FREE VIBRATION OF NONSYMMETRICAL THIN WALLED BEAMS

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Abstract. *Thin walled and open section beams are extensively used as structural components in different structures in Civil, Aeronautical and Mechanical Engineering fields. Free vibrations of doubly symmetrical beams or beams with one axis of symmetry are widely studied, in general by using Bernoulli-Navier theory. However, results about doubly unsymmetrical beams are rather limited. In this case, triple coupled flexural-torsional vibrations are observed.*

In this paper, a numerical study is presented about natural frequencies of doubly unsymmetrical thin-walled and open cross-section beams. The equations of motion are based on Vlasov's theory of thin-walled beams, which is modified to include the effects of shear flexibility, rotatory inertia in the stress resultants and variable cross-sectional properties. The formulation is also applicable to solid beams, constituting therefore a general theory of coupled flexure and torsion of straight beams. The differential equations are shown to be particularly suitable for analysis in the frequency domain using a state variables approach. A discussion related to vibration of unsymmetrical channel cross-section beams is presented. In this sense, relevant topics discussed in recent works are pointed out.

1 INTRODUCTION

The determination of natural frequencies and modes of vibration of undamped continuous beams and shafts is discussed in detail in Pestel and Leckie[1], who also describe the calculation of dynamic response to harmonic excitation. Ebner and Billington[2] employed numerical integration to study steady state vibrations of damped Timoshenko beams. Numerous other applications can be found in the literature concerning straight and curved beams, as well as arch and shell structures. On the other hand, the theory formulated by Vlasov[3] has been extensively used in the dynamic analysis of thin-walled, open section beams, as exemplified by the studies of Christiano and Culver[4] and Yoo and Fehrenbach[5], in which theoretical predictions of natural frequencies and displacements induced in curved and straight beams by travelling loads, closely match experimentally determined values. In slender beams, modelling the structure with a large number of finite shell elements or other more refined approaches does not lead to any improvement in the correlation with experimental results, because the differences between the theoretical models are usually smaller than the experimental errors.

Nevertheless, although Vlasov's theory for open section beams is already firmly established, it presents some limitations, namely: a) as in the common Bernoulli theory for flexure, it is assumed that shear strains do not contribute to the beam flexibility. Consequently, important errors should be expected in the analysis of deep beams or in the dynamic response associated to higher vibrations modes, even in the case of slender beams (Timoshenko and Young[6]). b) The influence of rotatory inertia in the stress resultants is also neglected and c) Vlasov's fourth-order equations are valid only for beams with uniform cross-section. In previous papers[7,8], the authors proposed a modified theory, which is based on Vlasov's formulation, but it accounts the effects mentioned above. This formulation, using the so-called state variables approach in the frequency domain, lends itself to efficient numerical treatment, which on account of generality and precision can be very useful in a variety of applications.

Other theories that also account for coupling between bending and torsion in beams are presented in Gere and Lin[9] that derive a simplified equation for uniform open section beams and Muller[10] that formulates a general theory that include all coupling effects between the equations of motions, but it is not easy to handle in applications. Most other contributions in the field are restricted to particular cases. For example, Aggarwal and Cranch[11] and Yaman[12] deal with channel-section beams and Ali Hasan and Barr[13] with equal angle-sections.

More recently, Tanaka and Bercin[14] extend the approach of Bishop et al.[15] to study triply coupling of uniform beams using Mathematica. The governing differential equations presented by the mentioned authors[14] exhibit a confusion of co-ordinate system that was clarified by Arpaci and Bozdag[16]. However, the equations presented in the last paper[16] correcting those presented by Tanaka and Bercin[14] are verified comparing results with a case which neglects the same terms of Tanaka and Bercin[14] making it impossible to verify the accuracy of the both theories. In this paper, a numerical study is presented based on the equations developed in previous papers[7,8] and the results of Arpaci and Bozdag[16] are discussed. Moreover, the field matrix of the state variables approach is presented for the case

in which the warping is neglected. These equations lead to a more simple theory that is used for comparison purposes in this paper.

