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SOLVING CONTACT PROBLEMS WITH FREE SURFACES USING A LAGRANGIAN MESHLESS SCHEME

F. Del Pin[*], S.R. Idelsohn[†], E. Oñate[†] and N. Calvo[*]

[†]International Center for Computational Methods in Engineering (CIMEC)
Universidad Nacional del Litoral and CONICET, Santa Fe, Argentina
e–mail: sergio@ceride.gov.ar

[*]International Center for Numerical Methods in Engineering (CIMNE)
Universidad Politécnica de Cataluña, Barcelona, Spain
e–mail: onate@cimne.upc.es

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Abstract. *A combination between a meshless method and a Lagrangian formulation will be presented. The capability of the meshless methods for solving complicated geometries and the natural way that Lagrangian formulations have to represent problems with big deformations of the domain will be put together in this work. The resulting approach showed to be robust from the CFD point of view and easy to implement from the computational point of view. All contacts and free surface calculations are embedded in the meshless method, making their computation straight forward. Lagrangian formulations also have the advantage of solving the convection of velocity by moving the material points, thus avoiding the tedious convection terms in the Navier-Stokes equations. As a meshless method, the Meshless Finite Element Method has been implemented. This method conserves the useful properties of the Finite Element Method so adding the benefits of the meshless methods. The connection between material points is calculated using the Extended Delaunay Tessellation. To solve Finite Element, simplicial elements are used in most of the domain, except when the element quality is not good enough, then polyhedrons are taken automatically as elements. The Non-Sibsonian shape function adapts to the polyhedrons keeping the good properties that Finite Element shape functions have. The presented scheme has found to have remarkable results in a large variety of contact problems with free surfaces.*

1. INTRODUCTION

Recently *Particle Methods,* where each fluid particle is followed in a lagrangian manner, have been used [1,2] to model fluid mechanics problems. The first ideas on this approach were proposed by Monaghan[1] for the treatment of astrophysical hydrodynamic problems with the so called *Smooth Particle Hydrodynamics Method* (SPH).
On the other hand, a family of methods called *Meshless Methods* have been developed both for structural [6] and fluid mechanics problems [7].
The authors[5] presented the numerical solution for the fluid mechanics equations using a lagrangian formulation and a meshless method called the Finite Point Method. Lately, the meshless ideas were generalized to take into account the finite element type approximations in order to obtain the same computing time in mesh generation as in the evaluation of the meshless connectivities[6]. This method was called the Meshless Finite Element Method (MFEM) and uses the Extended Delaunay Tessellation to build the mesh in a computing time which is linear with the number of nodal points.
In this paper new ideas and results for the solution of a particle method in the field of Fluid-Structure Interaction (FSI) using the Meshless Finite Element Method are presented. A more general formulation is used in which all the classical advantages of the FEM for the evaluation of the unknown functions and derivatives are preserved.
Depending on the degree of coupling between the equations for the fluid and the structure, two cases can be distinguished[7] to solve FSI problems. The first one occurs when there is a strong coupling between the fluid flow and the elastic deformation of the structure[7,8]. The second case occurs when there is a weak interaction between the fluid and the rigid deformation of the structure. Both cases of FSI are more easily studied with a lagrangian formulation of the fluid equations, which can be seen as a solid with a small shear coefficient or vice versa.
The lagrangian fluid flow equations for the Navier-Stoke problem and the Meshless Finite Element Method (MFEM) will be revised in the next sections and both techniques will be used to solve some FSI problems for rigid solids.

2. THE MESHLESS FINITE ELEMENT METHOD

A full description of the MFEM may be found in Ref.[6]. Nevertheless and for the sake of completeness a summary is presented in this section.

The MFEM combines a particular finite element subdivision in polyhedral shape called the Extended Delanay Tessellation and ad-hoc shape functions for this kind of polyhedra.

The Extended Delaunay Tessellation (EDT)

*Definition***: The Extended Delaunay tessellation** within the set **N** is the unique partition of the convex hull Ω of all the nodes into regions Ω_i such that $\Omega = \cup\, \Omega_i$, where each Ω_i is the polyhedron defined by all the nodes laying on the same Voronoï sphere.

The main difference between the traditional Delaunay tessellation and the Extended Delaunay tessellation is that, in the latter, all the nodes belonging to the same Voronoï sphere define a unique polyhedron. With this definition, the domain Ω is divided into tetrahedra and other polyhedra.

