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ERROR ESTIMATION IN HIERARCHICAL FINITE ELEMENT METHOD APPLIED TO ACOUSTIC CAVITIES

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Abstract. *The main objective of this paper is to present an error estimator applied to the adaptive refinement process in acoustic dynamic analysis. A posteriori error estimator based on Fribergs formulations is developed, and the numerical solutions is obtained using triangular hierarchical finite elements. The procedure is tested using analytical solutions of 2-dimensional acoustic cavities. The acoustic frequencies and mode shapes are compared with the analytical solutions for different initial mesh. The influence of mesh distortions is also evaluated.*

1 INTRODUCTION

There is a large bibliography on the error analysis applied to Finite Element method, a good introduction about this subject can be found in Noor.[1] There is two main techniques from which was developed a great number of particular methods to specific problems, one of this is called recovery method also known as ZZ or Z^2 or Zienkiewicz Zhu. This method is the base for different versions in different applications[2,9] of the finite element method, its basic principle can be found on papers by Zienkiewicz and Zhu.[4–7] Another method is the residual method and its principle can be found in Babuška.[10,11] Both methods are considered *a posteriori*, it means that this techniques use information obtained during the solutions process, in addition to some assumptions about the results, and both are local methods computed on elements of the discretized domain.

The error analysis of the Finite Element method applied to external acoustic problems received great attention in the last decade, and I. Babuška gave significant contributions to this topic.[13–18] Part of that work was published by F. Ihlenburg,[19] where an additional bibliography can be found. The most important result from that research was the discover of the numerical pollution on infinite domain acoustic problems. The pollution error comes from non local influences on the local error.[2,13]

On the other hand, in eigenvalue problems, contributions from N.-E. Wiberg *et. al.*[9] in vibrations and Fuenmayor *et. al.*[20] in internal acoustics has utilized a *h-adaptive* procedure based on a error estimator derived from the recovery method. The results published by Fuenmayor to internal acoustics problem proved that the procedure is effective, but it conducts to a high degree of mesh refinement.

This work applies to internal acoustic problem a error estimator developed to p version of Finite Element method. The error estimator was presented by O. Friberg[21] and it is based on the energy norm. This estimator was derived from the property of the matrices obtained from hierarquical shape functions where the new eigenpair can be evaluate from previous results.[21,22,26] This particular aspect from what the error estimator was obtained means that the estimator uses the whole information of the matricial system, not only local information as the traditional estimators do.

This document presents results from the application of the Friberg's error estimator on internal acoustic problem and the influence of the mesh properties on the reliability of the finite element solutions at the end of adaptive process. A comparison between theoretical modes and post-processed eigenvetors achieved at the end of the iterative routine is presented too.

2 FORMULATION OF THE PROBLEM

The Equation (1) is the Helmholtz equation and it describes the two-dimensional acoustic domain:[23,24]

$$\nabla^2 P - \frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} = 0 \tag{1}$$

P represents the pressure oscillation from equilibrium, and c is the acoustic velocity. Multiplying the Equation (1) by a weight function , using Galerkin and applying Green, the Helmholtz equation can be written in the weak formulation as:

$$\int_{\Omega}\left(\frac{\partial P}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial P}{\partial y}\frac{\partial v}{\partial y}\right)d\Omega+\int_{\Omega}v\frac{1}{c^2}\frac{\partial^2 P}{\partial t^2}d\Omega=\oint_{\Gamma}v\frac{\partial P}{\partial n}d\Gamma \tag{2}$$

Assuming that the solution can be written as a product of two functions of time and space, discretizating the domain and approximating the solution by shape functions, the generalized formulation can be expressed on typical eigenvalue problem:

$$[H_f]\{P\}-\frac{\omega^2}{c^2}[E_f]\{P\}=0 \tag{3}$$

In this expression $[H_f]$ and $[E_f]$ will be called respectively the volumetric and compressibility matrix. The shape functions employed to approximate the solution are conventional shape functions in area coordinates and the hierarchical shape functions are those suggested by Peano.[26] The boundary condition are of the null pressure over the boundary $P = 0$.

3 ERROR ESTIMATOR

The error estimator is based on the properties of the hierarchical matrix which comes from the approximation by the finite element technique using hierarchical shape functions. The hierarchical shape functions are polynomials of order p which are a subset of polynomials of order $(p + 1)$, so that the matrix produced by finite element method from approximation polynomials of order p is also a subset of a matrix from the $(p + 1)$ polynomials. Using the energy norm and this hierarchical property a error estimator was derived by P. O. Friberg.[21] The error estimator which is applied to the Equation (3) can be expressed by:

$$\eta_{i,j}^{k}=\frac{\{[H_{n+1,n}-\lambda_i^n E_{n+1,n}]\{P_i^n\}\}^2}{k_i\,[H_{n+1,n+1}-\lambda_i^n E_{n+1,n+1}]} \tag{4}$$

where $\eta_{i,j}^{k}$ is the estimate error to i th frequency obtained by increasing the hierarchical order from n to $n+1$ on the j th degree of freedom of the k th element. For a $n \times n$ system, λ_i^n represents the calculated eigenvalue and $\{P_i^n\}$ the correspondent computed eigenvetor. The term k_i is determined by $k_i = \{P_i^n\}^T[H]_{n,n}\{P_i^n\}$.

