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GREEN'S FUNCTIONS OF A NON-SELF ADJOINT OPERATOR APPLIED TO SOUND RADIATION IN A CIRCULAR DUCT WITH AXIAL FLOW

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Abstract. *This paper presents a mathematical model to simulate the sound radiation from a moving boundary of a lined circular duct in the presence of a convective axial flow. The model is based on finding a new closed form solution for the Green's functions of the convected wave equation inside a soft wall duct, using the eigenfunctions method. Using the Divergence Theorem, this closed form solution allows to find expressions for the sound field generated by a rectangular shaped piston source with uniform velocity. This formulation can be applied to model discontinuities in acoustic liners for turban engines such as embedded actuators used in active noise control, the scattering effects of liner splices near the fan and so forth. By properly selecting the piston velocity or strength, the different discontinuities in the liner can be modeled. An example consisting of a circumferential array of rigid patches mounted on the wall of the lined duct is described.*

1 INTRODUCTION

The use of acoustic liners is the most successful technique to reduce turbofan engine noise. Acoustic liners are porous materials typically placed on the wetted surfaces of the engine nacelle and their efficiency is proportional to the effective length or area of the applied treatment. These liner systems can be classified as the absorber type, the resonator type, and a type which has a combination of both of these characteristics. Absorber liners consist of a thick layer of porous material and in general attenuate broadband noise. However, these are not particularly suitable for attenuating large amplitude components at discrete frequencies. Resonant liners consist of a thin sheet of perforated facing material separated from an impervious surface by a cavity divided into compartments by a honeycomb spacer structure; this forms an array of resonators which effectively attenuate a predominantly narrow frequency band of noise. Lastly, a liner combining the essential features of both types, consisting of a thin porous absorptive facing material backed by resonant cavities, has good attenuation characteristics over a wide range of frequencies.

In general, these liners have uniform properties that have the effect of minimizing reflection and scattering of energy between modes. Unfortunately, liner properties are not uniform when mounted on real engine inlets. Due to many construction requirements, the surface of the liner is discontinued by the presence of other devices. These discontinuities will always lead to reflection of acoustic energy back towards the fan and scatter of energy among both circumferential and radial propagating modes. Also, the presence of liner splices of different impedance may produce the necessary disturbance at the wall to produce the mentioned effects. The behavior of these discontinuities is similar to those of noise sources located at the duct boundary producing disturbances that add to the present sound field. Interesting cases are when these portions of the wall have actually no penetration properties (*hard wall patches*), or they are a complete sector embedded in a different liner.

This paper presents a mathematical formulation to solve for the Green's functions of the convected wave equation in a soft wall circular duct. The solution will be found using the eigenfunction method, which leads to a closed form expression. Using the Divergence Theorem, this closed form solution allows to find expressions for the sound field generated by a rectangular shaped piston source with uniform velocity. It is important to mention that the operator that defines the differential equation, i.e. the convected wave equation, is not self adjoint. As a consequence, in the process of extending the Green's function to a finite piston source, the reciprocity principle must be applied to the adjoint solution, i.e. the complex conjugate solution, rather than to the direct solution.

An example consisting of a circumferential array of rigid patches mounted on the wall of the lined duct is described.

2 ANALYTICAL MODEL

The problem to solve is that of the propagation of acoustic modes in a circular lined duct with a moving boundary or discrete source, i.e. section of the wall having a prescribed motion. It is assumed that the duct is of infinite length and it has radius "*a*". There is also a uniform mean flow with positive Mach number $\boldsymbol{M}$ in the direction of the z coordinate. A schematic of the model is shown in *Figure 1* in conjunction with the cylindrical coordinate system $\vec{r} = (r, \theta, z)$. The liner is assumed to be locally reactive.

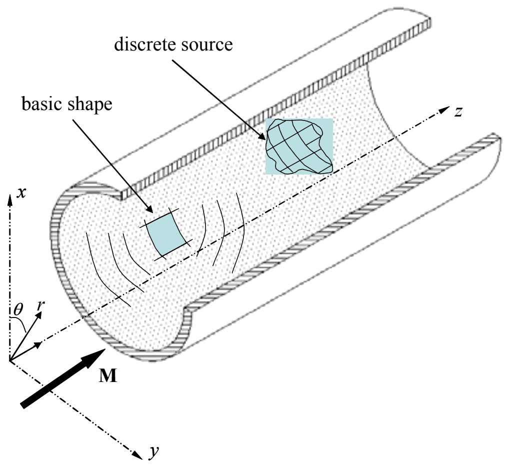

Figure 1: Model of Sound Radiation from the Boundaries of a Lined Duct and source shapes that lead to close form solutions

The source is modeled as a sector of the wall of the duct moving with a certain velocity. The vibrating wall sector is assumed to have a constant radial velocity distribution, i.e. a piston like motion. A close form solution for the sound pressure field can be found by assuming that the shape of the moving surfaces is defined by lines of constant z and θ coordinates (*see Figure 1*). Nevertheless, sources with complex shapes and velocity distributions can also be simulated using this model by breaking the area down into several basic rectangular shapes.