The Meshless Finite Element shape functions

In order to define the shape functions inside each polyhedron the non-Sibsonian interpolation is used.
Let $\mathbf{P} = \{\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_m\}$ be the set of nodes belonging to a polyhedron. The shape function $N_i(\mathbf{x})$ corresponding to the node $\mathbf{n}_i$ at an internal point $\mathbf{x}$ is defined by building first the Voronoï cell corresponding to $\mathbf{x}$ in the tessellation of the set $\mathbf{P} \cup \{\mathbf{x}\}$ and then by computing:

$$N_i(\mathbf{x}) = \frac{\dfrac{s_i(\mathbf{x})}{h_i(\mathbf{x})}}{\displaystyle\sum_{j=1}^{m} \frac{s_j(\mathbf{x})}{h_j(\mathbf{x})}}$$

where $s_i(\mathbf{x})$ is the surface of the Voronoï cell face corresponding to node the node $\mathbf{n}_i$ and $h_i(\mathbf{x})$ is the distance between point $\mathbf{x}$ and the node **n.**
Non-Sibsonian interpolations have the following properties[19].

1) $0 \le N_i(\mathbf{x}) \le 1$
2) $\Sigma_i N_i(\mathbf{x}) = 1$
3) $N_i(\mathbf{n}_j) = \delta_{ij}$
4) $\mathbf{x} = \Sigma_i N_i(\mathbf{x})\, \mathbf{n}_i$

The algorithm steps for the MFEM are:

1) For a set of nodes, compute all the empty spheres with 4 nodes.
2) Generate all the polyhedral.
3) Calculate the shape functions and their derivatives at all the Gauss points necessary to evaluate the integrals of the weak form.

3. GOVERNING EQUATIONS

Now that we understand more clearly the numerical method that will be used, the equation of motion will be presented. The mass and momentum conservation equations can be written in a lagrangian formulations as:

Mass conservation:

$$\frac{D\rho}{Dt} + \rho \frac{\partial u_i}{\partial x_i} = 0 \qquad (1)$$

Momentum conservation:

$$\rho \frac{Du_i}{Dt} = -\frac{\partial}{\partial x_i} p + \frac{\partial}{\partial x_j} \tau_{ij} + \rho f_i \qquad (2)$$

where ρ is the density u_i are the Cartesian components of the velocity field, p the pressure, τ_{ij} the deviator stress tensor, f_i the source term (normally the gravity) and $\frac{D\phi}{Dt}$ represents the total or material time derivative of a function ϕ.

Boundary conditions

On the boundaries, the standard boundary conditions for the Navier-Stokes equations are:

$$\tau_{ij}\nu_j - p\nu_i = \overline{\sigma}_{ni} \text{ on } \Gamma_\sigma \qquad (3)$$

$$u_i\nu_i = \overline{u}_n \text{ on } \Gamma_n \qquad (4)$$

$$u_i\zeta_i = \overline{u}_t \text{ on } \Gamma_t \qquad (5)$$

where ν_i and ζ_i are the components of the normal and tangent vector to the boundary.

4. THE TIME SPLITTING

A fractional step method is proposed[9] which consists in splitting each time step in 2 steps as follows:

Split of the momentum equations

$$\frac{Du_i}{Dt} \approx \frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{u_i^{n+1} - u_i^* + u_i^* - u_i^n}{\Delta t} = (-\frac{1}{\rho}\frac{\partial}{\partial x_i} p + \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j} + f_i)^{n+\theta} \qquad (6)$$

where $\Delta t = t^{n+1} - t^n$ is the time step; $u_i^n = u_i(x^n, t^n)$; $u_i^{n+1} = u_i(x^{n+1}, t^{n+1})$ and u_i^* are fictitious variables defined by the split

A) $$u_i^* = u_i^n + f_i \,\Delta t + \frac{\Delta t}{\rho}\frac{\partial}{\partial x_j}\tau^n{}_{ij} \qquad (7)$$

C) $$u_i^{n+1} = u_i^* - \frac{\Delta t}{\rho}\frac{\partial}{\partial x_i} p^{n+1}. \tag{8}$$