From Equation (4), if the mesh has q elements the estimate error for the i th frequency is the sum of the estimate error of each degree of freedom for each one of the q elements: $\eta_i = \sum_{k=1}^{q}\sum_{j=1}^{m}\eta_{i,j}^{k}$, considering m degrees of freedom for the k th element.

4 NUMERICAL RESULTS FOR INTERNAL CLOSED ACOUSTIC CAVITY

This section presents the numerical results from the p adaptive procedure using the error estimator give by Equation (4). To do that, natural frequencies and mode shapes of

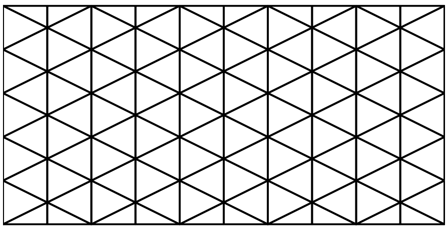


Figure 1: Mesh *F110*, 110 elements, 71 nodes, mean element quality $\alpha_{med} = 0.928$, $h_{med} = 0.635$, 90 elements with $\alpha = 0.99$, 20 elements with $\alpha = 0.693$.

a rectangular cavity are analyzed at the end of the adaptive process. The results comes from a cavity that has L=3,048m height , and H=6,096m length, and it is fulfilled with a fluid that exhibits a density ρ_0=999,21Kg/m^3 and a sound velocity c=1524m/s. The boundary condition is null pressure on the borders, and it is the same as presented on the reference Y. S. Shin and M. K. Chargin[28] where the others properties and dimensions comes by. To show the influence of the element quality on the results, two mesh with different degree of distortion is included.

The quality of one element can be defined as:[29]

$$\alpha_i = \frac{4\sqrt{3}A_i}{l_1^2 + l_2^2 + l_3^2} \tag{5}$$

where l_1, l_2 and l_3 are sides of the triangular element and the variable A_i represents the element area. The mesh quality can be expressed by:

$$\alpha_{med} = \frac{\sum_{i=1}^{q} \alpha_i}{q} \tag{6}$$

To express the refinement degree of the triangular mesh with q elements the characteristic dimension h_{med} is employed:

$$h_{med} = \frac{\sum_{i=1}^{q} \left(\frac{1}{3} \sum_{j=1}^{3} l_j \right)}{q} \tag{7}$$

The mesh *F110*, Figure (1), and mesh *F105d*, Figure (2), show two mesh qualities and different distribution of the distorted element on the discretized domain. This two meshes have almost the same number of nodes and elements and its main characteristics can be found on Figures (1) and (2) and on Table (1).

The analytical solution of the internal acoustic problem as above formulated is expressed by:[24]

$$\omega_{(m,n)} = c\sqrt{\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{H}\right)^2} \tag{8}$$

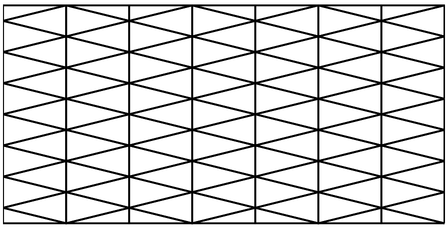


Figure 2: Mesh *F105d*, 105 elements, 68 nodes, mean element quality α_{med} =0.675, h_{med} = 0.733 m, 91 elements with α =0.729, 14 elements with α =0.408.

In this equation the variables c, L and H are respectively the speed of sound in the fluid, length and height of the cavity, $m = 1, 2, 3 \ldots \infty$ and $n = 1, 2, 3 \ldots \infty$. The mode shapes are described by the expression:[23]

$$P_{m,n}(x, y) = \sin\left(\frac{m\,\pi x}{L}\right) \sin\left(\frac{n\,\pi y}{H}\right) \tag{9}$$

Knowing the exact frequency $\omega_{(m,n)}$ it's possible to determine the finite element solution error, this error will be called computed error or actual error to distinguish it from the estimate error calculated by Equation (4).

