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SOME STUDIES OF THE LOPI MULTIQUADRICS MESHLESS METHOD

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Abstract. An improvement of a new multiquadric collocation meshless method for solving PDEs is presented. The scheme proposed here is a simple recipe for dealing with the shape parameter of the multiquadric at different scales of the size of the grid. Other technical improvements in the implementation of the method are described and a framework for an h adaptive theory is established. Numerical experiments are provided to demonstrate the utility and robustness of the proposed scheme.

1 INTRODUCTION

The concept of solving PDEs using *radial basis functions* (RBFs) was introduced by Kansa in 1990.[6,7] He implemented this approach using the *multiquadrics* radial basis function (MQ RBF) proposed by Hardy[5] for interpolation of scattered data. Kansa's MQ method for solving PDEs is a meshless collocation method with global basis functions, which leads to finite-dimensional problems with full matrices. In[21] we have introduced a meshless local interpolation scheme (LOPI) that compares well with Hardy's interpolation. LOPI is $r-$*reproducing* and has compactly supported basis. Worth to be mentioned, it also verifies the delta-Kronecker property, simplifying greatly the introduction of essential boundary conditions. Roughly speaking, the scheme consists of local interpolation based in Optimal Point interpolants,[9] pasted together with help of the Partition of Unity concept. In[19,23] we began the study of the use of LOPI for solving PDEs in a meshless collocation method that produces very accurate results.

It is already known that the shape parameter c_q in radial basis functions interpolators can meaningfully affect the quality of the interpolator. An $h-c$ adaptive method is proposed by Kansa et al.[1] which tries to take advantage of this fact. In Section 5 we discuss more largely this issue. The goal of this paper is to propose a multiquadric meshless method to overcome the problem of the shape parameter optimization. The method is easy to implement, stable, and produces accurate solutions. We also begin here a discussion of an h adaptive methodology for LOPI. Several numeric experiments and tests are given. The last Section provides some concluding remarks.

2 LOPI INTERPOLANTS

Let $||\,.\,||$ denote the Euclidean norm in n-dimensional space $\mathbb{R}^n$ and $B(\mathbf{y};r)$ the open ball $\{\,\mathbf{x}=(x_1,\ldots,x_n)\in\mathbb{R}^n : ||\mathbf{x}-\mathbf{y}||<r\,\}$ with center $\mathbf{y}$ and radius r. We use standard multi-index notation. In particular, given any multi-index $\nu=(\nu_1,\ldots,\nu_n)\in\mathbb{R}^n$, $|\nu|$ denotes the sum $\nu_1+\cdots+\nu_n$, and $D^\nu f$ the partial derivative $\frac{\partial^{|\nu|}}{\partial x_1^{\nu_1}\ldots\partial x_n^{\nu_n}}f$ for any sufficiently smooth function f.

Let $\mathbf{f}=(f_\beta)$, $\beta=0,1,\ldots,K$, be the set of given values at distinct points $\mathcal{X}=\{\mathbf{x}_0,\mathbf{x}_1,\ldots,\mathbf{x}_K\}$, and g the *multiquadrics* (MQ) radial C^∞ function :

$$g(\mathbf{x})=\sqrt{c_q^2+\,||\mathbf{x}||^2},\quad \mathbf{x}\in\mathbb{R}^n \tag{1}$$

for a given constant $c_q>0$ called the *shape* parameter. We may construct an exact interpolant of data $\mathbf{f}$ as the linear combination

$$T_{\mathcal{X}}(\mathbf{f})=\sum\nolimits_{\beta=0}^{K} a_\beta\cdot g(\mathbf{x}-\mathbf{x}_\beta)$$

of translates of the basic function g by using the *Vandermondian* (or *Grammian*) of g

$$V = \begin{bmatrix} g(\mathbf{x}_0 - \mathbf{x}_0) & \cdots & g(\mathbf{x}_K - \mathbf{x}_0) \\ \vdots & \ddots & \vdots \\ g(\mathbf{x}_0 - \mathbf{x}_K) & \cdots & g(\mathbf{x}_K - \mathbf{x}_K) \end{bmatrix}$$

Coefficients a_β may be computed using

$$\mathbf{a} = V^{-1}\mathbf{f}$$

whenever the Vandermondian V is invertible.

An interpolant $\mathcal{O}$ is $r - reproducing$ iff $O[\mathbf{P}] = P \quad \forall P$ polynomial of degree r. Interpolants constructed in this way are not *reproducing*. This property can be retrieved by augmentation, in a Boolean sum of operators. In order to do this, we must build a projector Q onto the space of polynomials we wish to preserve. For simplicity, we shall describe only the case $r = 1$. Therefore, the space we consider is the linear space $\mathcal{P}_1$ of polynomials of degree ≤ 1.