Split of the mass conservation equations

$$\frac{D\rho}{Dt} \approx \frac{\rho^{n+1} - \rho^n}{\Delta t} = \frac{\rho^{n+1} - \rho^* + \rho^* - \rho^n}{\Delta t} = -\rho \frac{\partial (u_i^{n+1} - u_i^* + u_i^*)}{\partial x_i} \tag{9}$$

where ρ^* is a fictitious variable defined by the split

$$\frac{\rho^* - \rho^n}{\Delta t} = -\rho \frac{\partial u_i^*}{\partial x_i} \tag{10}$$

$$\frac{\rho^{n+1} - \rho^*}{\Delta t} = -\rho \frac{\partial (u_i^{n+1} - u_i^*)}{\partial x_i} \tag{11}$$

Coupled equations

From eqs.(8) and (11) the coupled mass-momentum equation becomes:

B) $$\frac{\rho^{n+1} - \rho *}{\Delta t^2} = \frac{\partial^2}{\partial x_i^2} p^{n+1} \tag{12}$$

Taking into account eq.(10) then:

B) $$\frac{\rho^{n+1} - \rho^n}{\Delta t^2} + \frac{\rho}{\Delta t}\frac{\partial u_i^*}{\partial x_i} = \frac{\partial^2}{\partial x_i^2} p^{n+1} \tag{13}$$

Where by the incompressibility conditions

$$\rho^{n+1} = \rho^n = \rho^0 = \rho \tag{14}$$

5. SPATIAL DISCRETIZATION

Using the MFEM[6], the unknown functions are approximated using an equal order interpolation for all variables as (in matrix form)

(15)

$$u_i = N_i^T U = \begin{bmatrix} N^T & & \\ & N^T & \\ & & N^T \end{bmatrix} U$$

$$p = N_p^T P = N^T \ P \qquad (16)$$

$$\rho = N_\rho^T \underline{\rho} = N^T \underline{\rho} \qquad (17)$$

where N^T are the MFEM shape functions and U, P, $\underline{\rho}$ the nodal values of the three components of the unknown velocity, the pressure and the density respectively.
Using the Galerkin weighted residual method to solve equations (7), (13) and (8) with boundary conditions (3-5):

$$\text{A)} \int_V N_i \{(u_i^* - u_i^n)\frac{\rho}{\Delta t} - f_i \ \rho - \mu \frac{\partial \ \tau_{ij}^n}{\partial x_j}\} dV - \int_{\Gamma_\sigma} N_i (\overline{\sigma}_{ni}{}^n - \tau_{ij}{}^n \nu_j) d\Gamma = 0 \qquad (18)$$

$$\text{B)} \int_V N_p \left\{ \frac{\rho}{\Delta t}\frac{\partial}{\partial x_i} u_i^* - \frac{\partial^2}{\partial x_i^2} p^{n+1} \right\} dV + \frac{\rho}{\Delta t} \int_{\Gamma u} N_p (\overline{u}_i^{n+1} \nu_i \ - u_i^{n+1} \nu_i) d\Gamma = 0 \qquad (19)$$

$$\text{C)} \int_V N_i \{(u_i^{n+1} - u_i^*)\frac{\rho}{\Delta t} + \frac{\partial}{\partial x_i} p^{n+1}\} dV - \int_{\Gamma_\sigma} N_i p^{n+1} \nu_i d\Gamma = 0 \qquad (20)$$

where the boundary conditions have been also split.
Integrating by parts some of the terms:

$$\text{A)} \int_V N_i (u_i^* - u_i^n - f_i \ \Delta t)\frac{\rho}{\Delta t} dV + \mu \int_V \frac{\partial N_i}{\partial x_i}\frac{\partial u_i^n}{\partial x_i} dV - \int_{\Gamma\sigma} N_i \overline{\sigma}_{ni}{}^n d\Gamma = 0 \qquad (21)$$

$$\text{B)} \int_V \frac{\partial N_p}{\partial x_i} (\frac{\rho}{\Delta t} u_i^* - \frac{\partial p^{n+1}}{\partial x_i}) dV + \frac{\rho}{\Delta t} \int_{\Gamma_u} N_p \ \overline{u}_n{}^{n+1} d\Gamma = 0 \qquad (22)$$

$$\text{C)} \int_V N_i \{(u_i^{n+1} - u_i^*)\frac{\rho}{\Delta t} + \frac{\partial}{\partial x_i} p^{n+1}\} dV - \int_{\Gamma_\sigma} N_i p^{n+1} d\Gamma = 0 \qquad (23)$$

