51}$$

In flow computations, the τs calculated for the element nodes or element equations would be used in interpolating the values of τs at the integration points.

Remark 4 *The concept of calculating a separate τ for each element node or equation can be extended to calculating a separate τ for each global node or equation. This can be accomplished by first representing a global matrix or vector in terms of its row matrices or subvectors associated with the global nodes or equations, and then by calculating the components of τ using the norms of these global row matrices or subvectors. With this approach, applying the class of stabilization techniques described in this paper to element-free methods would become more direct.*

Remark 5 *We can also calculate a separate τ for each integration point by using for that integration point the ratios of the norms of the element matrices or vectors contributed by that integration point. For example, a separate τ_{S1} or τ_{SV1} for each element integration point l would be calculated by using the expression*

$$(\tau_{S1})_l = \frac{\|\mathbf{c}_l\|}{\|\tilde{\mathbf{k}}_l\|}, \quad l = 1, 2, \ldots, n_{int} \tag{52}$$

or

$$(\tau_{SV1})_l = \frac{\|(\mathbf{c}_V)_l\|}{\|(\tilde{\mathbf{k}}_V)_l\|}, \quad l = 1, 2, \ldots, n_{int}. \tag{53}$$

Here n_{int} is the number of integration points, $\mathbf{c}_l$ and $\tilde{\mathbf{k}}_l$ are the element matrices contributed by the integration point l, and $(\mathbf{c}_V)_l$ and $(\tilde{\mathbf{k}}_V)_l$ are the element vectors contributed by the integration point l.

7 UGN-BASED STABILIZATION PARAMETERS FOR NAVIER-STOKES EQUATIONS

For the purpose of comparison, we define here also the stabilization parameters that are based on an earlier definition of the length scale h:[11]

$$h_{\mathrm{UGN}} = 2\ \|\mathbf{u}^h\| \left(\sum_{a=1}^{n_{en}} |\mathbf{u}^h \cdot \boldsymbol{\nabla} N_a| \right)^{-1}, \tag{54}$$

where N_a is the interpolation function associated with node a. The stabilization parameters are defined as follows:

$$\begin{aligned}
\tau_{\mathrm{SUGN1}} &= \frac{h_{\mathrm{UGN}}}{2\|\mathbf{u}^h\|}, & (55)\\
\tau_{\mathrm{SUGN2}} &= \frac{\Delta t}{2}, & (56)\\
\tau_{\mathrm{SUGN3}} &= \frac{h_{\mathrm{UGN}}^2}{4\nu}, & (57)\\
(\tau_{\mathrm{SUPG}})_{\mathrm{UGN}} &= \left(\frac{1}{\tau_{\mathrm{SUGN1}}^2} + \frac{1}{\tau_{\mathrm{SUGN2}}^2} + \frac{1}{\tau_{\mathrm{SUGN3}}^2} \right)^{-\frac{1}{2}}, & (58)\\
(\tau_{\mathrm{PSPG}})_{\mathrm{UGN}} &= (\tau_{\mathrm{SUPG}})_{\mathrm{UGN}}, & (59)\\
(\tau_{\mathrm{LSIC}})_{\mathrm{UGN}} &= \frac{h_{\mathrm{UGN}}}{2}\|\mathbf{u}^h\|\ z. & (60)
\end{aligned}$$

Here z is given as follows:

$$z = \begin{cases} \left(\frac{Re_{\mathrm{UGN}}}{3}\right) & Re_{\mathrm{UGN}} \leq 3, \\ 1 & Re_{\mathrm{UGN}} > 3, \end{cases} \tag{61}$$

where $Re_{\mathrm{UGN}} = \frac{\|\mathbf{u}^h\| h_{\mathrm{UGN}}}{2\nu}$.

Comparisons between the performances of these earlier stabilization parameters and the ones proposed here can be found in.[14] These comparisons show that, especially for special element geometries, the performances are similar.