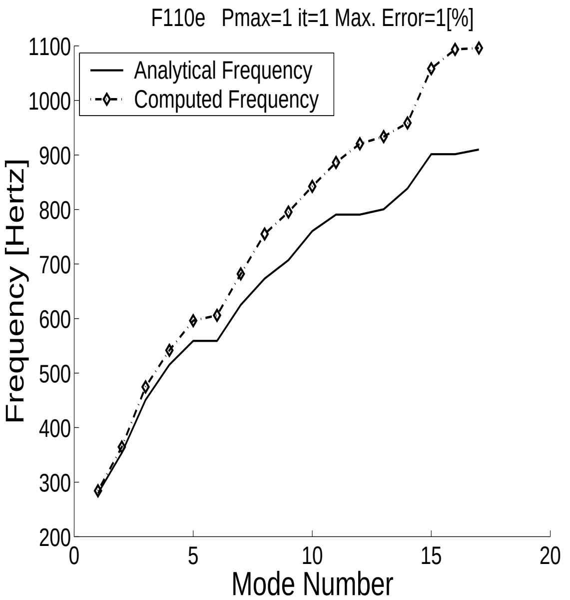
The finite element solution is obtained at the end of the iterative process in which the error is under a prescribed value for a frequency range. In all cases reported here the upper frequency limit is 1110 $Hertz$ and the error acceptable should be under 1,0%.

To the mesh *F110* the iterative process can be seen on Figures (3) and (4). The Figure shows the evolution of the analytic and computed frequencies in function of the mode number in each iteration of the p adaptive process. The Figure (4) presents the behavior of the actual or computed error and estimate error associate to frequency against the mode number to the same process.

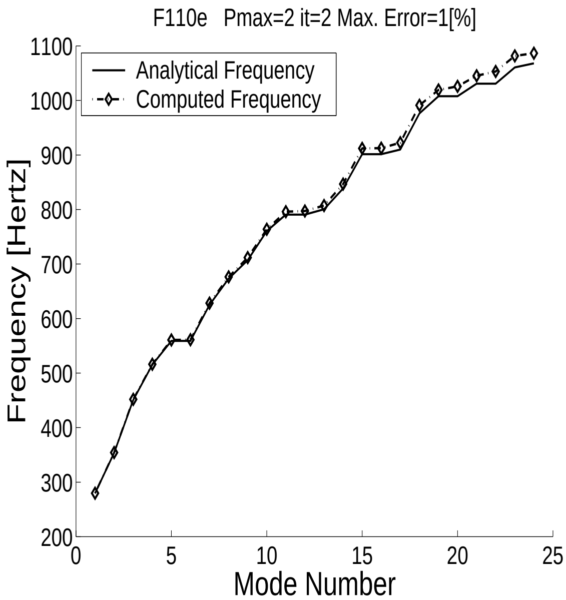
It should be emphasized that the error estimator employed is a eigenvalue estimator, it only give a guarantee that the frequency or eigenvalue error is under a specified value, but it does not assure settling the eigenpair. For instance, on the Figure (3) the modes 15, 16 and 17 have very close frequencies, if the difference between two neighbor frequencies is

mesh	*nodes*	*elements*	α_{med}	h_{med}	*Eq. of system*
F110	*71*	*110*	*0,928*	*0,635*	*41*
F105d	*68*	*105*	*0,675*	*0,733*	*39*

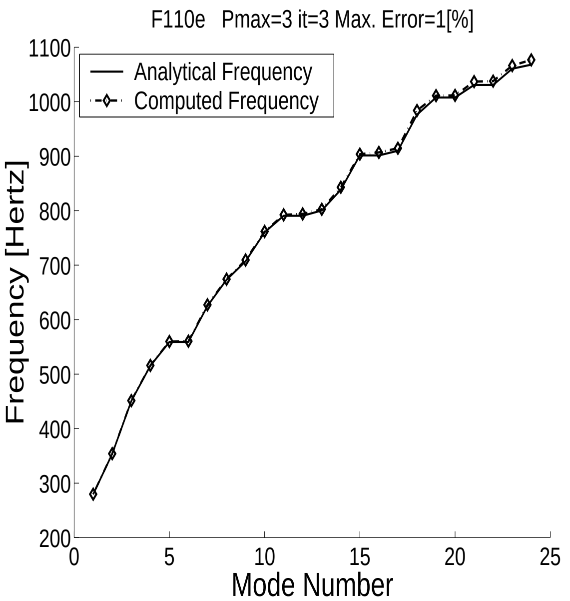
Table 1: Properties of the employed meshes.