In the following, we shall consider $\mathbf{x}_0$ as a *center node*, with nodes $\mathbf{x}_1, \ldots, \mathbf{x}_K$ sprinkled around. We think of $\mathcal{P}_1$ as generated by a basis of monomials centered at $\mathbf{x}_0$; that is, $\mathcal{P}_1$ is generated by

$$\{b_0 = 1, b_1, \ldots, b_p\} = \{1, x_1 - x_{0,1}, \ldots, x_n - x_{0,n}\}$$

Note that $p \equiv n$ since the degree of polynomials is limited to $r = 1$.

We shall search for a weighted least square approximation to data $\mathbf{f}$, using a ('Taylor expansion') linear polynomial of the form

$$P_1(\mathbf{x}) = \sum_{0 \leq |k| \leq 1} a_k \, (\mathbf{x} - \mathbf{x}_0)^k,$$

where $k \in \mathbb{R}^n$, and involving a matrix

$$W = (W(\mathbf{x}_i, \mathbf{x}_0)) = \begin{pmatrix} W(\mathbf{x}_1, \mathbf{x}_0) & & 0 \\ & \ddots & \\ 0 & & W(\mathbf{x}_K, \mathbf{x}_0) \end{pmatrix}$$

with a radially decreasing weight function $W(.,.)$ that has still to be chosen. Coefficients a_k in P_1 are given by

$$\mathbf{a} = QT_{(\mathcal{X})}\,\mathbf{f}$$

where

$$QT_{(\mathcal{X})} = (B^T_{(\mathcal{X})} W B_{(\mathcal{X})})^{-1} B^T_{(\mathcal{X})} W$$

and where matrix $B_{(\mathcal{X})}$ is written

$$B_{(\mathcal{X})} = \begin{pmatrix} 1 & x_{1,1} - x_{0,1} & \dots & x_{1,n} - x_{0,n} \\ \vdots & \vdots & & \vdots \\ 1 & x_{K,1} - x_{0,1} & \dots & x_{K,n} - x_{0,n} \end{pmatrix}.$$

In order to verify the *r-reproducing* property, some geometrical conditions on $\mathcal{X}$ are needed to assure that $B^T_{(\mathcal{X})} W B_{(\mathcal{X})}$ is a non singular matrix. In[22] it was proved that a necessary and sufficient condition is $rank(B_{(\mathcal{X})}) = \dim(\mathcal{P}_r)$.

By setting now

$$PT_{(\mathcal{X})} = V^{-1}\left(I - B_{(\mathcal{X})} Q T_{(\mathcal{X})}\right) \tag{2}$$

together with vector functions

$$v(\mathbf{x}) = (g(\mathbf{x} - \mathbf{x}_0), \dots, g(\mathbf{x} - \mathbf{x}_K))$$

and

$$b(\mathbf{x}) = (1, b_1(\mathbf{x}), \dots, b_p(\mathbf{x}));$$

then, the local interpolating operator may be defined as follows :

$$\mathcal{IL}_{(\mathcal{X})}[\mathbf{f}](\mathbf{x}) = < v(\mathbf{x}), PT_{(\mathcal{X})}\mathbf{f} > + < b(\mathbf{x}), QT_{(\mathcal{X})}\mathbf{f} >, \qquad \mathbf{x} \in \mathbb{R}^n$$

where $< \cdot, \cdot >$ is the Euclidean scalar product of vectors.

Operator $\mathcal{IL}_{(\mathcal{X})} : \mathbf{f} \to \mathcal{IL}_{(\mathcal{X})}[\mathbf{f}]$ is clearly linear. Then, writing $\mathbf{f} = \sum f_\alpha \mathbf{e}_\alpha$ in term of standard basis, we have

$$\mathcal{IL}_{(\mathcal{X})}[\mathbf{f}] = \sum_\alpha f_\alpha \phi_\alpha \tag{3}$$

where $\phi_\alpha = \mathcal{IL}_{(\mathcal{X})}[\mathbf{e}_\alpha]$. Basic properties of the family of functions $\{\phi_\alpha\}$ are:

- $\phi_\alpha(x_\beta) = \delta_{\alpha\beta}$. That is, the $\delta - property$ is satisfied.
- $P = \sum_\alpha P(x_\alpha)\phi_\alpha$, for every polynomial of degree r (r is the chosen degree of reproducibility).

This squeme was introduced by Zuppa,[21] and it is inspired in Local Optimal Point Interpolations of Lancaster and Salkauskas (see 11.5 of[9]). It looks a little more complicated than interpolators of the form:

$$TP_{\mathcal{X}} = \sum a_\beta \cdot g(\mathbf{x} - \mathbf{x}_\beta) + \sum c_\gamma b_\gamma(\mathbf{x}) \tag{4}$$

treated in the theory of kriging interpolants and coefficients (a_β, c_γ) are obtained in a different way (see for example[12]). Interpolators $TP_{\mathcal{X}}$ have the same properties than $\mathcal{IL}_{(\mathcal{X})}$. When $r = 0$, it is required that $\sum a_\beta + c_0 = 0$ and this is the scheme used by Kansa in his global approach.[1,6,7]

3 APPROXIMATE SOLUTION TO PDES BY COLLOCATION

Let $\mathcal{Q}^N = \{\mathbf{x}_1, ..., \mathbf{x}_N\} \subset \overline{\Omega} \subset \mathbb{R}^n$ be a given set of nodes (Ω a bounded domain). We assume that, for every node $\mathbf{x}_\alpha \in \mathcal{Q}^N$, we have selected in some way a subset $S_\alpha \subset \mathcal{Q}^N$, called the *star* of $\mathbf{x}_\alpha$, such that S_α is the set of neighbor nodes of $\mathbf{x}_\alpha$. We assume of course that $\mathbf{x}_\alpha \in S_\alpha$, and $\mathbf{x}_\alpha$ is the first element. For later use, we shall denote d_α the diameter of S_α.