It was pointed out in[16,17] that the expression for τ_{SUGN1} can be written more directly as

$$\tau_{\text{SUGN1}} = \left(\sum_{a=1}^{n_{en}} |\mathbf{u}^h \cdot \boldsymbol{\nabla} N_a| \right)^{-1}, \tag{62}$$

and based on that, the expression for h_{UGN} can be written as

$$h_{\text{UGN}} = 2\|\mathbf{u}^h\| \; \tau_{\text{SUGN1}}. \tag{63}$$

A rationale for τ_{SUGN1} given by Eq. (62) was provided in.[17]

8 DISCONTINUITY-CAPTURING DIRECTIONAL DISSIPATION (DCDD)

As a potential alternative or complement to the LSIC stabilization, we proposed in[4,15,17] the Discontinuity-Capturing Directional Dissipation (DCDD) stabilization. In describing the DCDD stabilization, we first define the unit vectors $\mathbf{s}$ and $\mathbf{r}$:

$$\mathbf{s} = \frac{\mathbf{u}^h}{\|\mathbf{u}^h\|}, \qquad \mathbf{r} = \frac{\boldsymbol{\nabla}\|\mathbf{u}^h\|}{\|\; \boldsymbol{\nabla}\|\mathbf{u}^h\| \;\|}, \tag{64}$$

and the element-level matrices and vectors $\mathbf{c}_{\text{r}}$, $\tilde{\mathbf{k}}_{\text{r}}$, $(\mathbf{c}_{\text{r}})_{\text{V}}$, and $(\tilde{\mathbf{k}}_{\text{r}})_{\text{V}}$:

$$\mathbf{c}_{\text{r}} : \quad \int_{\Omega^e} \mathbf{w}^h \cdot \rho(\mathbf{r} \cdot \boldsymbol{\nabla}\mathbf{u}^h) d\Omega \qquad : (\mathbf{c}_{\text{r}})_{\text{V}} \;, \tag{65}$$

$$\tilde{\mathbf{k}}_{\text{r}} : \quad \int_{\Omega^e} (\mathbf{r} \cdot \boldsymbol{\nabla}\mathbf{w}^h) \cdot \rho(\mathbf{r} \cdot \boldsymbol{\nabla}\mathbf{u}^h) d\Omega \quad : (\tilde{\mathbf{k}}_{\text{r}})_{\text{V}} \;. \tag{66}$$

Then the DCDD stabilization is defined as

$$S_{\text{DCDD}} = \sum_{e=1}^{n_{el}} \int_{\Omega^e} \rho \nu_{\text{DCDD}} \boldsymbol{\nabla}\mathbf{w}^h : \left(\left[\mathbf{rr} - (\mathbf{r} \cdot \mathbf{s})^2 \mathbf{ss} \right] \cdot \boldsymbol{\nabla}\mathbf{u}^h \right) d\Omega, \tag{67}$$

where the element-matrix-based and element-vector-based DCDD viscosities are:

$$\nu_{\text{DCDD}} \;=\; |\mathbf{r} \cdot \mathbf{u}^h| \frac{\|\mathbf{c}_{\text{r}}\|}{\|\tilde{\mathbf{k}}_{\text{r}}\|} \;, \tag{68}$$

$$(\nu_{\text{DCDD}})_{\text{V}} \;=\; |\mathbf{r} \cdot \mathbf{u}^h| \frac{\|(\mathbf{c}_{\text{r}})_{\text{V}}\|}{\|(\tilde{\mathbf{k}}_{\text{r}})_{\text{V}}\|} \;. \tag{69}$$

An approximate version of the expression given by Eq. (68) can be written as

$$\nu_{\text{DCDD}} = |\mathbf{r} \cdot \mathbf{u}^h| \frac{h_{\text{RGN}}}{2}, \tag{70}$$

where

$$h_{\mathrm{RGN}} = 2 \left(\sum_{a=1}^{n_{en}} |\mathbf{r} \cdot \boldsymbol{\nabla} N_a| \right)^{-1} . \tag{71}$$

A different way of determining ν_{DCDD} can be expressed as

$$\nu_{\mathrm{DCDD}} = \tau_{\mathrm{DCDD}} \|\mathbf{u}^h\|^2, \tag{72}$$

where

$$\tau_{\mathrm{DCDD}} = \frac{h_{\mathrm{DCDD}}}{2\|\mathbf{U}\|} \frac{\|\ \boldsymbol{\nabla}\|\mathbf{u}^h\|\ \|\ h_{\mathrm{DCDD}}}{\|\mathbf{U}\|} . \tag{73}$$