The essential and natural boundary conditions of equations (22) are:

$$p = 0 \ on \ \Gamma_\sigma \tag{24}$$

$$\overline{u}^{n+1} \cdot \nu = 0 \ on \ \Gamma_u \tag{25}$$

Discrete equations

Using the approximations (15), (16) and (17) the discrete equations become:

A)

$$\int_V N_i N_i^T dV \, U_i^* = \int_V N_i N_i^T dV \, U_i^n + \Delta t \int_V N_i f_i \, dV - \frac{\Delta t \mu}{\rho} \int_V \frac{\partial N_i}{\partial x_i} \frac{\partial N_i^T}{\partial x_i} \, dV \, U_i^n + \frac{\Delta t}{\rho} \int_{\Gamma_\sigma} N_i \overline{\sigma_n^n} d\Gamma \tag{26}$$

In compact form:

$$\text{A)} \quad M_u U^* = M_u U^n + \Delta t \ F - \frac{\Delta t \mu}{\rho} K U^n \tag{27}$$

In the same way:

$$\text{B)} \qquad -\frac{\rho}{\Delta t} \int_V (\frac{\partial N_p}{\partial x_i} N_i^T) dV \ U^* + \frac{\rho}{\Delta t} \int_{\Gamma_u} N_p \ \overline{u}_n^{\ n+1} d\Gamma = -\int_V (\frac{\partial N_p}{\partial x_i} \frac{\partial N_p^T}{\partial x_i}) dV \ P^{n+1} \tag{28}$$

In compact form:

$$\text{B)} \qquad \frac{\rho}{\Delta t} B \quad U^* - \frac{\rho}{\Delta t} \widehat{U} = S \ P^{n+1} \tag{29}$$

and:

$$\text{C)} \int_V N_i N_i^T dV \ U^{n+1} = \int_V N_i N_i^T dV \ U^* - \frac{\Delta t}{\rho} \int_V N_i \frac{\partial N_p^T}{\partial x_i} dV \ P^{n+1} + \int_{\Gamma_\sigma} N_i N_p^T d\Gamma \ P^{n+1} \tag{30}$$

In compact form (noting that $p = 0 \ on \ \Gamma_\sigma$):

$$\text{C)} \quad M_u U^{n+1} = M_u U^* - \frac{\Delta t}{\rho} B^T P^{n+1} \tag{31}$$

Where the matrices are:

$$M = \begin{bmatrix} M_p & 0 & 0 \\ 0 & M_p & 0 \\ 0 & 0 & M_p \end{bmatrix} \tag{32}$$

$$M_p = \int_V N N^T dV \tag{33}$$

$$B = \left[\int_V \left(\frac{\partial N}{\partial x} N^T\right) dV \; ; \int_V \left(\frac{\partial N}{\partial y} N^T\right) dV \; ; \int_V \left(\frac{\partial N}{\partial z} N^T\right) dV \right] \tag{34}$$

$$S = \int_V \left(\frac{\partial N}{\partial x}\frac{\partial N^T}{\partial x} + \frac{\partial N}{\partial y}\frac{\partial N^T}{\partial y} + \frac{\partial N}{\partial z}\frac{\partial N^T}{\partial z}\right) dV \tag{35}$$

$$\widehat{U} = \int_{\Gamma_u} N \;\; \overline{u}_n^{\,n+1} d\Gamma \tag{36}$$

$$K = \begin{bmatrix} S & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & S \end{bmatrix} \tag{37}$$

$$\begin{aligned} F^T &= \left[\int_V N^T f_x \; dV; \quad \int_V N^T f_y \; dV; \quad \int_V N^T f_z \; dV \right] + \\ &\frac{1}{\rho}\left[\int_V N^T \overline{\sigma}_{nx} \; dV; \quad \int_V N^T \overline{\sigma}_{ny} \; dV; \quad \int_V N^T \overline{\sigma}_{nz} \; dV \right] \end{aligned} \tag{38}$$

6. BOUNDARY SURFACES

The use of the MFEM with the Extended Delaunay partition makes it easier to recognize boundary nodes.