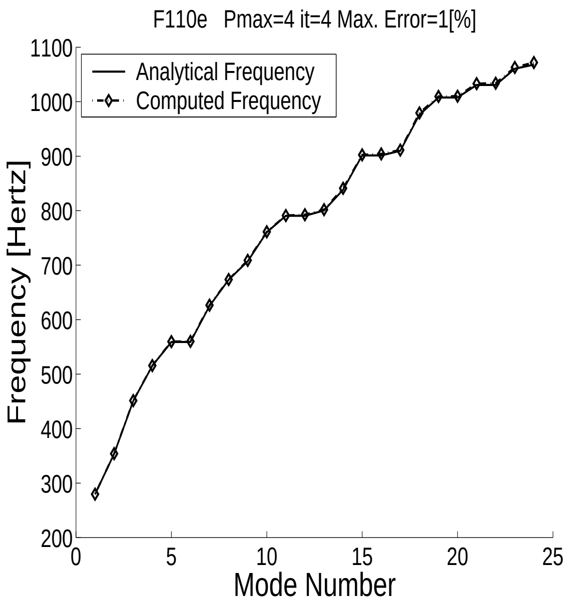
(A) First iteraction.



(B) Second iteraction.



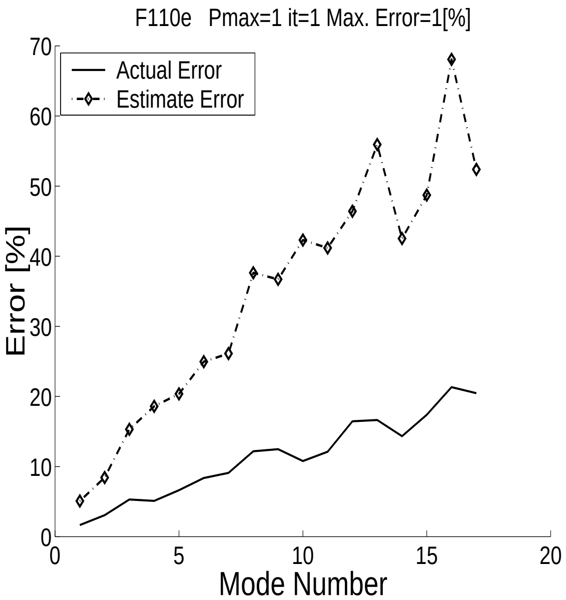
(C) Third iteraction.



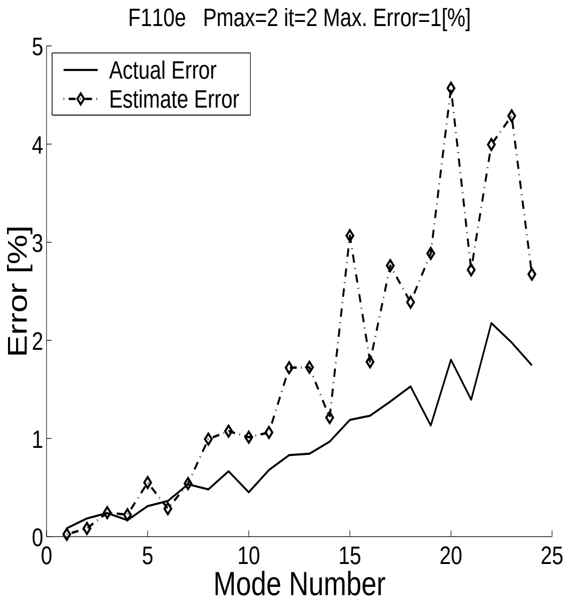
(D) Fourth iteraction.

Figure 3: Computed and actual frequencies as a function of the mode number. Mesh *F110*.

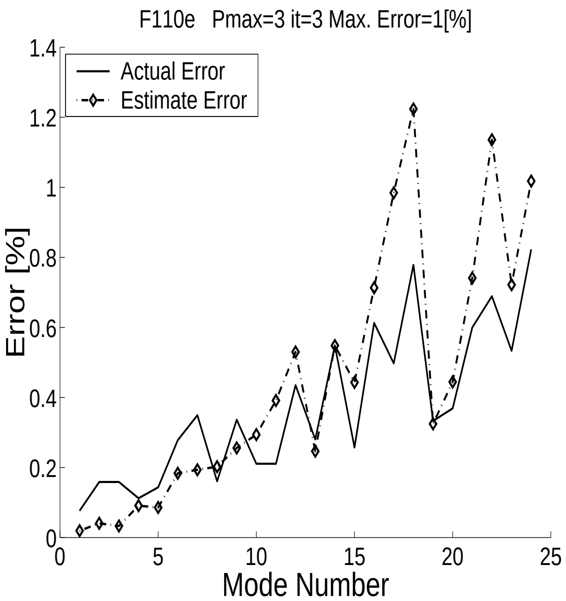
under the specified error value, then it could occur a change in the aigenvalue order. But independently of the order, the error of the frequencies will be under desired value.