The source radial velocity or source strength is considered to be known whose value depends on the specific application of the model. Based on the application, the source strength will have different physical interpretations which will lead to different methods to compute it.

2.1 General duct-acoustics with liners

The sound field that propagates inside the duct is obtained by solving the homogeneous acoustic wave equation in a moving media [1]:

$$\nabla^2 p = \frac{1}{c^2}\left(\frac{\partial}{\partial t} + cM\frac{\partial}{\partial z}\right)^2 p \tag{1}$$

subjected to the lined wall boundary condition

$$-\left.\frac{\partial p}{\partial r}\right|_{r=a} = i\omega\rho v_r\left(a,\theta,z\right) + \rho c M \left.\frac{\partial v_r}{\partial z}\right|_{r=a} \tag{2}$$

where $\nabla^2(.)$ is the Laplacian operator in cylindrical coordinates, $p(r,\theta,z)$ is the acoustic pressure, $v_r(a,\theta,z)$ is the radial component of the particle velocity at the duct wall, ρc is the fluid characteristic impedance, and *M* is the flow Mach number.

The solution to (1) and (2) is expressed as a linear combination of propagating acoustic modes present in the duct as follows

$$p(\vec{r}) = \sum_m \sum_n A_{mn}^{(+)} \Phi_{mn}^{(+)}(r,\theta) e^{-ik_z^{(+)} z} e^{i\omega t} + \sum_m \sum_n A_{mn}^{(-)} \Phi_{mn}^{(-)}(r,\theta) e^{-ik_z^{(-)} z} e^{i\omega t} \qquad (3)$$

where the superscripts (+) and (-) indicate variables associated to positive and negative z-direction propagation, respectively. Thus, $A_{mn}^{(+)}$ and $A_{mn}^{(-)}$ are the complex modal amplitudes, $k_z^{(+)}$ and $k_z^{(-)}$ are the mode axial wavenumbers, and $\Phi_{mn}^{(+)}$ and $\Phi_{mn}^{(-)}$ are the acoustic modes corresponding the mn^{th} mode propagating in the positive and negative z-direction, respectively. The subscripts *m* and *n* refer to the circumferential and radial mode order, respectively.

2.2 Eigenproblem

The wall of the duct is embedded in a liner with impedance Z_w, commonly expressed in terms of the specific admittance β_w, i.e. $Z_w = \rho c / \beta_w$. The modes Φ_{mn} satisfy the boundary condition at the wall. This condition is the equilibrium equation (2) applied at the wall, i.e. r = *a*, including the effect of the mean flow, and is expressed as [2, 3]:

$$\left.\frac{\partial \Phi_{mn}^{(\ell)}}{\partial r}\right|_{r=a} = -i\beta_w \frac{\left(k_0 - k_z^{(\ell)} M\right)^2}{k_0} \Phi_{mn}^{(\ell)} \qquad \ell = +,- \qquad (4)$$

where $k_0 = \omega/c$ is the free field wavenumber and the superscript $\ell = +$ or - is used to indicate variables associated to positive and negative propagation, respectively.

The modes that satisfy expressions (1) through (4) are given in terms of the first kind complex Bessel functions $J_m(.)$:

$$\Phi_{mn}^{(\ell)}(r,\theta) = \cos(m\theta) J_m\left(k_{mn}^{(\ell)} r\right) \qquad \ell = +,- \qquad (5)$$

where $k_{mn}^{(\ell)}$ is the mn^{th} complex root (eigenvalues) of equation (4) after replacing expression (5). It is important to remark that there exist two sets of solutions to equation (4), i.e. two characteristic equations, corresponding to the two propagation directions. These lead to different expressions for the positive and negative traveling mode shapes, i.e. $\Phi_{mn}^{(+)}$ and $\Phi_{mn}^{(-)}$, eigenvalues $k_{mn}^{(\ell)}$, and axial wavenumbers $k_z^{(\ell)}$. Also notice that the modes expressed in (5) are not orthogonal with respect to the radial integration.

The propagation of each mode in the duct depends directly on the values of $k_{mn}^{(\ell)}$. Indeed, the axial wavenumber of a propagating mode is given by the following expressions [2]:

$$k_z^{(+)} = \frac{-k_0 M + \sqrt{k_0^2 - \left(1-M^2\right) k_{mn}^{2(+)}}}{\left(1-M^2\right)}$$

$$k_z^{(-)} = \frac{-k_0 M - \sqrt{k_0^2 - \left(1-M^2\right) k_{mn}^{2(-)}}}{\left(1-M^2\right)} \tag{6}$$