The procedure will be illustrated with elliptic PDEs. For the sake of simplicity, we assume the interior and boundary operators, $\mathcal{P}, \mathcal{B}_N$ and $\mathcal{B}_D$, respectively to be linear and that they define a well-posed elliptic boundary value problem:

$$\begin{array}{l} \mathcal{P}u(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in \Omega \\ \mathcal{B}_N u(\mathbf{x})|_{\Gamma_N} = s_N(\mathbf{x}) \\ \mathcal{B}_D u(\mathbf{x})|_{\Gamma_D} = s_D(\mathbf{x}) \end{array} \tag{5}$$

Here, B_D is a pure Dirichlet operator and $\mathcal{B}_N$ a Neumann or mixed operator. After reordering, we can partitioned the set of *nodes* $\mathcal{Q}^N$ in the form

$$\mathcal{Q}^N = \{(\mathbf{x}_\beta)|_{\beta=1,...,M_1} \subset \Omega, (\mathbf{x}_\beta)|_{\beta=M_1+1,...,M_2} \subset \Gamma_N, (\mathbf{x}_\beta)|_{\beta=M_2+1,...,N} \subset \Gamma_D$$

At every star S_α, we shall search for an approximate solution $\widehat{u}_\alpha$ of (5) in the form

$$\widehat{u} = \mathcal{IL}_\alpha[\mathbf{u}] = \sum_{\mathbf{x}_\beta \in S_\alpha \setminus \Gamma_D} u_\beta \phi_\beta + \sum_{\mathbf{x}_\gamma \in S_\alpha \cap \Gamma_D} u_\gamma \phi_\gamma \tag{6}$$

where $\mathcal{IL}_\alpha = \mathcal{IL}_{(S_\alpha)}$ and $u_\gamma = s_D(\mathbf{x}_\gamma)$.

Substituting $\widehat{u}$ into (5) and using collocation at nodes , we obtain the collocation system

$$\begin{aligned} \sum_{\mathbf{x}_\beta \in S_\alpha \setminus \Gamma_D} u_\beta \mathcal{P}[\phi_\beta](\mathbf{x}_\alpha) &= f(\mathbf{x}_\alpha) - \sum_{\mathbf{x}_\gamma \in S_\alpha \cap \Gamma_D} s_D(\mathbf{x}_\gamma) \mathcal{P}[\phi_\gamma](\mathbf{x}_\alpha), \\ \alpha &= 1, ..., M_0 \\ \sum_{\mathbf{x}_\beta \in S_\alpha \setminus \Gamma_D} u_\beta \mathcal{B}_N[\phi_\beta](\mathbf{x}_\alpha) &= s_N(\mathbf{x}_\alpha) - \sum_{\mathbf{x}_\gamma \in S_\alpha \cap \Gamma_D} b_D(\mathbf{x}_\gamma) \mathcal{B}_N[\phi_\gamma](\mathbf{x}_\alpha), \\ \alpha &= M_0 + 1, ..., M_1 \end{aligned}$$

In solving $\mathcal{P}[\phi_\beta](\mathbf{x}_\alpha)$, we are immediately faced with the problem of evaluating

$$\mathcal{P}[< v(\mathbf{x}_\alpha), PT\mathbf{e}_\beta > + < b(\mathbf{x}_\alpha), QT\mathbf{e}_\beta >]$$

By linearity, we can recast this equation as

$$< \mathcal{P}v(\mathbf{x}_\alpha), PT\mathbf{e}_\beta > + < \mathcal{P}b(\mathbf{x}_\alpha), QT\mathbf{e}_\beta >$$

The first term in the sum is

$$< \mathcal{P}v(\mathbf{x}_\alpha), V^{-1}(I - BQT)\mathbf{e}_\beta > = < V^{-1}(\mathcal{P}v(\mathbf{x}_\alpha)), (I - BQT)\mathbf{e}_\beta >$$

Therefore, instead of getting the inverse matrix V^{-1}, it is a little less expensive and more stable to solve equation $V\mathbf{y} = \mathcal{P}v(\mathbf{x}_\alpha)$.[23]

3.1 Neumann problems

The direct implementation of boundary Neumann condition, although formally straightforward, creates some effects that may reduce the accuracy of the collocation method in its most straightforward form. This is because:

1. Discretization of the boundary condition at a boundary node precludes discretization of the principal governing equation at this node (one equation is generated per degree of freedom). Thus, only internal nodes support the governing equation.