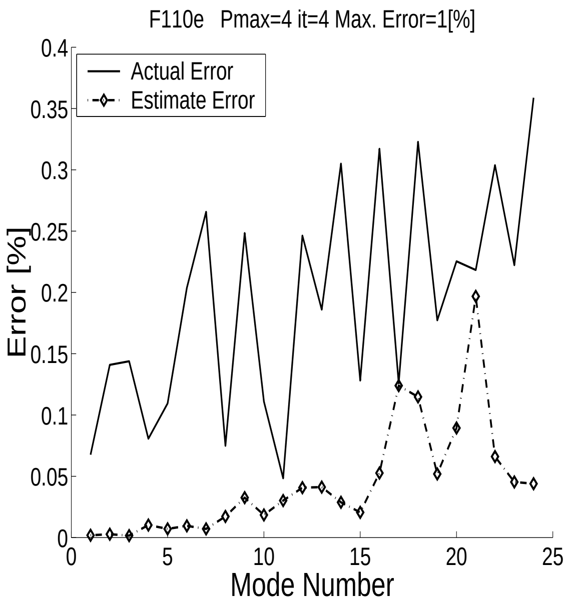
(A) First iteraction.



(B) Second iteraction.



(C) Third iteraction.



(D) Fourth iteraction.

Figure 4: Estimate and actual error as a function of the mode number. Mesh *F110*.

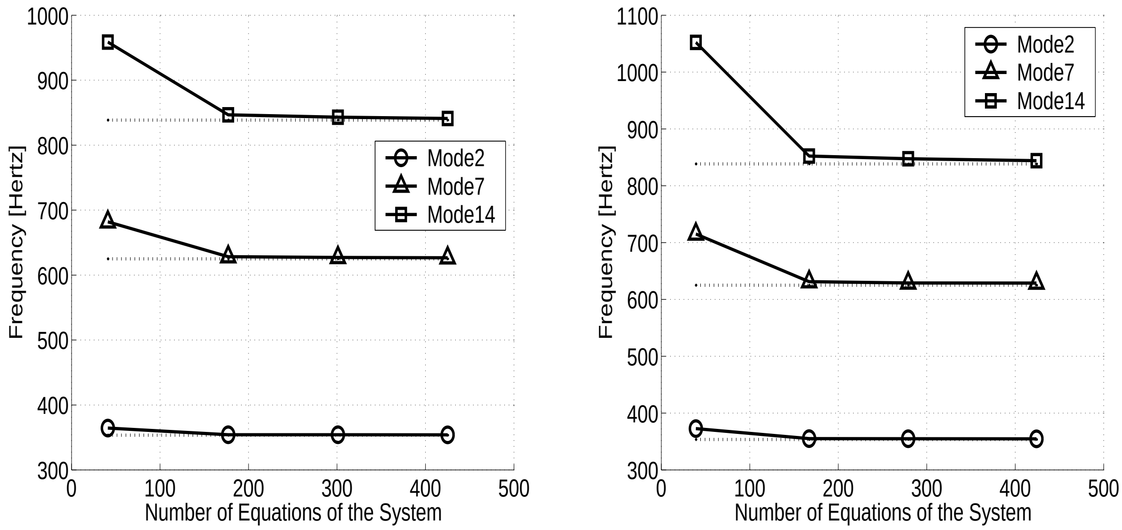
4.1 The Internal Closed Acoustical Cavity Using Different Mesh Quality

Triangular elements in area coordinate are used in the numerical approximation to the 2-dimensional acoustic problem. The elementary matrix is computed in area coordinate with a subsequent coordinate transformation to global domain. This coordinate transformation is supposed to be linear (Jacobian of transformation is constant), but it is linear only if the elements were not deformed or if the angles did not change.[11] Therefore the triangular elements always exhibit a degree of distortion with error on result except for perfectly equilateral elements.

The Figures (5-A), (5-B) and Figure (7) are included to disclose the influence of the mesh quality on the adaptive process efficiency. The Figure (5) exhibits the frequency evolution as a function of the number of equations of the system for successive iterations. This graphics are drawn for the $2\,th$, $7\,th$ and $14\,th$ modes, this shape modes are shown on the Figures (8) and (9) for the meshes *F110* and *F105d*.

The graphics (5-A) and (5-B) show very close behavior during the iterative process although the quality coefficient of the mesh α_{med} has been changed from 0.928 to 0.675. This Figure also shows that the curves for each mode shape evolution are similar. The Figure (5) does not reveal the necessary resolution to compare the influence of the mesh on error estimator once the greatest error is under 1.0%.

The estimate error is the work variable of the adaptive process, so it is chosen in conjunction to actual error to compare the behavior of the iterative process of the two meshes. To show up the influence of the mesh quality, the graphic of error as function of number of equations of the system are displayed on Figure (6) for the $2\,th$, $7\,th$ and $14\,th$ mode shapes for each one of the meshes. On this graphics the intersection point of error estimate curve and computed error curve represents the limit of confidence of the error estimate procedure. Under this confidence point the error estimator loses the ability to



(A) (B)

Figure 5: Frequencies versus number of equations of the system. Modes 2, 7 and 14 for the meshes *F110* (a), and mesh *F105d* on graphic (b).