2.3 Green's functions

The closed form sound radiation from a point source inside the lined duct is investigated in this section. The Green's functions are the solution to the Laplace transformed non-homogeneous wave equation in cylindrical coordinates under the soft wall boundary condition [4]:

$$\frac{\partial^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} + \frac{\partial^2 g}{\partial z^2}\left(1-M^2\right) - 2iMk_0 \frac{\partial g}{\partial z} + k_0^2 g = \delta(r-r_0)\frac{\delta(\theta-\theta_0)}{r}\delta(z-z_0) \tag{7}$$

where $\delta(.)$ is the delta Dirac function and $\left(r_0, \theta_0, z_0\right)$ is the location of the point source. The boundary condition is:

$$\left.\frac{\partial g}{\partial r}\right|_{r=a} = -i\beta_w \frac{\left(k_0 - k_z M\right)^2}{k_0} g \tag{8}$$

The solution to equations (7) and (8) was previously investigated by Zorumski using approximations with series of circumferential inverse Fourier transforms.[4] The approach taken here is based on the eigenfunction method similar to the hard wall case.[5] One of the main advantages of this method is that it is possible to explicitly satisfy that the Green's function is continuous at the source location plane. Another benefit is that it is a simpler and closed form formulation.

First, the solution of the equations (7) and (8) is assumed as a linear combination of the modes inside the duct:

$$g^{(+)} = \sum_{m=0}^{M}\sum_{n=0}^{N} A_{mn}^{(+)} \Phi_{mn}^{(+)} e^{-ik_z^{(+)}(z-z_0)} \qquad z \geq z_0$$

$$g^{(-)} = \sum_{m=0}^{M}\sum_{n=0}^{N} A_{mn}^{(-)} \Phi_{mn}^{(-)} e^{-ik_z^{(-)}(z-z_0)} \qquad z \leq z_0 \tag{9}$$

where M and N are the maximum number of terms to be included in the expansion. It is now convenient to express the Green's functions for the complete domain in terms of the Heaviside functions $H\left(z-z_0\right)$ (i.e. $H(z)=1$ $(z>0)$; $H(z)=0$ $(z<0)$ and $H(0)=1/2$)

$$g\left(\vec{r}\middle|\vec{r_0}\right) = g^{(+)}H\left(z-z_0\right) + g^{(-)}\left[1-H\left(z-z_0\right)\right] \tag{10}$$

The problem is now reduced to find the complex amplitudes $A_{mn}^{(+)}$ and $A_{mn}^{(-)}$ that define the Green's functions in (9). To this end, the continuity of the Green's functions at $z=z_0$ is first explicitly imposed as follows [6]:

$$g^{(+)}\Big|_{z=z_0} = g^{(-)}\Big|_{z=z_0} \tag{11}$$

For the sake of clarity, a compact notation is used for the linear operator $L(.) = \frac{\partial^2(.)}{\partial r^2} + \frac{1}{r}\frac{\partial(.)}{\partial r} + \frac{1}{r^2}\frac{\partial^2(.)}{\partial\theta^2}$, which does not depend on the z coordinate. Then, equation (7) can be rewritten as:

$$k_0^2 g + Lg + \frac{\partial^2 g}{\partial z^2}\left(1-M^2\right) - 2iMk_0\frac{\partial g}{\partial z} = \delta(r-r_0)\frac{\delta(\theta-\theta_0)}{r}\delta(z-z_0) \tag{12}$$

Replace equation (10) into (12) as:

$$\begin{aligned}&\left(k_0^2+L\right)\left[g^{(+)}H\left(z-z_0\right)+g^{(-)}\left[1-H\left(z-z_0\right)\right]\right]+\left(1-M^2\right)\frac{\partial^2}{\partial z^2}\left[g^{(+)}H\left(z-z_0\right)+\right.\\&\quad\left.+g^{(-)}\left[1-H\left(z-z_0\right)\right]\right]-2iMk_0\frac{\partial}{\partial z}\left[g^{(+)}H\left(z-z_0\right)+g^{(-)}\left[1-H\left(z-z_0\right)\right]\right]=\frac{\delta(\vec{r}|\vec{r}_0)}{r}\end{aligned} \tag{13}$$

Considering that the derivative of the Heaviside function is $H'\left(z-z_0\right)=\delta\left(z-z_0\right)$, equation (13) can be rearranged as follows

$$\begin{aligned}&\left(k_0^2+L\right)\left[g^{(+)}H\left(z-z_0\right)+g^{(-)}\left[1-H\left(z-z_0\right)\right]\right]+\left(1-M^2\right)\left[\frac{\partial^2 g^{(+)}}{\partial z^2}H\left(z-z_0\right)+\right.\\&\quad\left.+\frac{\partial^2 g^{(-)}}{\partial z^2}\left[1-H\left(z-z_0\right)\right]+2\left(\frac{\partial g^{(+)}}{\partial z}-\frac{\partial g^{(-)}}{\partial z}\right)\delta\left(z-z_0\right)+\left(g^{(+)}-g^{(-)}\right)\delta'\left(z-z_0\right)\right]-\\&\quad-2iMk_0\left[\frac{\partial g^{(+)}}{\partial z}H\left(z-z_0\right)+\frac{\partial g^{(-)}}{\partial z}\left[1-H\left(z-z_0\right)\right]+\left(g^{(+)}-g^{(-)}\right)\delta\left(z-z_0\right)\right]=\frac{\delta(\vec{r}|\vec{r}_0)}{r}\end{aligned} \tag{14}$$