Here $\mathbf{U}$ represents a global velocity scale, and h_{DCDD} can be calculated by using the expression

$$h_{\mathrm{DCDD}} = 2 \frac{\|\mathbf{c}_{\mathrm{r}}\|}{\|\tilde{\mathbf{k}}_{\mathrm{r}}\|} , \tag{74}$$

or the approximation

$$h_{\mathrm{DCDD}} = h_{\mathrm{RGN}} . \tag{75}$$

Combining Eqs. (72) and (73), we obtain

$$\nu_{\mathrm{DCDD}} = \frac{1}{2} \left(\frac{\|\mathbf{u}^h\|}{\|\mathbf{U}\|} \right)^2 (h_{\mathrm{DCDD}})^2\ \|\ \boldsymbol{\nabla}\|\mathbf{u}^h\|\ \| . \tag{76}$$

9 UGN/RGN-BASED STABILIZATION PARAMETERS FOR NAVIER-STOKES EQUATIONS

In,[4,17] we proposed to re-define τ_{PSPG} and provided the reason for doing that. We described how we re-define τ_{PSPG} by modifying the definitions of τ_{P3} and τ_{PV3} given by Eqs. (42) and (46). We proposed to accomplish that by using the expressions

$$\tau_{\mathrm{P3}} = \tau_{\mathrm{P1}} \frac{\|\mathbf{c}\|}{\nu\ \|\tilde{\mathbf{k}}_{\mathrm{r}}\|} , \qquad \tau_{\mathrm{PV3}} = \tau_{\mathrm{PV1}} \frac{\|\mathbf{c}\|}{\nu\ \|\tilde{\mathbf{k}}_{\mathrm{r}}\|} , \tag{77}$$

or the approximations

$$\tau_{\mathrm{P3}} = \tau_{\mathrm{P1}}\ Re \left(\frac{h_{\mathrm{RGN}}}{h_{\mathrm{UGN}}} \right)^2 , \qquad \tau_{\mathrm{PV3}} = \tau_{\mathrm{PV1}}\ Re \left(\frac{h_{\mathrm{RGN}}}{h_{\mathrm{UGN}}} \right)^2 . \tag{78}$$

In,[4] we further stated that these modifications can also be applied to $\tau_{\rm S3}$ and $\tau_{\rm SV3}$ given by Eqs. (21) and (25). In,[17] we wrote those expressions explicitly as follows:

$$\tau_{\rm S3} = \tau_{\rm S1} \, \frac{\|\mathbf{c}\|}{\nu \, \|\tilde{\mathbf{k}}_{\rm r}\|} \; , \qquad \tau_{\rm SV3} = \tau_{\rm SV1} \, \frac{\|\mathbf{c}\|}{\nu \, \|\tilde{\mathbf{k}}_{\rm r}\|} \; , \tag{79}$$

$$\tau_{\rm S3} = \tau_{\rm S1} \; Re \left(\frac{h_{\rm RGN}}{h_{\rm UGN}} \right)^2 \; , \qquad \tau_{\rm SV3} = \tau_{\rm SV1} \; Re \left(\frac{h_{\rm RGN}}{h_{\rm UGN}} \right)^2 \; . \tag{80}$$

We noted in[17] that if we are dealing with just an advection-diffusion equation, rather than the Navier-Stokes equations of incompressible flows, then the definition of the unit vector $\mathbf{r}$ changes as follows:

$$\mathbf{r} = \frac{\boldsymbol{\nabla} |\phi^h|}{\| \, \boldsymbol{\nabla} |\phi^h| \, \|} \; . \tag{81}$$

We also proposed in[16,17] to re-define $\tau_{\rm SUGN3}$ given by Eq. (57) as follows:

$$\tau_{\rm SUGN3} = \frac{h_{\rm RGN}^2}{4\nu} \; . \tag{82}$$

Furthermore, we proposed in[16,17] to replace $(\tau_{\rm LSIC})_{\rm UGN}$ given by Eq. (60) as follows:

$$(\tau_{\rm LSIC})_{\rm UGN} = (\tau_{\rm SUPG})_{\rm UGN} \, \|\mathbf{u}^h\|^2 \; . \tag{83}$$

We further commented in[16,17] that the "element length"s $h_{\rm UGN}$ (given by Eq. (54)) and $h_{\rm RGN}$ (Eq. (71)) can be viewed as the local length scales corresponding to the advection- and diffusion-dominated limits, respectively.