2 THEORY

2.1 Equations of motion

Following Vlasov's convention, the left-handed rectangular global coordinates system (x, y, z) shown in Figure 1 was adopted. The associated displacements are designated ξ, η, and ζ. The basic concepts needed to introduce the effects of shear strains, rotatory inertia and variable cross-sectional properties within the framework of Vlasov's theory are described by the authors in previous papers[7,8], in which a complete derivation of the equations of motion for free and forced vibrations may be found, as well as comparisons with other continuum formulations and a thorough discussion of the definition of shear coefficients. The present paper is more oriented to practical applications and only the derivation of the differential equations for free vibrations in the state variables method is given.
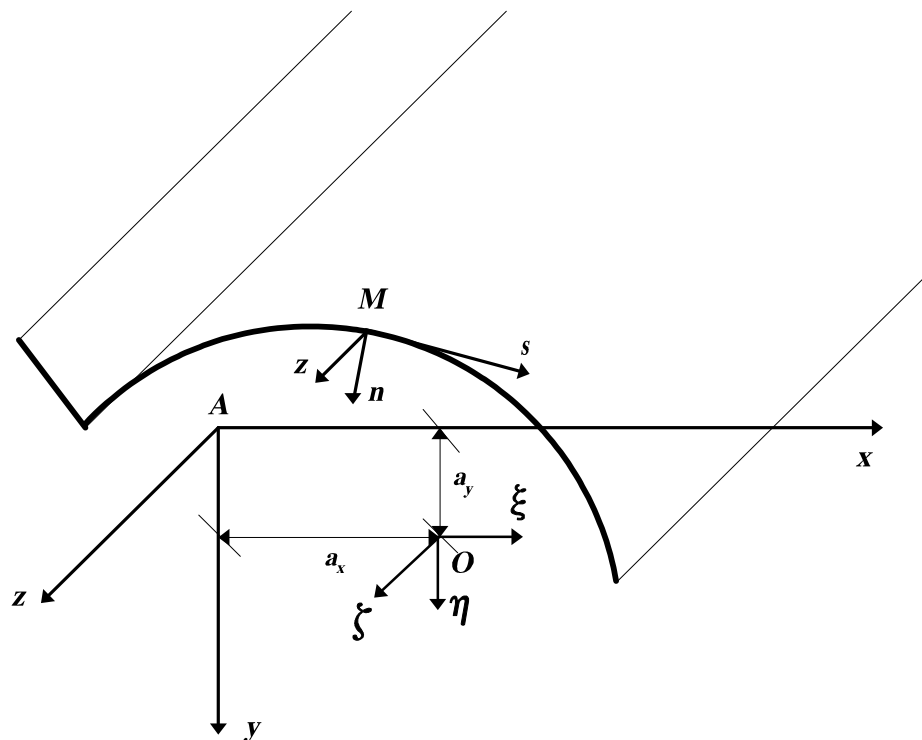

Figure 1: Definition of terms.

In Figure 1 A represents the centroid and O the shear center. For the case of free vibrations, the physical model is formed by the following three fourth order partial differential equations in the generalised displacements ξ, η, and θ:

$$E\left[J_y(z)\left(\frac{\partial^4\xi}{\partial z^4}-\frac{\partial^3\gamma_{mx}}{\partial z^3}\right)+2\left(\frac{\partial^3\xi}{\partial z^3}-\frac{\partial^2\gamma_{mx}}{\partial z^2}\right)\frac{dJ_y(z)}{dz}\right]-\rho J_y(z)\left(\frac{\partial^4\xi}{\partial z^2\partial t^2}-\frac{\partial^3\gamma_{mx}}{\partial z\partial t^2}\right)\cdot$$
$$-\rho\frac{dJ_y(z)}{dz}\left(\frac{\partial^3\xi}{\partial z\partial t^2}-\frac{\partial^2\gamma_{mx}}{\partial t^2}\right)+\rho F_T(z)\left(\frac{\partial^2\xi}{\partial t^2}+a_y\frac{\partial^2\theta}{\partial t^2}\right)=0 \tag{1a}$$

$$E\left[J_x(z)\left(\frac{\partial^4\eta}{\partial z^4}-\frac{\partial^3\gamma_{my}}{\partial z^3}\right)+2\left(\frac{\partial^3\eta}{\partial z^3}-\frac{\partial^2\gamma_{my}}{\partial z^2}\right)\frac{dJ_x(z)}{dz}\right]-\rho J_x(z)\left(\frac{\partial^4\eta}{\partial z^2\partial t^2}-\frac{\partial^3\gamma_{my}}{\partial z\partial t^2}\right)\cdot$$
$$-\rho\frac{dJ_x(z)}{dz}\left(\frac{\partial^3\eta}{\partial z\partial t^2}-\frac{\partial^2\gamma_{my}}{\partial t^2}\right)+\rho F_T(z)\left(\frac{\partial^2\eta}{\partial t^2}+a_x\frac{\partial^2\theta}{\partial t^2}\right)=0 \tag{1b}$$