In order to find the modal amplitudes for the expressions in (9), eq.(14) needs to be pre multiplied by the acoustic modes defined in the complete domain as $\left(\Phi_{er}^{(+)}H\left(z-z_o\right)+\Phi_{er}^{(-)}\left[1-H\left(z-z_o\right)\right]\right)$, and integrated over a small volume as shown in *Figure 2*. The axial dimension of this volume is defined as 2ε, where $\varepsilon\to 0$. After solving the integral in the z coordinate and taking the limit $\varepsilon\to 0$, it can be shown that most terms of equation (14) vanish, given that the continuity condition in (11) is imposed. This procedure leads to

$$\int_0^a\int_0^{2\pi}\lim_{\varepsilon\to 0}\int_{z_o-\varepsilon}^{z_o+\varepsilon}\left(1-M^2\right)\left\{\left(\frac{\partial g^{(+)}}{\partial z}-\frac{\partial g^{(-)}}{\partial z}\right)\left(\Phi_{er}^{(+)}H(z-z_o)+\Phi_{er}^{(-)}\left[1-H(z-z_o)\right]\right)\delta\left(z-z_0\right)\right\}dz\,r\,d\theta\,dr=$$
$$=\int_0^a\int_0^{2\pi}\lim_{\varepsilon\to 0}\int_{z_o-\varepsilon}^{z_o+\varepsilon}\left\{\delta(r-r_o)\delta(\theta-\theta_o)\delta(z-z_o)\left(\Phi_{er}^{(+)}H(z-z_o)+\Phi_{er}^{(-)}\left[1-H(z-z_o)\right]\right)\right\}dz\;d\theta\,dr \tag{15}$$

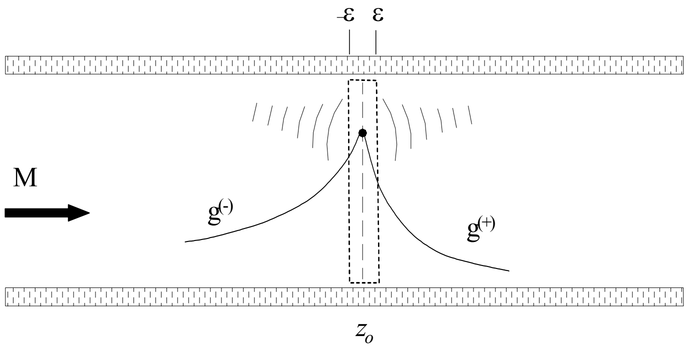

Figure 2: Schematic of the Green's functions calculation

Finally, replacing the expanded solution (9) into (15) and solving the integrals yields:

$$\sum_{n=0}^{N}\left\{k_z^{(+)}A_{mn}^{(+)}\int_0^{2\pi}\int_0^a\left(\frac{\Phi_{mr}^{(+)}+\Phi_{mr}^{(-)}}{2}\right)\Phi_{mn}^{(+)}\,r\,d\theta\,dr-k_z^{(-)}A_{mn}^{(-)}\int_0^{2\pi}\int_0^a\left(\frac{\Phi_{mr}^{(+)}+\Phi_{mr}^{(-)}}{2}\right)\Phi_{mn}^{(-)}\,r\,d\theta\,dr\right\}=$$
$$=i\,\frac{\Phi_{mr}^{(+)}\left(r_0,\theta_0\right)+\Phi_{mr}^{(-)}\left(r_0,\theta_0\right)}{2\left(1-M^2\right)}$$
$$m=0,1,2,3,........ \tag{16}$$

where the factor 2 in the denominator of the right hand side appears as a consequence of solving the Dirac delta integral on the axial location of the source. Note that the orthogonality of the modes in the circumferential direction has been used in (16). However, the system of equations is fully coupled because the modes are not orthogonal in the radial direction.

Equation (16) is a system of $m\times n$ equations with $2\left(m\times n\right)$ unknowns, i.e. $A_{mn}^{(+)}$ and $A_{mn}^{(-)}$, since the positive traveling modal amplitudes are different from the negative traveling ones. The remaining set of $m\times n$ equations is obtained using the continuity condition in (11). Pre-multiplying equation (11) by $(\Phi_{mr}^{(+)}+\Phi_{mr}^{(-)})/2$ and integrating over the duct cross section yields

$$\sum_{n=0}^{N}\left\{A_{mn}^{(+)}\int_0^{2\pi}\int_0^a\left(\frac{\Phi_{mr}^{(+)}+\Phi_{mr}^{(-)}}{2}\right)\Phi_{mn}^{(+)}\,r\,d\theta\,dr-A_{mn}^{(-)}\int_0^{2\pi}\int_0^a\left(\frac{\Phi_{mr}^{(+)}+\Phi_{mr}^{(-)}}{2}\right)\Phi_{mn}^{(-)}\,r\,d\theta\,dr\right\}=0$$
$$m=0,1,2,3,........ \tag{17}$$

and like the system in (16), this system of equation is also fully coupled.