Considering that the node follows a variable $h(x)$ distribution, where $h(x)$ is the minimum distance between two nodes, the following criterion has been used:

All nodes on an empty sphere with a radius $r(x)$ *bigger than* $\alpha\, h(x)$, *are considered as boundary nodes.*

7. NUMERICAL TESTS

Water column collapse

Figure 1 shows the point positions at different time steps. Agreement with the experimental results of ref.[4] both in the shape of the free surface as well as in the time development are excellent.

Moving ship with known velocity
In this case (Fig. 2), the same ship of the previous example is now moving at a fixed velocity

Solid falling into a recipient with water.
In this example the fluid is interacting with a solid that is totally free, without any imposed velocity. Figure 3 represents a free cube falling down into a recipient full of water. The solid cube was modeled by introducing a high viscosity parameter.

8. CONCLUSIONS

Lagrangian formulation and the Meshless Finite Element Method are an excellent combination to solve fluid mechanic problems, especially fluid-structure interactions with moving free-surface and contact problems.
Breaking waves, collapse problems, and contact problems can be solved easily without any additional constraint.
Furthermore, the Meshless Finite Element Method presented, as opposed to other methods, has the advantages of a good meshless method concerning the easy introduction of the nodes connectivity in a bounded time of order n. The method proposed also shares some advantages with the FEM such as: a) the simplicity of the shape functions, b) C_0 continuity between elements, c) an easy introduction of the boundary conditions, and d) symmetric matrices.
Both the lagrangian formulation and the MFEM are the key ingredients to solve fluid-structure interaction problems including with free-surface, breaking waves and collapse situations.