2. The boundary *stars* tend to be of strongly unsymmetrical shape, with the center of mass shifted towards the inside of the domains.

The accuracy of the solution is improved in finite difference communities using the following strategy:

For every *node* $x \in \Gamma_N$, an associated external node in the direction of the outward normal of Γ_N, was added, at a distance that agree with some measure of the size of the grid (and the geometry of $\partial\Omega$, of course). In this form, the number of nodes was increased, one for every node $x \in \Gamma_N$, and two equations at all boundary nodes in Γ_N were imposed: one equation resulting from the boundary condition, and one from the PDE.

A different technique is also implemented in:[20,23] the number of basis functions is augmented at every Neumann node instead.

In a node $\mathbf{x}_\gamma \in S_\alpha \cap \Gamma_N$, we define function ψ_γ by

$$\psi_\gamma(\mathbf{x}) =< n(\mathbf{x}_\gamma), \mathbf{x} - \mathbf{x}_\gamma > \cdot \phi_\gamma(\mathbf{x})$$

where $n(\mathbf{x}_\gamma)$ is the normal vector in $\mathbf{x}_\gamma \in \Gamma_N$. It can be observed that $\psi_\gamma(\mathbf{x}_\beta) = 0$ for all $\mathbf{x}_\beta \in \mathcal{Q}^N$ and $\frac{\partial}{\partial n}\psi_\gamma(\mathbf{x}_\gamma) = 1$.

If $S_\alpha \cap \Gamma_N \neq \emptyset$, we improve the approximation (6) by

$$\widehat{u} = \sum_{\mathbf{x}_\beta \in S_\alpha \setminus \Gamma_D} u_\beta \phi_\beta + \sum_{\mathbf{x}_\gamma \in S_\alpha \cap \Gamma_N} u_\gamma \psi_\gamma + \sum_{\mathbf{x}_\delta \in S_\alpha \cap \Gamma_D} s_D(\mathbf{x}_\delta)\phi_\delta \tag{7}$$

and, as before, two equations are imposed at every node $\mathbf{x}_\alpha \in \Gamma_N$.

4 STARS SELECTION

A good selection of *stars* is of fundamental importance in meshless method in general and in this method in particular. On must assure that *stars* tied together and there are enough *nodes* in S_α, but the use of large *star* increases the bandwidth of the global matrix.

4.0.1 Stars by Delaunay triangulation

The nearest-site search is efficiently resolved by Delaunay triangulations (Voronoy diagrams). Given a Delaunay triangulation of $\mathcal{Q}^N$, *stars* can be defined by chosen the nodes

connected to a given one at first order, second order, etc. In[20] two families of *stars* were considered in numerical experiments:

$V1:$ $S_\alpha^1 = \{\text{first neighbor nodes of } \mathbf{x}_\alpha\}$,
$V2:$ $S_\alpha^2 = \{\text{first and second neighbor nodes of } \mathbf{x}_\alpha\}$,

4.0.2 The sphere method

In[19,23] *stars* was defined by

$$S_\alpha = \mathcal{Q}^N \cap \omega_\alpha$$

where $\omega_\alpha = B(\mathbf{x}_\alpha, rh_\alpha)$, h_α is some measure of the mesh spacing in $\mathbf{x}_\alpha$, and $r > 0$. In[19,23] and in this work r will be equal to 2.1 and the diameter of the *star* d_α is then $2.1h_\alpha$. It is also assumed that $\overline{\Omega} \subset \cup_{\alpha=1}^N \omega_\alpha$.

The sphere method can be of difficult application in very irregular grids, but it has several advantages:

1. There is a rapidly growing body of literature concerned with generating good quality node sets for meshless methods using biting and sphere packing.[15–18] The radii of spheres are related with a mesh spacing function or the size of spheres are defined by a sizing function $h : \overline{\Omega} \to \mathbb{R}_+$.

2. Let $\mathcal{S}_N := \{\mathcal{W}_\alpha\}_{\alpha=1}^N$ be a positive C^s *partition of unity* subordinate to the open covering $\{\omega_\alpha\}$ and let $\mathcal{F} := \mathbb{R}^N$ be the set of possible values $\mathbf{f} = (f_\alpha)_{\alpha=1}^N$ of functions at nodes $\{\mathbf{x}_\alpha\}$. A global interpolator $\mathcal{O} : \mathcal{F} \to C^s(\mathbb{R}^n)$ can be defined by

$$\mathcal{O}[\mathbf{f}](\mathbf{x}) = \sum_{\alpha=1}^{N} \mathcal{IL}_\alpha[\mathbf{f}_\alpha](\mathbf{x}) \cdot \mathcal{W}_\alpha(\mathbf{x}) \tag{8}$$

   where $\mathbf{f}_\alpha$ is the restriction of $\mathbf{f}$ to the set of indexes corresponding to nodes in S_α. For the details, see.[21]

Using $\mathcal{O}$, solution of (5) can be defined over all $\overline{\Omega}$ and a more global error could be considered. Besides, this methodology can be applied in adaptive schemes in order to obtain a global sizing function which is used by biting method for building a new set of nodes and an associated new family $\{\omega_\alpha\}$.