The system of equations for the modal amplitudes $A_{mn}^{(\ell)}$ in (16) and (17) can be written in matrix form as

$$\left[\begin{array}{c|c} \left[\Lambda_{m,nr}^{(+)}\right]\left[k_z^{(+)}\right] & -\left[\Lambda_{m,nr}^{(-)}\right]\left[k_z^{(-)}\right] \\ \hline \left[\Lambda_{m,nr}^{(+)}\right] & -\left[\Lambda_{m,nr}^{(-)}\right] \end{array}\right]\left\{\begin{array}{c} A_{mn}^{(+)} \\ \hline A_{mn}^{(-)} \end{array}\right\} = \left\{\begin{array}{c} \psi_r \\ \hline 0 \end{array}\right\}$$

$$m = 0,1,2,3,........ \tag{18}$$

where

$$\{\Psi_r\} = i\frac{\left(\Phi_{mr}^{(+)}(r_0,\theta_0) + \Phi_{mr}^{(-)}(r_0,\theta_0)\right)}{2\pi a^2\left(1-M^2\right)} \tag{19}$$

The matrices $\left[\Lambda_{m,nr}^{(\ell)}\right]$ are fully populated and their components are defined by

$$\Lambda_{m,nr}^{(\ell)} = \frac{1}{\pi a^2}\int_0^{2\pi}\int_0^a \frac{\Phi_{mr}^{(+)} + \Phi_{mr}^{(-)}}{2}\Phi_{mn}^{(\ell)}\, r\, dr\, d\theta \qquad \ell = +,- \tag{20}$$

On the other hand, the matrices $\left[k_z^{(\ell)}\right]$ are diagonal and contain the axial wavenumber for each mode.

2.3 Finite source radiation

The Green's functions found in (9) can now be used to find the sound field due to a motion of the boundary. Here the simple case of a finite piston source is modeled as a sector of the wall vibrating with a known uniform velocity V_p, i.e. piston source defined by constant z and θ coordinates, as shown in *Figure 3*. This piston source is referred as the radiating surface of the duct. The rest of the duct surface is referred as non-radiating surface. The derivation in this section follows the approach taken by Morse and Ingard for radiation from boundaries of lined duct without flow.[1] However, the formulation here is extended to the case with flow.

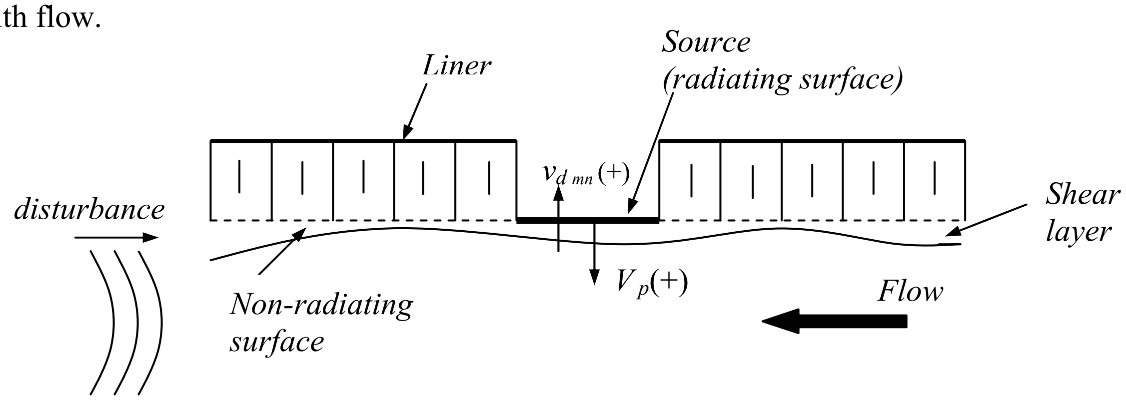

Figure 3: Schematic of the shear layer at the lined duct walls

In this analysis, it is first convenient to review the behavior of the non-radiating surface, i.e. liner. Although the non-radiating areas of the duct boundary do not move, the radial particle velocity $v_{r_{mn}}$ on the lined wall does not vanish. The relationship between the particle velocity $v_{r_{mn}}$ in the radial direction and modal pressure p_{mn} over the liner (non-radiating surface) in the flow is as follows [1]:

$$v_{r_{mn}} = -\frac{1}{i\rho c\left(k_0 - k_z M\right)} \frac{\partial p_{mn}}{\partial r}\bigg|_{r=a} = \frac{\beta_\omega}{\rho c}\frac{\left(k_0 - k_z M\right)}{k_0} p_{mn}\bigg|_{r=a} \tag{21}$$

The first two terms of the equality in (21) are related to the equilibrium relation between the pressure gradient and particle velocity, i.e. Euler's equation. The last two terms relate the pressure and its gradient using the definition of the liner specific acoustic admittance. Then, from equation (21) the difference between the radial particle velocity and the term $\left(1/\rho c k_0\right)\beta_\omega\left(k_0 - k_z M\right)p_{mn}\big|_{r=a}$ must vanish.