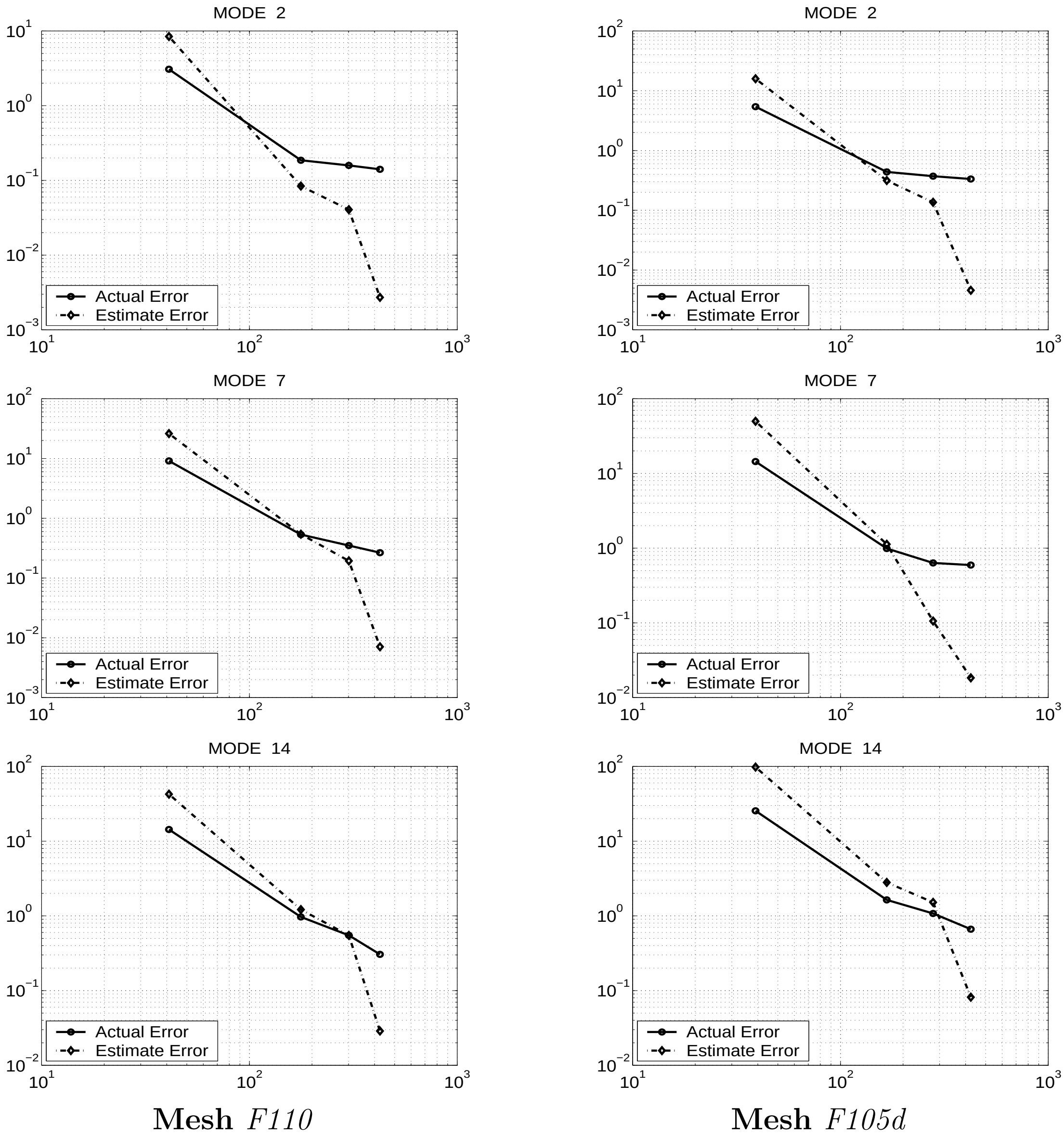


Figure 6: Estimate and actual error versus number of equations of system. Modes 2, 7 and 14 meshes *F110* and *F105d*.

identify properly the error.

Comparing the evolution of the mode shapes of the mesh *F110* in Figure (6) and the equivalent graphics of the mesh *F105d* one can observe that the latest has a higher confidence points than the former. The mesh distortion looks like to become the results a little more inaccurate. But, to the chosen mode shapes show on Figure (6) the confidence point for the two meshes seems to be very close.