5 OPTIMIZATION OF THE SHAPE PARAMETER

Several authors[7,14] have noticed that the shape parameter c_q in radial basis functions interpolators can meaningfully affect the quality of the interpolator. In certain cases, the deviation can be significantly decreased by increasing c_q and it is generally believed that this is always the case (see[10]). In[1] Kansa *et al.* developed an $h - c$ method along this line. But this brings us to one unpleasant feature of these interpolators: the matrices

which must be inverted become poorly conditioned. It has been argued that this, and the roundoff error, become the ultimate barriers for the interpolation to reach higher accuracy: there is an optimal large value of c_q before the solution breakdown. This is the Schaback's uncertainty principle.[14]

I think that this issue remain a challenging problem in RBF interpolators. Madych's proof concerns the approximation of a given function, not the solution of a PDE. Such a proof so far does not exist. In fact, a numerical experiment with model 1 of section 7 with a uniform mesh of size $h = 0.125$, gives the evolution of the maximal error shown in Figure 1. At all the values of c_q displayed in the figure, the condition number of the matrix are of order $O(1.0e + 7)$ and the numerical calculations do not breakdown at all.

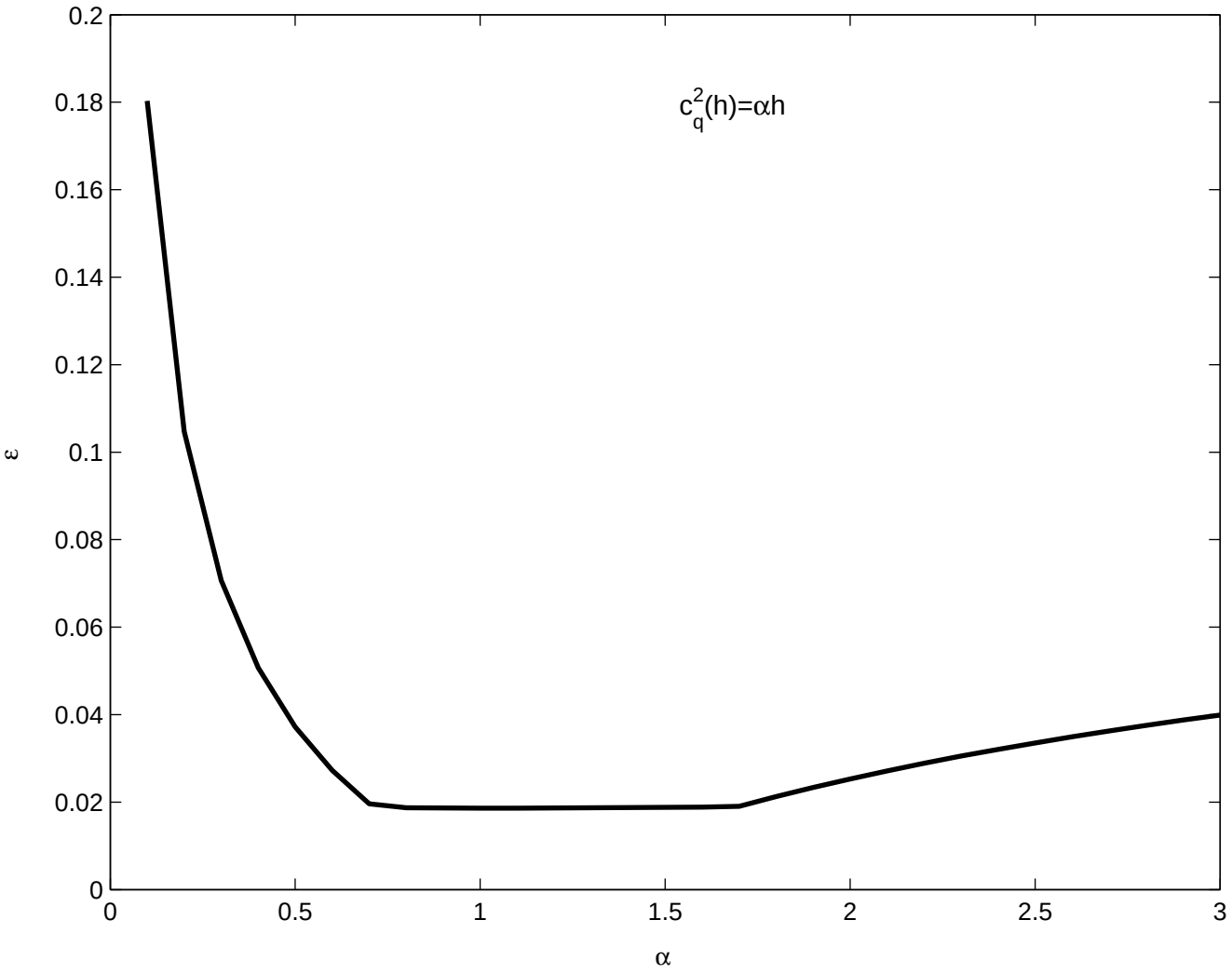

Figure 1: Maximal error vs. c_q

Following the same line of numerical experiments, we have obtained a fitting curve of optimal values of the shape parameter equal to

$$c_q(d) \approx 0.6251719 \cdot d^{0.12679} \tag{9}$$

where $d = 2.1h$.