10 DEFORMING-SPATIAL-DOMAIN/STABILIZED SPACE-TIME (DSD/SST) FORMULATION

In the DSD/SST method,[1] the finite element formulation of the governing equations is written over a sequence of N space-time slabs Q_n, where Q_n is the slice of the space-time domain between the time levels t_n and t_{n+1}. At each time step, the integrations involved in the finite element formulation are performed over Q_n. The space-time finite element interpolation functions are continuous within a space-time slab, but discontinuous from one space-time slab to another. The notation $(\cdot)_n^-$ and $(\cdot)_n^+$ denotes the function values at t_n as approached from below and above. Each Q_n is decomposed into elements Q_n^e, where $e = 1, 2, \ldots, (n_{el})_n$. The subscript n used with n_{el} is for the general case in which the number of space-time elements may change from one space-time slab to another. The Dirichlet- and Neumann-type boundary conditions are enforced over $(P_n)_{\rm g}$ and $(P_n)_{\rm h}$, the complementary subsets of the lateral boundary of the space-time slab. The finite element trial function spaces $(\mathcal{S}_{\mathbf{u}}^h)_n$ for velocity and $(\mathcal{S}_p^h)_n$ for pressure, and the test function spaces $(\mathcal{V}_{\mathbf{u}}^h)_n$ and $(\mathcal{V}_p^h)_n = (\mathcal{S}_p^h)_n$ are defined by using, over Q_n, first-order polynomials in both space and time. The DSD/SST formulation[1,16,17] is written as follows: given $(\mathbf{u}^h)_n^-$, find $\mathbf{u}^h \in (\mathcal{S}_{\mathbf{u}}^h)_n$ and $p^h \in (\mathcal{S}_p^h)_n$ such that $\forall \mathbf{w}^h \in (\mathcal{V}_{\mathbf{u}}^h)_n$ and $q^h \in (\mathcal{V}_p^h)_n$:

$$
\begin{aligned}
& \int_{Q_n} \mathbf{w}^h \cdot \rho \left(\frac{\partial \mathbf{u}^h}{\partial t} + \mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{u}^h - \mathbf{f}^h \right) dQ + \int_{Q_n} \boldsymbol{\varepsilon}(\mathbf{w}^h) : \boldsymbol{\sigma}(p^h, \mathbf{u}^h) dQ \\
- & \int_{(P_n)_{\mathrm{h}}} \mathbf{w}^h \cdot \mathbf{h}^h dP + \int_{Q_n} q^h \boldsymbol{\nabla} \cdot \mathbf{u}^h dQ + \int_{\Omega_n} (\mathbf{w}^h)_n^+ \cdot \rho \left((\mathbf{u}^h)_n^+ - (\mathbf{u}^h)_n^- \right) d\Omega \\
+ & \sum_{e=1}^{(n_{el})_n} \int_{Q_n^e} \frac{1}{\rho} \left[\tau_{\mathrm{SUPG}} \rho \left(\frac{\partial \mathbf{w}^h}{\partial t} + \mathbf{u}^h \cdot \nabla \mathbf{w}^h \right) + \tau_{\mathrm{PSPG}} \nabla q^h \right] \cdot \left[\text{Ł}(p^h, \mathbf{u}^h) - \rho \mathbf{f}^h \right] dQ \\
+ & \sum_{e=1}^{n_{el}} \int_{Q_n^e} \tau_{\mathrm{LSIC}} \boldsymbol{\nabla} \cdot \mathbf{w}^h \rho \boldsymbol{\nabla} \cdot \mathbf{u}^h dQ = 0,
\end{aligned} \tag{84}
$$

where

$$
\text{Ł}(q^h, \mathbf{w}^h) = \rho \left(\frac{\partial \mathbf{w}^h}{\partial t} + \mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{w}^h \right) - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(q^h, \mathbf{w}^h), \tag{85}
$$

and τ_{SUPG}, τ_{PSPG} and τ_{LSIC} are the stabilization parameters (see[16,17]). This formulation is applied to all space-time slabs $Q_0, Q_1, Q_2, \ldots, Q_{N-1}$, starting with $(\mathbf{u}^h)_0^- = \mathbf{u}_0$. For an earlier, detailed reference on the DSD/SST formulation see.[1]