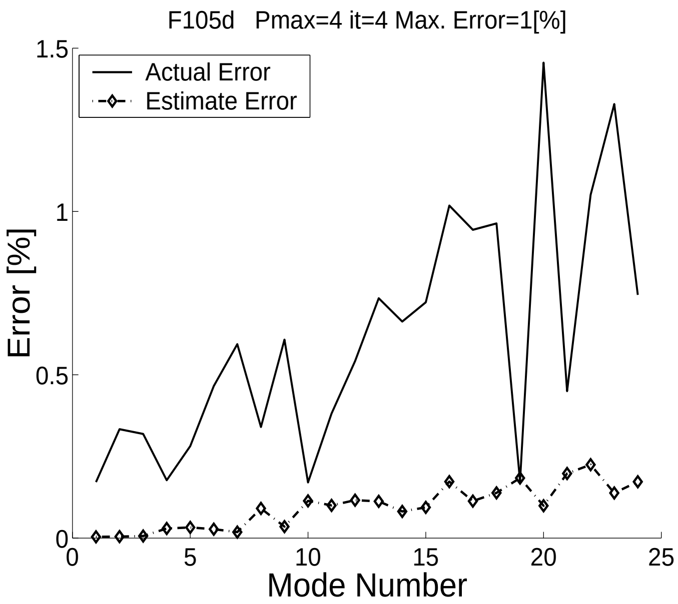
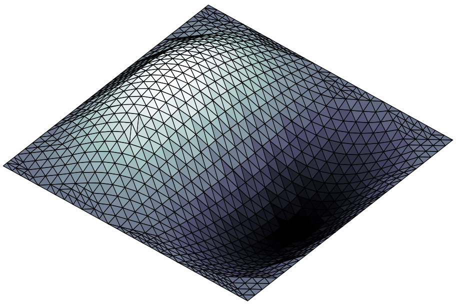


Figure 7: Estimate and actual error relative to the frequencies versus mode number. Fourth iteration, mesh *F105d*.

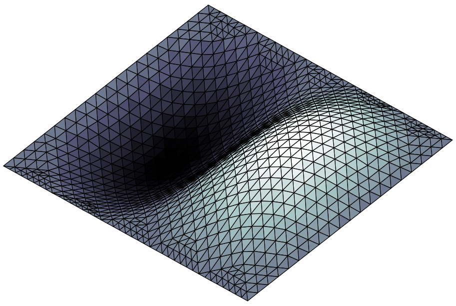
The error graphic of latest iterative process to the mesh *F105d*, showed on Figure (7), is more enlightenment. In this figure the estimate error curve is under the actual error curve, and for the higher frequencies of the range of interest the actual error curve overcome the prescribed error that is fixed in 1.0%. The adaptive process attempts to put the error under a determined value so if the routine is unable to estimate the correct order of the error there is no confidence in the process. In this case is more suitable to consider that the mesh *F105d* do not show a discretization quality enough to warrant a error under 1.0% for frequencies below 1110 Hz.

The analysis made on Figure (6) results from the post processing the eigenvectors, and gathering the similar ones in each iteration to put the results on figure (6). So this figure arise from some some physical interpretation of the results taken from adaptive process allowing to validate and evaluate its accuracy and its utility. But, the adaptive routine does not do this, so in Figure (4) throughout the iterations might occur changing in the order of the shape modes due a error in the closest eigenvalues or due a big error in a eigenvalue.

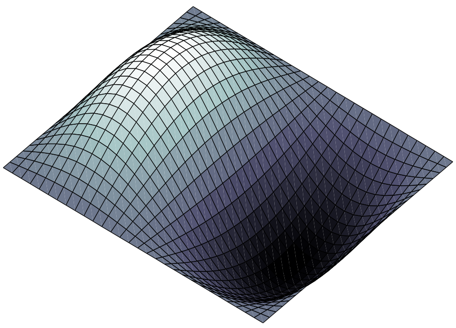
The Figure (8) and Figure (9) show up the shape modes 2, 7, and 14 taken from the eigenvectors at the end of the iterative process to the meshes *F110* and *F105d*. This shape modes are included post-processed with the analytical shapes modes to show the ability of the *p*-formulation to detail the shape modes. The analytical shape modes are obtained from the Equation (9). As pointed out, the error indicator is a eigenvalue error estimator, but it computes the eigenvalue error using the eigenvectors, so the result is the



(a)



(b)



(c)

Figure 8: Mode 2 post-processed for mesh *F105d* (a) and mesh *F110e* (b). Exact solution on graphic (c).

improvement of the eigenvectors accuracy too. The eigenvector post-processing enhances the shape mode description as it uses the expression:

$$P = P_1N_1 + P_2N_2 + P_3N_3 + a_1N_4 + a_2N_5 + \ldots + a_nN_n \tag{10}$$

where N_i represents the physical and hierarchical shape functions, P_j is the nodal pressure, and a_i the hierarchical coefficients. P_j and a_i are taken from the eigenvector. In practical problems the shape mode of interest are those of low order, hence the eigenvector post-processing gives enough accuracy to depicts this shape mode.