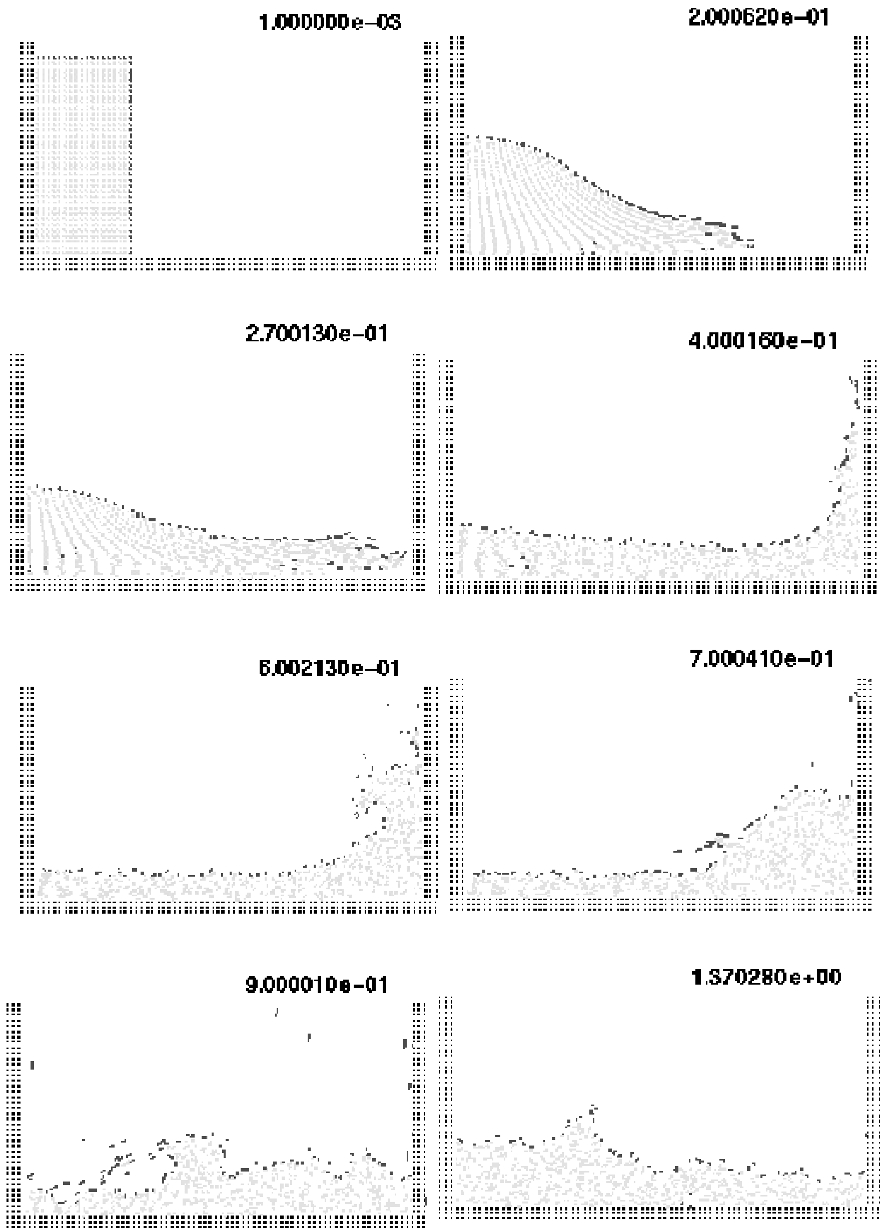

Figure 1: Water column collapse at different time steps.

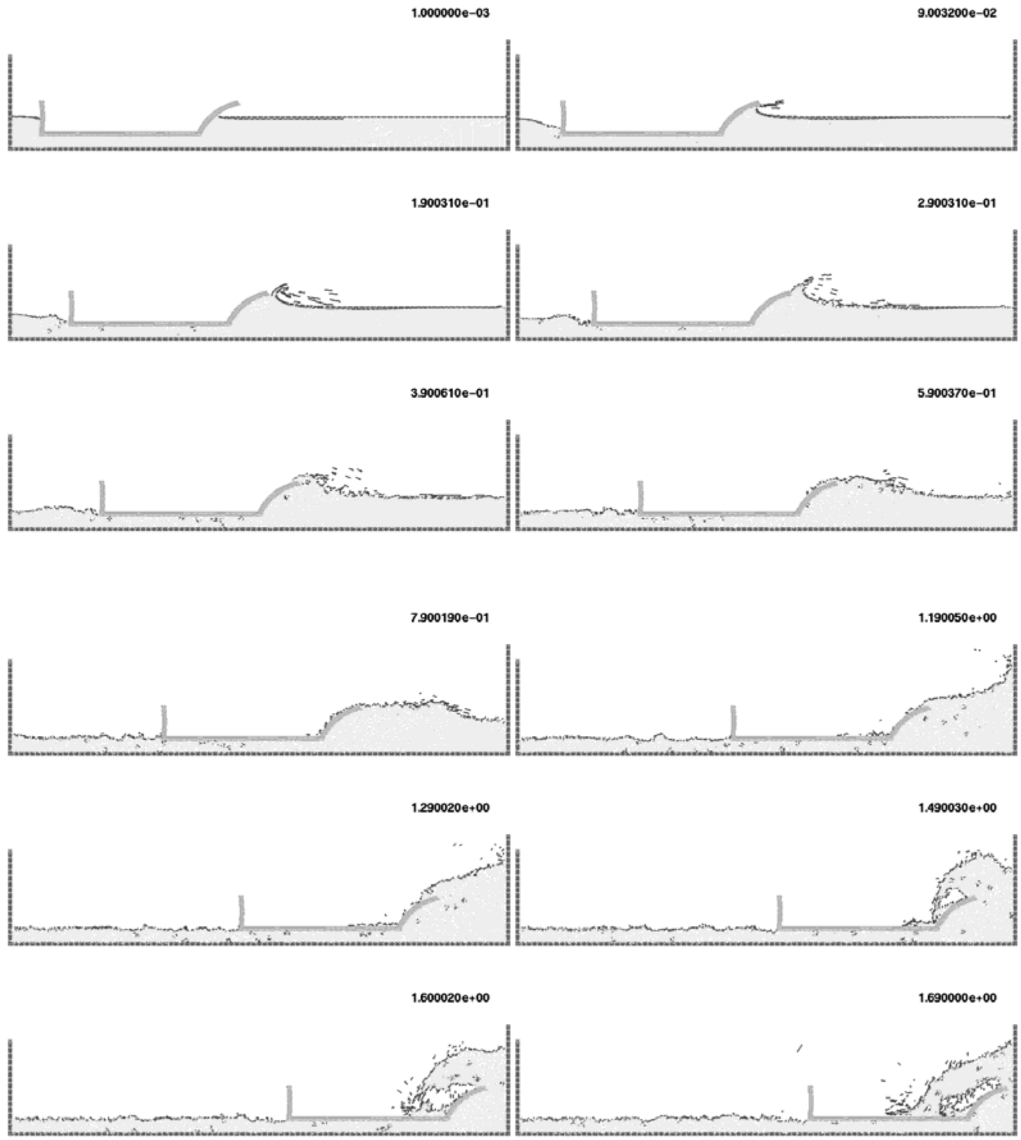

Figure 2. Moving ship with known velocity.