Multiple attempts have been made to provide formulae for optimal c_q (see[13]), but I am not aware of any efficient squeme which can be used without caution. As a matter of fact, in solving PDEs using RDBs, the selection of the optimal shape parameter can be related with the methodology employed in discretizing the problem in preparation for numerical implementation.

Also, it should be pointed out that when $h \approx 0.031$, for example, we have obtained condition numbers of matrix V of order $1.e + 13$. Beyond that limit, the matrix solver needs to carry more and more floating points to maintain the accuracy of the solution. Fornberg and Wright[4] method extracts the singularity before the assemblage of the matrix in general RBF interpolators, thus alleviating the roundoff error problem.

Despite the lack of theoretical error estimates for the use of radial basis function in this asymmetric collocation method, it is useful to get an idea of what we should expect. Let f be a function defined in the convex hull CS_α of the *star* S_α with diameter d_α and suppose $\delta_\alpha = d_\alpha/2 = \max_{\mathbf{y}\in CS_\alpha}(\inf_{\mathbf{x}_\beta \in S_\alpha} ||\mathbf{y} - \mathbf{x}_\beta||)$. We want to estimate

$$\sup_{\mathbf{x}\in CS_\alpha} |f(\mathbf{x}) - \mathcal{IL}_\alpha[\mathbf{f}](\mathbf{x})| \tag{10}$$

By a change of coordinates $\mathbf{x} = \mathbf{x}_\alpha + (c_q(d_\alpha))^{-1}\mathbf{z}$, we can recast (10) as

$$\sup_{\mathbf{z}\in B(0,1)} |\widetilde{f}(\mathbf{z}) - \widetilde{\mathcal{IL}}[\widetilde{\mathbf{f}}](\mathbf{z})|$$

where the interpolator $\widetilde{\mathcal{IL}}$ uses now the standard multiquadric $h_1(\mathbf{z}) = \sqrt{1 + ||\mathbf{z}||^2}$, and the new δ has the value $\delta = (c_q(d_\alpha))^{-1}(d_\alpha/2) \approx 0.79 \cdot d_\alpha^{0.87}$. According to Madych's result,[10,11] we have

$$\sup_{\mathbf{z}\in B(0,1)} |\widetilde{f}(\mathbf{z}) - \widetilde{\mathcal{IL}}[\widetilde{\mathbf{f}}](\mathbf{z})| = O\left(\lambda^{\frac{1}{\delta}}\right)$$

with $0 < \lambda < 1$, if $\delta \to 0$.

This considerations should allow us to conclude the exponential convergence of the interpolator. Unfortunately, Madych's result requires that the function f (in fact, an extension over all $\mathbb{R}^n$) be in $L^2(\mathbb{R}^n, (|\xi|^2\widehat{h}_1(\xi))^{-1}d\xi)$, where $\widehat{h}_1$ is the Fourier transform of h_1. That is, f must be extremely good. We do remind the reader however that Madych's estimates concerns the approximation of a given function and not the solution of a PDE.

6 AN ADAPTIVE METHOD FOR LOPI

What make linear finite element method so attractive is its simplicity. The problem of optimizing the shape parameter in radial basis function approximations is, in my opinion, a burden that destroys the special appeals that this method could have. As we have remarked, a good fitting of optimal values of the shape parameter is an issue that remains to be solved. Simple recipes for the shape parameter are then especially attractive.

One simple recipe for the shape parameter that produce good numerical solutions was introduced in[23] . In this paper we introduce another recipe whose major attraction is its extreme simplicity and which appears to produce better results.

We shall use LOPI interpolants in $B(\mathbf{x}_\alpha, 1)$ with a fixed multiquadrics

$$g = \sqrt{A + ||\,\mathbf{z}\,||^2} \tag{11}$$

where $A > 0$ is a constant that has still to be chosen.

Given a *star* S_α centered in $\mathbf{x}_\alpha$ and with diameter d_α, the scaling function is the dilation $T : B(\mathbf{x}_\alpha, d_\alpha) \to B(\mathbf{x}_\alpha, 1)$ given by $T(\mathbf{x}) = \gamma(\mathbf{x} - \mathbf{x}_\alpha)$, where $\gamma = \frac{1}{d_\alpha}$. An interpolation problem in S_α is transformed into an interpolation problem in $B(x_\alpha, 1)$ via the change of coordinates given by T. Also, all necessary calculations in discretizing PDEs are made at this fixed scale.

The good behavior of constant $A = 35$ has been determined experimentally and has been used in our numerical examples. By scaling, the condition number of matrix V has always an $O(1.0e + 12)$ behavior.