In the radiating areas, this difference cannot vanish because of the presence of the piston motion. This difference has to be the perturbation velocity in the flow produced by the piston motion. Define it as

$$v_{d_{mn}} = -\left(\frac{1}{i\rho c\left(k_0 - k_z M\right)} \frac{\partial p_{mn}}{\partial r} + \frac{\beta_w}{\rho c}\frac{\left(k_0 - k_z M\right)}{k_0} p_{mn}\right)_{r=a} \tag{22}$$

The particle velocity in the flow due to the radiating piston, $v_{d_{mn}}$, needs to be related to the piston velocity at the wall. As shown in *Figure 3*, there is a shear layer separating the lined wall and the flow. Applying particle displacement continuity, the relation between the piston velocity and the radial particle velocity outside the shear layer (in the flow) is given by[7]

$$v_{d_{mn}} = -\frac{\left(k_0 - k_z M\right)}{k_0} V_p \tag{23}$$

where the negative sign is used to change the positive velocity convention to be inwards.

The radiation from the source is now obtained by applying the Green's Divergence Theorem with the adjoint solution of the Green's functions, i.e. the complex conjugate. The reason for this is the fact that the linear operator defining the convected wave equation with soft wall boundary condition is not self-adjoint. Then, although the direct solution is commonly used to solve this problem, the adjoint one must be considered. The application of the Divergence Theorem will lead to integration over the duct surface. As mentioned above, only the radiating surface will be moving with velocity V_p. The integral over the non-radiating surface will then vanish because the Green's functions satisfy the soft wall boundary condition. Thus, only the integral over the radiating surface, i.e. piston source, needs to be solved

Using the Divergence Theorem with the adjoint solution of the Green's functions that satisfy eq. (8), the modal pressure can be obtained as:

$$p_{mn}\left(\vec{r}\right)=-\int_S\left(\left.\frac{\partial p_{mn}}{\partial r}\right|_{r=a}\overline{g_{mn}}-p_{mn}\overline{\left.\frac{\partial g_{mn}}{\partial r}\right|_{r=a}}\right)dS=-\int_S\left(\left.\frac{\partial p_{mn}}{\partial r}\right|_{r=a}\overline{g_{mn}}+ip_{mn}\frac{\beta}{k_0}\left(k_0-k_zM\right)^2\overline{g_{mn}}\right)dS$$

$$=-i\rho c\left(k_0-k_zM\right)\int_S\left(\frac{1}{i\rho c\left(k_0-k_zM\right)}\frac{\partial p_{mn}}{\partial r}+\frac{\beta}{\rho c}\frac{\left(k_0-k_zM\right)}{k_0}p_{mn}\right)\overline{g_{mn}}\,dS \tag{24}$$

Then, replacing the factor in the integral of eq.(24) by the perturbation velocity $v_{d_{mn}}$ in eq.(22) leads to

$$p_{mn}\left(\vec{r}\right)=i\rho c\left(k_0-k_zM\right)v_{d_{mn}}\int_{S_R}\overline{g_{mn}\left(\vec{r_0}\middle|\vec{r}\right)}dS \tag{25}$$

Note that in the last expression the integral is over the radiating surface S_R only, and since the piston velocity is uniform, $v_{d_{mn}}$ was taken outside the integral.

Finally, using eq. (23), the pressure inside the duct due to the motion of a finite source is given by:

$$\begin{aligned}p^{(+)}\left(\vec{r}\middle|\vec{r_0}\right)&=-i\rho c\,V_p\sum_m\sum_n\frac{\left(k_0-k_z^{(+)}M\right)^2}{k_0}\int_{S_R}g_{mn}^{(+)}\left(\vec{r}\middle|\vec{r_0}\right)dS\\ p^{(-)}\left(\vec{r}\middle|\vec{r_0}\right)&=-i\rho c\,V_p\sum_m\sum_n\frac{\left(k_0-k_z^{(-)}M\right)^2}{k_0}\int_{S_R}g_{mn}^{(-)}\left(\vec{r}\middle|\vec{r_0}\right)dS\end{aligned} \tag{26}$$

Note that eq. (26) is given for the two directions of propagation of the noise field. Moreover, the relative location between the source and the receiver was changed using the reciprocity principle between the adjoint and the direct Green's function, i.e. $\overline{g_{mn}\left(\vec{r}_o\middle|\vec{r}\right)}=g_{mn}\left(\vec{r}\middle|\vec{r}_o\right)$.