$$E\left[J_\varphi(z)\frac{\partial^4\theta}{\partial z^4}+2\frac{\partial^3\theta}{\partial z^3}\frac{dJ_\varphi(z)}{dz}\right]-\rho J_\varphi(z)\frac{\partial^4\theta}{\partial z^2\partial t^2}-\rho\frac{dJ_\varphi(z)}{dz}\frac{\partial^3\theta}{\partial z\partial t^2}+$$
$$+\rho F_T(z)\left(a_y\frac{\partial^2\xi}{\partial t^2}-a_x\frac{\partial^2\eta}{\partial t^2}+r^2\frac{\partial^2\theta}{\partial t^2}\right)-GJ_d(z)\frac{\partial^2\theta}{\partial z^2}-G\frac{dJ_d(z)}{dz}\frac{\partial\theta}{\partial z}=0 \tag{1c}$$

In these equations, F_T is the cross-sectional area, J_x and J_y are the second moments of area of the cross-section in relation to the centroidal principal axes, J_φ the sectorial second moment of area, J_d the torsion modulus, a_x and a_y the coordinates of the shear centre. ρ denotes the mass density of the beam material. E and G are the Young's and the shear modulus respectively. Finally, γ_{mx} and γ_{my} represent the mean values of shear strains over a cross-section z = constant and

$$r^2=a_x^2+a_y^2+\frac{J_x+J_y}{F_T} \tag{2}$$

The system (1) represents a general model of non-uniform beams that take into account triply coupled flexural-torsional vibrations. It must be pointed out that the longitudinal vibration equation related to the generalised displacement ζ (Figure 1) is non-coupled with the rest of the system (1) and it was not taken in consideration in the analysis. In the case that the longitudinal vibrations are of interest, this equation can be treated independently.

2.2 State variables method

Using the Fourier transform, an equivalent system with twelve first order partial differential equations with twelve unknowns, in the frequency domain, is obtained. The scheme described above is known in the literature as 'state variables approach'. Six geometric and six static unknown quantities are selected as components of the state vector **v**: The displacements ξ and η, the bending rotations ϕ_x and ϕ_y, the normal shear stress resultants Q_x

and Q_y, the bending moments M_x and M_y, the torsional rotation θ and its spatial derivative θ', the total torsional moment M_T and the bimoment B.

$$\mathbf{v}(z,\omega) = \{\eta,\ \phi_y,\ Q_y,\ M_x,\ \xi,\ \phi_x,\ Q_x,\ M_y,\ \theta,\ \theta',\ M_T,\ B\}^{\mathrm{T}} \tag{3}$$

in which, M_T denotes the total torsional moment given by:

$$M_T = H_\varphi + H_k \tag{4}$$

with $H_k = GJ_d\theta'$ = Saint Venant torsion moment. The system is:

$$\frac{\partial \mathbf{v}}{\partial z} = \mathbf{A}\mathbf{v} \tag{5}$$

$$
\mathbf{A} = \begin{bmatrix}
0 & 1 & \dfrac{1}{k'_y FG} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\dfrac{1}{EJ_x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\rho F\omega^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho F\omega^2 a_x & 0 & 0 & 0 \\
0 & \rho J_x\omega^2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & \dfrac{1}{k'_x FG} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dfrac{1}{EJ_y} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\rho F\omega^2 & 0 & 0 & 0 & -\rho F\omega^2 a_y & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\rho J_y\omega^2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\dfrac{1}{EJ_\varphi} \\
\rho F\omega^2 a_x & 0 & 0 & 0 & -\rho F\omega^2 a_y & 0 & 0 & 0 & -\rho F\omega^2 r^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B\theta' & 1 & 0
\end{bmatrix} \tag{6}
$$

In which $\mathbf{A}$ is the system matrix, while k'_x and k'_y denote the Cowper's shear coefficients and:

$$B\theta' = \rho J_\varphi\omega^2 - GJ_d \tag{7}$$

The components of the 12-dimensional state vector $\mathbf{v}$ are designated "state variables". In the frequency domain, the state variables depend on the frequency ω and the longitudinal coordinate z. For simplicity, the same notation is being used for the state variables and their

Fourier transforms, since the domain can usually be identified by the indication of the function arguments. For example, $\eta(z, t)$ and $\eta(z, \omega)$ refer to the y-displacement in the time domain and to its Fourier transform, respectively. It is important to note that the present formulation constitutes a general theory of beams applicable to solid as well as thin-walled beams.