An a posteriori error estimate compatible with Taylor expansion is given at each *node* $\mathbf{x}_\alpha$ by

$$Er(\mathbf{x}_\alpha) = d_\alpha^2 \cdot \max_{|\eta|=2} ||D^\eta \widehat{u}(\mathbf{x}_\alpha)||$$

This indicator is scaled over the domain

$$Ind(\mathbf{x}_\alpha) = \frac{Er(\mathbf{x}_\alpha)}{Erm}$$

where $Erm = mean_{\mathbf{x}_\alpha \in \mathcal{Q}^N} (Er(\mathbf{x}_\alpha))$.[8]

The implementation of an h refinement is achieved simply by inserting *nodes* in the region where $Ind > 1$. *Nodes* might be inserted following the algorithm described in.[3] Numerical experiments seem to hint however that grid size can not change too abruptly and perhaps this is a characteristic of collocation methods. Better results are obtained by using a smooth size function in biting method in order to build a new grid. The new size function can be built by interpolating function

$$f(\mathbf{x}_\alpha) = \frac{d_\alpha}{Ind(\mathbf{x}_\alpha)} \tag{12}$$

This squeme was used in model 2 of next section.

These two different approaches must be further investigated.

7 NUMERICAL EXAMPLES

We shall now perform several numerical tests to investigate the approximating properties of the new LOPI. We first make all settings in our experiments explicit. All of our examples are two dimensional and will measure error in the solution and, in some case, its derivatives.

We shall use the following error measures

$$\varepsilon l^2 = \frac{1}{\max_Q |u(\mathbf{x}_\beta)|}\sqrt{\frac{1}{|Q|}|u(\mathbf{x}_\beta) - \widehat{u}(\mathbf{x}_\beta)|^2}$$

$$\varepsilon xl^2 = \frac{1}{\max_Q |u_x(\mathbf{x}_\beta)|}\sqrt{\frac{1}{|Q|}|u_x(\mathbf{x}_\beta) - \widehat{u}_x(\mathbf{x}_\beta)|^2}$$

$$\begin{aligned} \varepsilon yl^2 &= \frac{1}{\max_Q |u_y(\mathbf{x}_\beta)|}\sqrt{\frac{1}{|Q|}|u_y(\mathbf{x}_\beta) - \widehat{u}_y(\mathbf{x}_\beta)|^2} \\ \varepsilon &= \max_Q |u(\mathbf{x}_\beta) - \widehat{u}(\mathbf{x}_\beta)| \end{aligned}$$

Test with both random and on uniformly spaced nodes were performed. In the former case, nodes were generated by adding a random perturbation of value $0.25h$ to a uniform grid with $h-$spacing. These kinds of grids will be called *uniform* and *random uniform* grids, respectively. Error displayed in figures in the randomly distributed points case, correspond to average over ten runs. It should be remarked that in all tests in uniform grids, *stars* were selected with the sphere method with $r = 2.1h$. In random grids the $V2$ method was used instead.

7.1 Model 1

This test example is a Poisson equation with an exponential source on a unit square domain. Mixed boundary conditions are imposed. This example was proposed by Aluru [2] to test performance of methods in presence of both high local gradient and mixed non-homogeneous boundary conditions.

The governing equation and imposed boundary conditions may be written:

$$-\Delta u = f$$

where

$$\begin{aligned} f(x,y) &= -6x - 8y \\ &\quad - \left[\frac{4}{\alpha^2} - 4\left(\frac{x-\beta}{\alpha^2}\right)^2 - 4\left(\frac{y-\beta}{\alpha^2}\right)^2\right] \exp\left[-\left(\frac{x-\beta}{\alpha}\right)^2 - \left(\frac{y-\beta}{\alpha}\right)^2\right] \end{aligned}$$

$$u(x=0) = -y^3 + \exp\left[-\left(\frac{\alpha}{\beta}\right)^2 - \left(\frac{y-\beta}{\alpha}\right)^2\right]$$

$$u(x=1) = -1 - y^3 + \exp\left[-\left(\frac{1-\beta}{\alpha}\right)^2 - \left(\frac{y-\beta}{\alpha}\right)^2\right]$$

$$u_y(y=0) = \frac{2\beta}{\alpha^2}\exp\left[-\left(\frac{\alpha}{\beta}\right)^2 - \left(\frac{x-\beta}{\alpha}\right)^2\right]$$

$$u_y(y=1) = -3 - 2\frac{1-\beta}{\alpha^2}\exp\left[-\left(\frac{x-\beta}{\alpha}\right)^2 - \left(\frac{1-\beta}{\alpha}\right)^2\right]$$

with $\alpha = 0.07$, $\beta = 0.5$.

Figure 4 displays the results in uniform grids for LOPI, kriging 0, kriging 1 and the shp scheme of Aluru, and shows the better behavior of the new LOPI scheme.