11 ELEMENT-MATRIX-BASED STABILIZATION PARAMETERS FOR THE DSD/SST FORMULATION

For extensions of the τ calculations based on matrix norms to the DSD/SST formulation, in[17] we defined the space-time augmented versions of the element-level matrices and vectors given by Eqs. (29), (33), and (37) as follows:

$$
\mathbf{c}_{\mathrm{A}} : \quad \int_{Q_n^e} \mathbf{w}^h \cdot \rho \left(\frac{\partial \mathbf{u}^h}{\partial t} + \mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{u}^h \right) dQ \quad : (\mathbf{c}_{\mathrm{A}})_{\mathrm{V}}, \tag{86}
$$

$$
\tilde{\mathbf{k}}_{\mathrm{A}} : \quad \int_{Q_n^e} \left(\frac{\partial \mathbf{w}^h}{\partial t} + \mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{w}^h \right) \cdot \rho \left(\frac{\partial \mathbf{u}^h}{\partial t} + \mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{u}^h \right) dQ \quad : (\tilde{\mathbf{k}}_{\mathrm{A}})_{\mathrm{V}}, \tag{87}
$$

$$
\gamma_{\mathrm{A}} : \quad \int_{Q_n^e} \boldsymbol{\nabla} q^h \cdot \left(\frac{\partial \mathbf{u}^h}{\partial t} + \mathbf{u}^h \cdot \boldsymbol{\nabla} \mathbf{u}^h \right) dQ \quad : (\gamma_{\mathrm{A}})_{\mathrm{V}}. \tag{88}
$$

The components of element-matrix-based τ_{SUPG} were defined in[17] as follows:

$$
\tau_{\mathrm{S12}} = \frac{\|\mathbf{c}_{\mathrm{A}}\|}{\|\tilde{\mathbf{k}}_{\mathrm{A}}\|}, \tag{89}
$$

$$
\tau_{\mathrm{S3}} = \tau_{\mathrm{S12}} \frac{\|\mathbf{c}_{\mathrm{A}}\|}{\nu \, \|\tilde{\mathbf{k}}_{\mathrm{r}}\|}, \tag{90}
$$

where $\tilde{\mathbf{k}}_{\mathrm{r}}$ is the space-time version (i.e. integrated over the space-time element domain Q_n^e) of the element-level matrix given by Eq. (66). To construct τ_{SUPG} from its components we proposed in[17] the form

$$\tau_{\mathrm{SUPG}} = \left(\frac{1}{\tau_{\mathrm{S12}}^r} + \frac{1}{\tau_{\mathrm{S3}}^r} \right)^{-\frac{1}{r}}. \tag{91}$$

The components of the element-vector-based τ_{SUPG} were defined in[17] as

$$\tau_{\mathrm{SV12}} = \frac{\|(\mathbf{c}_{\mathrm{A}})_{\mathrm{V}}\|}{\|(\tilde{\mathbf{k}}_{\mathrm{A}})_{\mathrm{V}}\|}, \tag{92}$$

$$\tau_{\mathrm{SV3}} = \tau_{\mathrm{SV12}} \frac{\|\mathbf{c}_{\mathrm{A}}\|}{\nu\ \|\tilde{\mathbf{k}}_{\mathrm{r}}\|}. \tag{93}$$

From these two components,

$$(\tau_{\mathrm{SUPG}})_{\mathrm{V}} = \left(\frac{1}{\tau_{\mathrm{SV12}}^r} + \frac{1}{\tau_{\mathrm{SV3}}^r} \right)^{-\frac{1}{r}}. \tag{94}$$

The components of element-matrix-based τ_{PSPG} were defined in[17] as follows:

$$\tau_{\mathrm{P12}} = \frac{\|\mathbf{g}^{\mathrm{T}}\|}{\|\gamma_{\mathrm{A}}\|}, \tag{95}$$

$$\tau_{\mathrm{P3}} = \tau_{\mathrm{P12}} \frac{\|\mathbf{c}_{\mathrm{A}}\|}{\nu\ \|\tilde{\mathbf{k}}_{\mathrm{r}}\|}, \tag{96}$$

where $\mathbf{g}^{\mathrm{T}}$ is the space-time version of the element-level matrix given by Eq. (32). To construct τ_{PSPG} from its components, we proposed in[17] the form

$$\tau_{\mathrm{PSPG}} = \left(\frac{1}{\tau_{\mathrm{P12}}^r} + \frac{1}{\tau_{\mathrm{P3}}^r} \right)^{-\frac{1}{r}}. \tag{97}$$