For the piston source defined by constant z and θ coordinates (rectangular shaped piston), as shown in *Figure 4*, the integrals in (26) have closed form expressions as

$$\begin{aligned}p_{piston}^{(\ell)}\left(\vec{r}\middle|\vec{r}_n\right)&=\ Z^{(\ell)}\left(\vec{r}\middle|\vec{r}_n\right)\cdot V_p\ =\\ &=-i\,\rho c\,V_p\sum_{m=0}^{M_g}\sum_{n=0}^{N_g}\frac{\left(k_0-k_z^{(\ell)}M\right)^2}{k_0}A_{mn}^{(\ell)}\ \cos\left[m\left(\theta-\theta_n\right)\right]J_m\left(k_{mn}^{(\ell)}a\right)\times\\ &\qquad\times\kappa_\theta(\alpha)\ e^{-ik_z^{(\ell)}(z-z_n)}\ \frac{\sin(k_z^{(\ell)}d)}{k_z^{(+)}d}\ 2d\end{aligned} \tag{27}$$

where $\kappa_\theta\left(\alpha\right)=\frac{2a\alpha\sin\left(m\alpha\right)}{m\alpha}$. The parameters α and d are defined in *Figure 4*, and the modal amplitudes $A_{mn}^{(\ell)}$ are found from the solution of the system of equations in (18). The function

$Z^{(\ell)}\left(\vec{r}\,|\,\vec{r}_n\right)$ is the pressure transfer function at any point in the duct due to a piston vibrating with velocity V_p. Also, in (27), $\ell = +$ is used for the sound field downstream of the source (positive z- direction), while $\ell = -$ is used for the upstream field (negative z-direction).

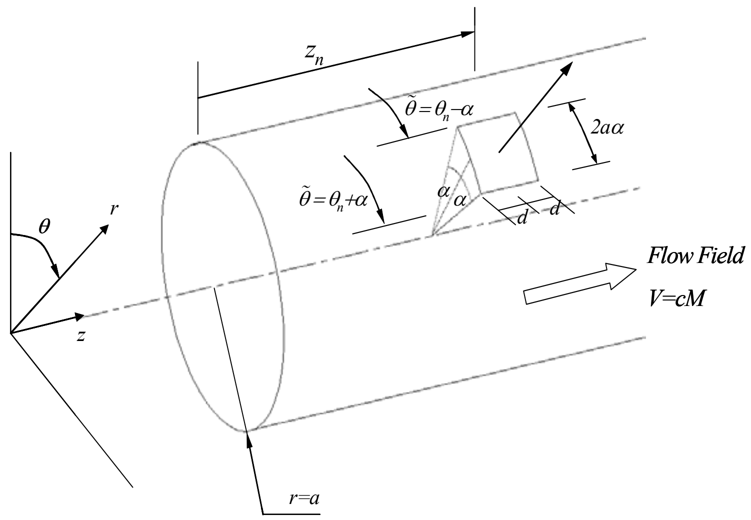

Figure 4: Model of the finite sources

3 APPLICATION EXAMPLE

The application example to illustrate the potential use of the formulation consists of modeling the effect of a circumferential array of rigid patches in a uniform liner. Liner splices are a clear example of rigid patches, i.e. local hard wall condition in a uniform liner. The moving boundary formulation developed above will be used to model the effect of these rigid patches as piston sources with appropriate velocities to simulate the correct rigid boundary condition. In this example, the disturbance incident noise field consists of positive propagating modes expressed using eq. (3) as:

$$p_{dist}\left(\vec{r}\right) = \sum_{m=0}^{M_d} \sum_{n=0}^{N_d} \left(A_{mn}^{(+)}\right)_{dist} \Phi_{mn}^{(+)}\left(r,\theta\right)\; e^{-ik_z^{(+)}z} e^{i\omega t} \tag{28}$$

Since the problem is linear and the effect of the rigid patches is modeled as piston sources, the resulting sound field will be due to the disturbance and the piston sources. For example, for a single rigid patch (piston source) the sound field in the duct will be as

$$p\left(\vec{r}\right) = p_{dist}\left(\vec{r}\right) + p_{piston}^{(+)} = p_{dist}\left(\vec{r}\right) + \; Z^{(+)}\left(\vec{r}\,|\,\vec{r}_n\right)\cdot V_p \tag{29a}$$

for the transmitted field (downstream) and,

$$p\left(\vec{r}\right) = p_{piston}^{(-)} = Z^{(-)}\left(\vec{r}\,|\,\vec{r}_n\right)\cdot V_p \tag{29b}$$

for the reflected field (upstream). The first term in (29*a*) represents the disturbance field in a uniform liner while the second represents the modification to the disturbance sound field due to the presence of the rigid patch. From knowledge of the pressure field, the radial particle velocity at the duct wall is found using (21) as follows

$$v_r = \frac{\beta_\omega}{\rho c} \sum_m \sum_n \frac{\left(k_0 - k_z M\right)}{k_0} p_{mn}\left(\vec{r}\right)\bigg|_{r=a} \tag{30}$$

To simulate the effect of a splice in this formulation, the piston velocity needs to be determined. The fact that the patch is rigid implies that the radial particle velocity must vanish at its surface. For this case, the piston velocity V_p must be the strength of a fictitious source required to cancel the existing particle velocity due to the disturbance on the surface of the patch. This condition implies satisfying the following expression

$$v_{r_{dist}}\left(\vec{r}_n\right) + {}^{v}Z\left(\vec{r}_n \middle| \vec{r}_n\right) \cdot V_p = 0 \tag{31}$$

where ${}^{v}Z$ is the particle velocity transfer function that can be derived by combining eqs. (27) and (30). Thus, the piston velocity is obtained from (31).