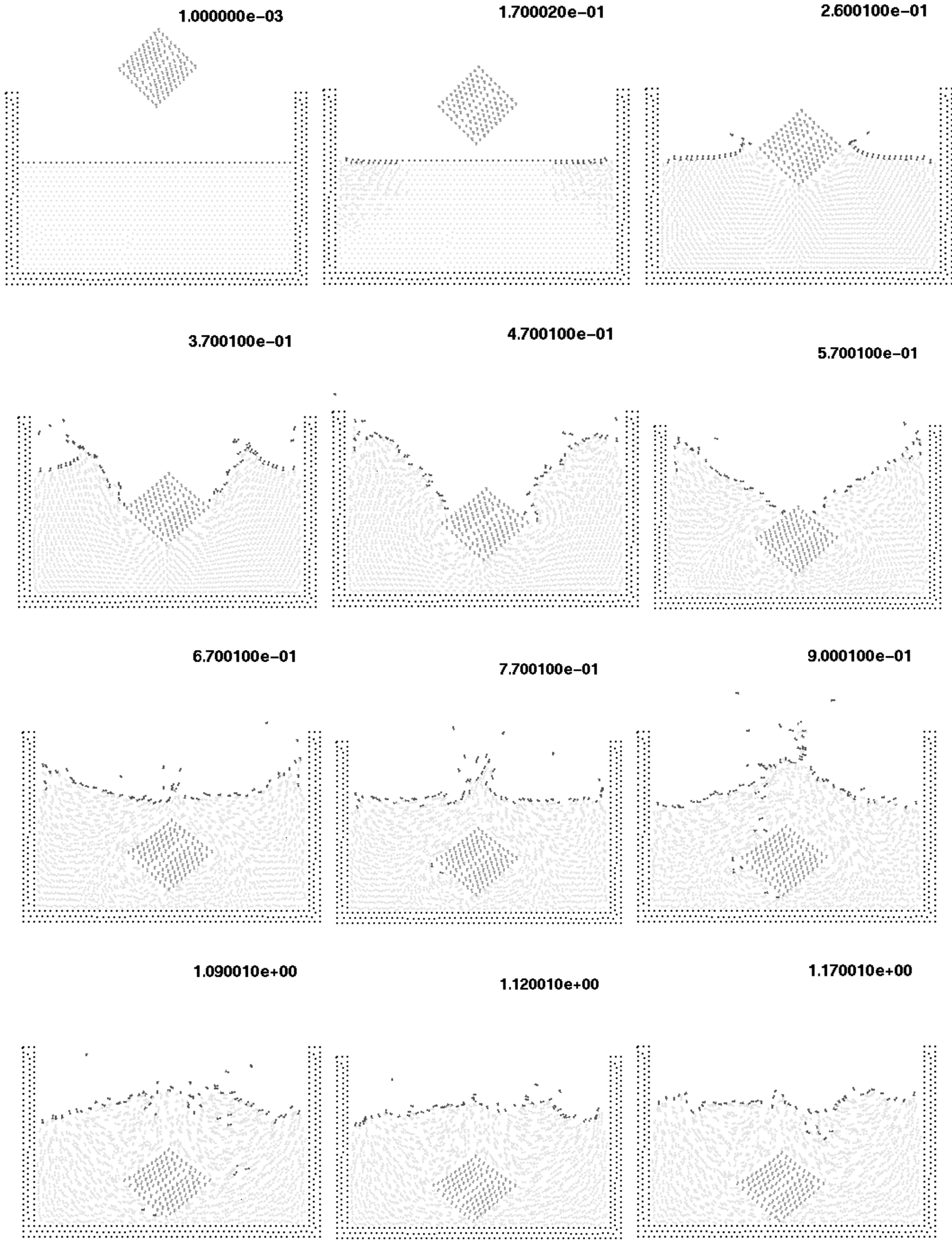

Figure 3. Solid cube falling into a recipient with water.

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