The components of the element-vector-based τ_{PSPG} were defined in[17] as follows:

$$\tau_{\mathrm{PV12}} = \frac{\|\mathbf{g}_{\mathrm{V}}^{\mathrm{T}}\|}{\|(\gamma_{\mathrm{A}})_{\mathrm{V}}\|}, \tag{98}$$

$$\tau_{\mathrm{PV3}} = \tau_{\mathrm{PV12}} \frac{\|\mathbf{c}_{\mathrm{A}}\|}{\nu\ \|\tilde{\mathbf{k}}_{\mathrm{r}}\|}. \tag{99}$$

From these components,

$$(\tau_{\mathrm{PSPG}})_{\mathrm{V}} = \left(\frac{1}{\tau_{\mathrm{PV12}}^r} + \frac{1}{\tau_{\mathrm{PV3}}^r} \right)^{-\frac{1}{r}}. \tag{100}$$

The element-matrix-based τ_{LSIC} was defined in[17] as

$$\tau_{\text{LSIC}} = \frac{\|\mathbf{c}_{\text{A}}\|}{\|\mathbf{e}\|}, \tag{101}$$

where $\mathbf{e}$ is the space-time version of the element-level matrix given by Eq. (39).

The element-vector-based τ_{LSIC} was defined in[17] as

$$(\tau_{\text{LSIC}})_{\text{V}} = \tau_{\text{LSIC}}. \tag{102}$$

12 UGN/RGN-BASED STABILIZATION PARAMETERS FOR THE DSD/SST FORMULATION

The space-time versions of τ_{SUGN1}, τ_{SUGN2}, τ_{SUGN3}, $(\tau_{\text{SUPG}})_{\text{UGN}}$, $(\tau_{\text{PSPG}})_{\text{UGN}}$, and $(\tau_{\text{LSIC}})_{\text{UGN}}$, given respectively by Eqs. (55), (56), (82), (58), (59), and (83), were defined in[16,17] as follows:

$$\tau_{\text{SUGN12}} = \left(\sum_{a=1}^{n_{en}} \left| \frac{\partial N_a}{\partial t} + \mathbf{u}^h \cdot \boldsymbol{\nabla} N_a \right| \right)^{-1}, \tag{103}$$

$$\tau_{\text{SUGN3}} = \frac{h_{\text{RGN}}^2}{4\nu}, \tag{104}$$

$$(\tau_{\text{SUPG}})_{\text{UGN}} = \left(\frac{1}{\tau_{\text{SUGN13}}^2} + \frac{1}{\tau_{\text{SUGN3}}^2} \right)^{-\frac{1}{2}}, \tag{105}$$

$$(\tau_{\text{PSPG}})_{\text{UGN}} = (\tau_{\text{SUPG}})_{\text{UGN}}, \tag{106}$$

$$(\tau_{\text{LSIC}})_{\text{UGN}} = (\tau_{\text{SUPG}})_{\text{UGN}} \left\|\mathbf{u}^h\right\|^2. \tag{107}$$

Here, n_{en} is the number of nodes for the space-time element, and N_a is the space-time interpolation function associated with node a.

13 CONCLUDING REMARKS

We described how we determine the stabilization parameters ("τ"s) and element length scales used in stabilized finite element formulations of flow problems. These stabilized formulations include the interface-tracking and interface-capturing techniques we developed for computation of flows with moving boundaries and interfaces. The interface-tracking techniques are based on the DSD/SST, where the mesh moves to track the interface. The interface-capturing techniques, typically used with non-moving meshes, are based on a stabilized semi-discrete formulation of the Navier-Stokes equations, combined with a stabilized formulation of an advection equation. The advection equation governs the time-evolution of an interface function marking the interface location. As specific stabilization methods, we SUPG and PSPG methods. For the Navier-Stokes equations and the advection equation, we described the element-matrix-based and element-vector-based τs designed for semi-discrete and space-time formulations. These τ definitions are expressed in terms of the ratios of the norms of the relevant matrices or vectors. They take into account the local length scales, advection field and the element-level Reynolds number. Based on these definitions, a τ can be calculated for each element, or for each element node or degree of freedom or element equation. Furthermore, based on these definitions, a τ can be calculated for each element integration point. We also described certain variations and complements of these new τs, including the approximate versions that are based on the local length scales for the advection- and diffusion-dominated limits.

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