2.3 Simplified theory

In order to compare the numerical results to those presented by Yaman[12] and Arpaci and Bozdag[16] which was one of the motivations of the present paper, a simplified theory is presented. This model excludes of the analysis the warping constraint, reducing the system (5) to ten first order partial differential equations with ten unknowns. In this case, the vector of state is:

$$\mathbf{v}^*(z, \omega) = \{ \eta, \phi_y, Q_y, M_x, \xi, \phi_x, Q_x, M_y, \theta, M_T^* \}^{\mathrm{T}} \tag{8}$$

in which, M_T^* denotes the torsional moment, in this case given by:

$$M_T^* = GJ_d\theta' \tag{9}$$

The new system is:

$$\frac{\partial \mathbf{v}^*}{\partial z} = \mathbf{A}^* \mathbf{v}^* \tag{10}$$

In which $\mathbf{A}^*$ is the system matrix given by.

$$\mathbf{A}^* = \begin{bmatrix} 0 & 1 & \dfrac{1}{k'_y FG} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\dfrac{1}{EJ_x} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\rho F\omega^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho F\omega^2 a_x & 0 \\ 0 & \rho J_x\omega^2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \dfrac{1}{k'_x FG} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dfrac{1}{EJ_y} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\rho F\omega^2 & 0 & 0 & 0 & -\rho F\omega^2 a_y & 0 \\ 0 & 0 & 0 & 0 & 0 & -\rho J_y\omega^2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dfrac{1}{GJ_d} \\ \rho F\omega^2 a_x & 0 & 0 & 0 & -\rho F\omega^2 a_y & 0 & 0 & 0 & -\rho F\omega^2 r^2 & 0 \end{bmatrix} \tag{11}$$

3 NUMERICAL PROCEDURE AND BOUNDARY CONDITIONS

The system (5) may be easily integrated using standard numerical procedures, such as the fourth order Runge-Kutta method, the predictor-corrector algorithm or other approaches. In order to solve the two-point value problem encountered both in the determination of natural frequencies and in dynamic response calculations, the latter must be transformed to an initial value problem as shown, for example, by Ebner and Billington[2]. The procedure is normally applied in the transfer matrix method (Pestel and Leckie[1]). Natural frequencies are determined by mean of the well-known Thomson's method.

The classical boundary conditions are considered in this paper: clamped, free or simply supported.

Clamped Boundary

$$\xi = \eta = 0, \ \phi_x = \phi_y = \theta = 0, \ \theta' = 0 \qquad (12)$$

Free Boundary

$$Q_y = Q_x = 0, \ M_x = M_y = M_T = 0, \ B = 0 \qquad (13)$$

Hinged Boundary

$$\xi = \eta = 0, \ M_x = M_y = 0, \ \theta = 0, \ B = 0 \qquad (14)$$

4 RESULTS AND DISCUSSION

As stated by Arpaci and Bozdag[16] in the theory developed by Tanaka and Bercin[14] although the flexural displacement and the offsets of the shear centre are determined with respect to the axes which are perpendicular and parallel to the web of unsymmetrical channel, the equations are formulated as if the principal axes are used. Effectively, in the equations (1)-(3) of Tanaka and Bercin[14] the product of inertia terms I_{xy} are not included.

However, because Tanaka and Bercin[14] do not give the value of product of inertia for the cross-section considered by them in the numerical example, the comparison carried out by Arpaci and Bozdag[16] was not made for the same cross-section. Moreover, although all properties of the unsymmetrical beams used in the examples 1-2 of Arpaci and Bozdag[16] are given in this reference, only the relative error between both papers was presented and consequently it was not possible to compare with the theory presented in this paper.

On the other hand, Arpaci and Bozdag[16] present another example in order to show the good agreement between their results and those obtained by Yaman[12] using the wave propagation approach. However, Arpaci and Bozdag[16] surprisingly uses the Yaman's results in which the product of inertia terms I_{xy} and the warping constraint are not included. In this way results make it impossible to verify if the theory presented by Arpaci and Bozdag[16] lead to right natural frequencies in the case of nonsymmetrical open cross-section beams. For this reason, the complete example presented by Yaman[12] is developed next.