7.2 Model 2

This is a Dirichlet problem with a singularity at the origin. The governing equation is

$$\Delta u = 0, \qquad \Omega = \{(x,y)|\ -1 < x < 1, 0 < y < 1\}$$

The Dirichlet condition $u|\partial\Omega = g$ is assumed in such a way that the exact solution is

$$u(x,y) = \sqrt{\sqrt{x^2+y^2} - x}$$

In a uniform grid with $h = 0.0625$, the following results were obtained

ε	εl^2
$3.356e-002$	$-6.526e+000$

The problem was recalculated in a new refined grid which was constructed by biting method and function size (12). The new results were

ε	εl^2
$8.084e-003$	$-7.984e+000$

8 CONCLUSIONS

Several computational features of the new method proposed are demonstrated:

- The numerical experiments performed show that the new LOPI scheme is a very efficient method. The method is highly flexible and produces accurate results. Moreover, it is truly meshless and of simple implementation.

- The method can be strongly related to the new biting and sphere packing methods to produce good quality point sets for meshless method.
- Much additional work remains to be done, specially for an $h-$adaptive theory. Nevertheless, the results presented in this paper are quite encouraging and show that the new LOPI scheme might have a great potential to become a very competitive method in some finite difference applications: advective problems, etc. We hope to show this claim in future works.

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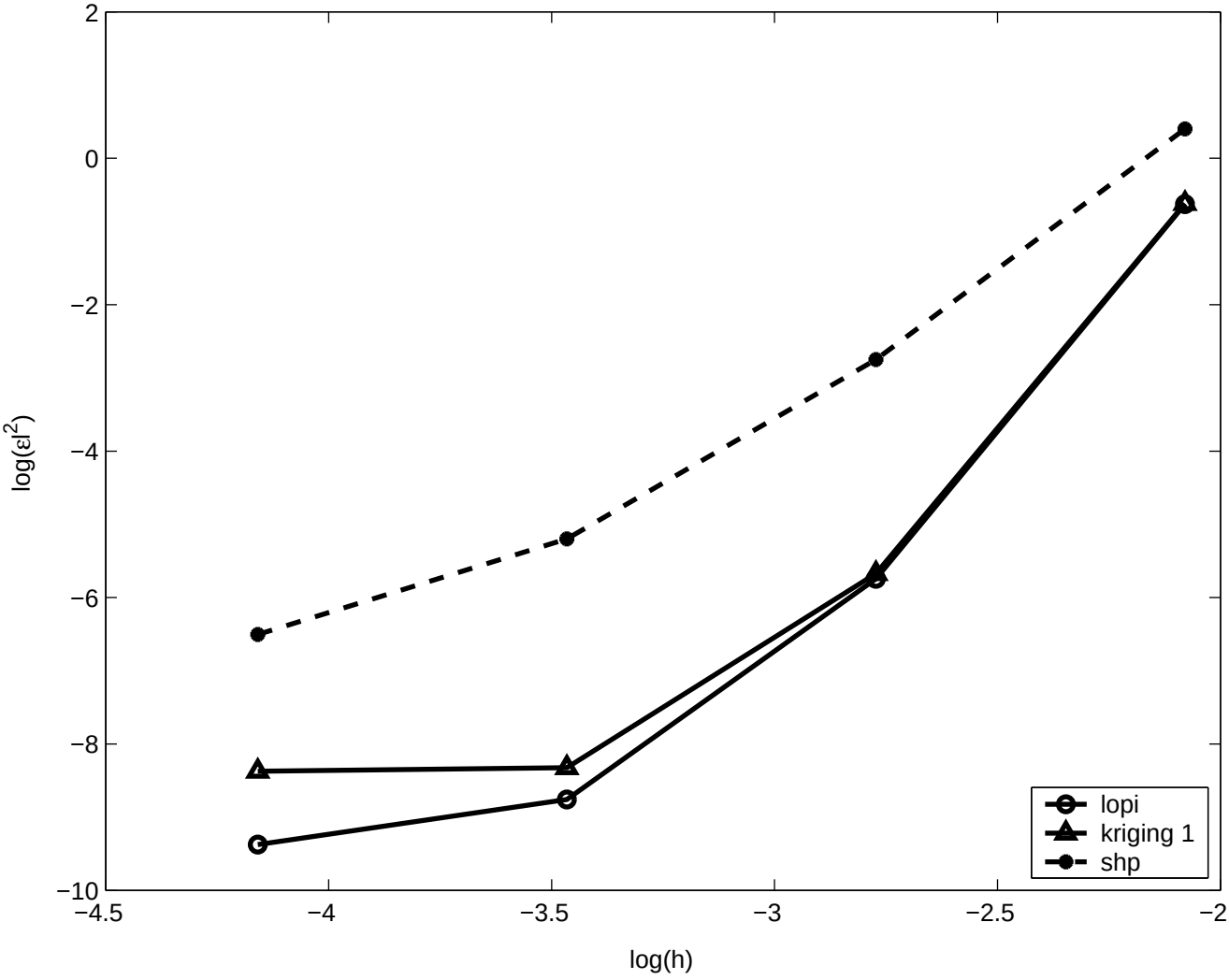

Figure 2: Convergence of LOPI and the shp scheme of Aluru