5 CONCLUSIONS

This work shows that the error estimator developed by Friberg can be used on p adaptive procedure to the dynamical internal acoustic problems. This error estimator is applied to the hierarchical finite element approximation to a 2-dimensional problem.

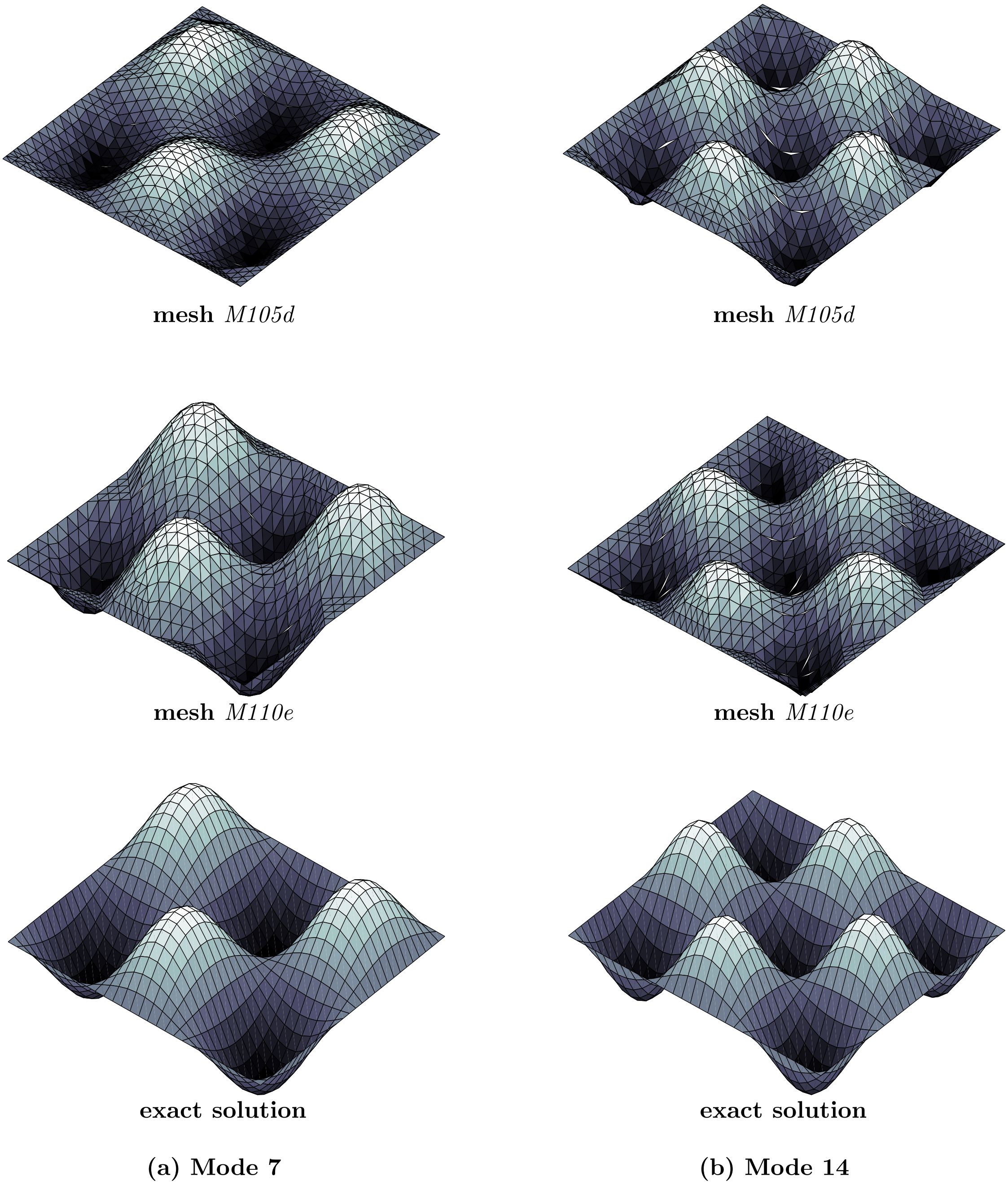


Figure 9: Modes 7 and 14 post-processed for Mesh *F105d* and mesh *F110e*. Exact solution from Equation (9) the last two graphics.

The results proved that the error estimator exhibits capability to recognize the error and improve the solution accuracy at the end of iterative routine.

The influence of the mesh quality on error estimator is also presented. The distortion of the elements conducts to a lost of accuracy on the error estimate. This lost of accuracy is observed for a small prescribed error and for very distorted elements.

The post-processed shape modes also demonstrate the ability of the hierarchical shape functions on describe it.

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