The theoretical model used by Yaman[12], which cross section is presented in figure 2, has

the following geometric and material properties:

$L = 1.00$ m $\qquad\qquad A = 9.68\ 10^{-5}$ m^2

$I_v = 5.08\ 10^{-9}$ m^4 $\qquad\qquad I_\xi = 2.24\ 10^{-8}$ m^4

$I_{v\xi} = 4.25\ 10^{-9}$ m^4

$\Gamma_0 = 7.11\ 10^{-12}$ m^6 $\qquad\qquad J = 5.20\ 10^{-11}$ m^4

$c_z = 9.09\ 10^{-3}$ m $\qquad\qquad c_y = 10.43\ 10^{-3}$ m

$E = 7.00\ 10^{10}$ N/m^2 $\qquad\qquad \rho = 2700$ kg/m^3

$G = 2.60\ 10^{10}$ N/m^2

In the notation of Yaman[12] L is the length of the beam, A the cross-sectional area, I_v and I_ξ are the second moments of area of the cross-section in relation to centroidal axes perpendicular and parallel to the web of the channel beam, Γ_0 the warping constant about shear centre, J the torsion constant and c_z and c_y the eccentricities between the centroid and the shear centre. The mechanical properties have the same notation that in this paper
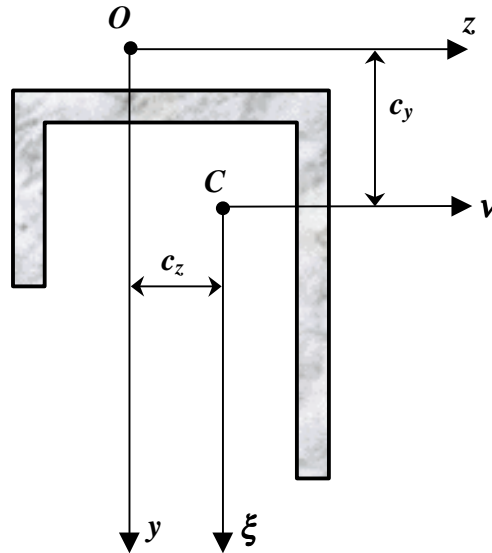

Figure 2: Cross-section of the beam of the example. Yaman's co-ordinate system

In order to use the theory presented in this paper the second moments of area of the cross-section in relation to the centroidal principal axes (Figure 3) must be determined as well as the coordinates of the shear centre. Then, the well-known following equations are applied

$$tg\ 2\alpha_0 = \frac{2I_{\eta\xi}}{I_\xi - I_\eta} \tag{15}$$

$$J_x = I_\eta\ cos^2\ \alpha_0 + I_\xi\ sen^2\ \alpha_0 - I_{\eta\xi}\ sen\ 2\alpha_0 \tag{16}$$

$$J_y = I_\xi \, cos^2 \, \alpha_0 + I_\eta \, sen^2 \, \alpha_0 + I_{\eta\xi} \, sen \, 2\alpha_0 \qquad (17)$$

Moreover, it can be demonstrated that

$$a_x = c_z \, cos \, \alpha_0 - c_y \, sen \, \alpha_0 \qquad (18)$$

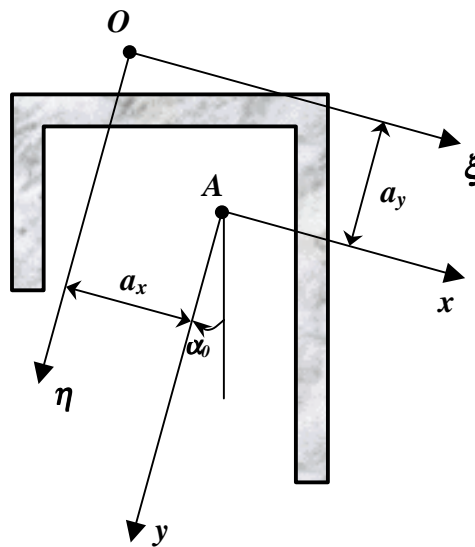$$a_y = -\left(c_y \, cos \, \alpha_0 + c_z \, sen \, \alpha_0 \right) \qquad (19)$$

Figure 3: Cross-section of the beam of the example. Principal centroidal axes

Then, the geometric and mechanical properties used in order to apply the theory presented in this paper are:

$l = 1.00$ m	$F_T = 9.68 \; 10^{-5}$ m^2
$J_x = 4.09 \; 10^{-9}$ m^4	$J_y = 2.34 \; 10^{-8}$ m^4
$J_{xy} = 0$	
$J_\varphi = 7.11 \; 10^{-12}$ m^6	$J_d = 5.20 \; 10^{-11}$ m^4
$a_x = 6.50 \; 10^{-3}$ m	$a_y = -12.22 \; 10^{-3}$ m
$E = 7.00 \; 10^{10}$ N/m^2	$\rho = 2700$ kg/m^3
$v = 0.346$	

The boundary condition for this example is considered simply supported. Three cases were analysed

a) Stiffness coupling terms and effects of warping constraint are ignored.

Due to the product of inertia terms couples the flexural vibration in the two directions considered, Yaman[12] named the product $EI_{v\xi}$ as coupling stiffness and, in first place, it presents the natural frequencies for the case in which the stiffness coupling terms and additionally the effects of warping constraint are ignored. In this case, the same error of

Tanaka and Bercin[14] is made. Obviously, these results cannot be used to verify a theory of nonsymmetrical thin walled beams.

In the context of this paper, the simplified theory presented in point 2.3 was used. Moreover, in order to ignore the stiffness coupling and to compare the results with those of Yaman[12], the following values were considered: $I_x = I_v$, $I_y = I_\xi$ and $I_{xy} = 0$. Table 1 shows the results obtained.

Table 1: Natural frequencies (Hz) of the beam of the example. Case a

Reference	Mode					
	1	2	3	4	5	6
Arpaci et al[16]	47.00	73.39	102.36	145.34	154.53	206.70
Yaman[12]	45.49	69.91	101.73	149.82	154.79	207.43
This paper	45.47	69.88	101.72	149.47	154.80	207.25

The frequencies corresponding to the modes 1, 3, 5 and 6 are torsion dominated resonance frequencies and those corresponding to the modes 2 and 4 are bending dominated resonance frequencies. There are complete agreement, for this case, between the simplified theory presented in this paper and the wave propagation theory[12].

b) Only effects of warping constraint are ignored.

In this case, the simplified theory presented in point 2.3 was used. Moreover, the values of I_x and I_y that correspond to centroidal principal axes were considered. Table 2 shows the results obtained.

Table 2: Natural frequencies (Hz) of the beam of the example. Case b

Reference	Mode			
	1	2	3	4
This paper. Case a	45.47	69.88	101.72	149.47
This paper. Case b	45.28	60.27	102.18	155.10

As demonstrated by Yaman[12] the consideration of the coupling stiffness is found to affect the bending dominated resonant frequencies but it has not significant effect on the values of the torsion dominated frequencies.

c) General case. All couplings and warping constraint effects are considered

In this case, the general theory presented in points 2.1 was used. Table 3 shows the results obtained. The frequencies corresponding to the modes 1 and 3 are torsion dominated resonance frequencies and those corresponding to the modes 2 and 4 are bending dominated resonance frequencies.

Table 3: Natural frequencies (Hz) of the beam of the example. Case c

Reference	Mode			
	1	2	3	4
Yaman[12]	51.87	114.79	207.41	263.88
This paper	51.39	96.48	188.84	204.81
Difference (%)	0.9	16.0	9.0	22.4

It may be seen that the first natural frequency is in close agreement. There are some discrepancies, however, in the higher modes especially in the flexural dominated ones. In this context, it would be useful to check the values in table 3 against experimental observations

5   CONCLUSIONS

In this paper, the equations of motion for thin-walled, variable open cross-section beams have been presented within the so-called state variables approach in the frequency domain. The equations take into account the influence of shear flexibility and rotatory inertias which are neglected in the original Vlasov's theory. The equations enable the analysis of practical problems using direct numerical integration in conjunction with techniques routinely applied in transfer matrix analyses. In addition, they may be resorted to in order to numerically evaluate transfer matrices or stiffness matrices for open section beam elements. Moreover, the proposed theory can also be used for solid beams in which coupling between bending and torsion occurs. Moreover, the field matrix of a simplified model which excludes the warping constraint is also presented.

A discussion related to vibration of doubly unsymmetrical channel cross-section beams is presented. In this sense, relevant topics discussed in recent works[12,14,16] are pointed out. Further studies, specially related to experimental tests of this type of nonsymmetrical sections are necessary.

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