The case of a circumferential array of rigid patches is of much more practical interest. The development for a single patch is then extended for the case of an array of rigid wall patches (*Figure 5*). As mentioned before, the condition to satisfy is that the particle velocity at all the patch surfaces must vanish. However, the particle velocity over a rigid patch is not uniform and thus it is better to force the "average" particle velocity over the patch surface to vanish. This implies that the formulation is limited to frequencies where the wavelength is smaller than the size of the patches, i.e. at high frequencies where the wavelength is smaller than the patch size the "average" particle velocity vanish.

To implement the formulation, the function ${}^{v}Z_{os}$ is defined as the average particle velocity transfer function over an "observer" piston due to the motion of another "source" piston. These functions are obtained by integrating ${}^{v}Z\left(\vec{r} \middle| \vec{r}_n\right)$ in eq. (31) on the surface of the observer piston as follows

$${}^{v}Z_{os} = \frac{1}{S_o} \int_{S_o} {}^{v}Z\left(\vec{r} \middle| \vec{r}_n\right) dS_o \tag{32}$$

where S_o is the observer piston area.

To consider the influence of all the sources over each rigid patch, the functions ${}^{v}Z_{os}$ can be arranged in a matrix form. Then, the zero average particle velocity condition over the patch surfaces (equivalent to the condition in eq. (31)) is extended to the case of an array of patches by solving the following system of equations:

$$\left\{\overline{v}_{r_{dist}}\right\}_N + \left[{}^{v}Z_{os}\right]_{N\times N} \cdot \left\{V_p\right\}_N = \left\{\ 0\ \right\}_N \tag{33}$$

where N is the number of patches and $\overline{v}_{r_{dist}}$ is the average particle velocity at the face of each piston due to the incoming noise disturbance. Solving the liner system of equations, the piston source velocities modeling the effect of the rigid patches are obtained. Upon the computation of these velocities, the sound field is found from the superposition of the disturbance and piston source responses which are expressed in terms of modes. The acoustic power can then be found to predict the performance of the system, i.e. noise attenuation. The expressions to compute the power are described in the Appendix.

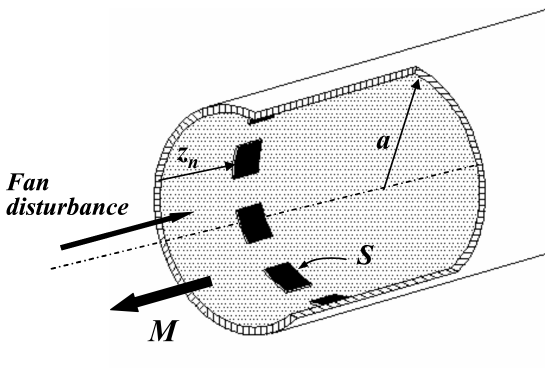

Figure 5: Lined duct with a single circumferential array of rigid patches.

4 CONCLUSIONS

A mathematical model to simulate the sound radiation due to a moving boundary inside a lined circular duct with uniform flow is developed. Closed form expressions of the Green's functions for a soft wall duct in the presence of a convective axial flow were developed. The Divergence Theorem was then used to find the radiation from a basic piston source of a simple rectangular shape which allows finding closed form expressions. However, the radiation from complex shapes with non-uniform velocity distribution can also be obtained from the basic piston source solution.

The mathematical formulation can be used to investigate many problems of practical significance. The main difference from one case to another lays in the way of computing the source velocity or strength required by the model. Some of the potential applications are in modeling of actuators in active noise control in ducts, liner splices, liners with non-uniform impedance distribution, and so forth.

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6 ACKNOWLEDGEMENT

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7 APPENDIX: SOUND POWER COMPUTATION

An expression for the sound power crossing any sector of the duct can be obtained by means of the acoustic intensity I_z in the z-direction given by [5]

$$I_z = \frac{1}{2}\mathrm{Re}\left[p v_z^* + \rho c |v_z|^2 M + \frac{|p|^2}{\rho c} M + v_z p^* M^2 \right] \tag{A1}$$

For positive traveling waves, the pressure can be written in terms of positive and negative spinning modes as follows

$$p = \sum_{m=0}^{M}\sum_{n=0}^{N} \left(A_{mn}^{(+)}\right)^{pos} J_m\left(k_{mn}^{(+)} r\right) e^{-im\theta} e^{-ik_z^{(+)}z} + \sum_{m=0}^{M}\sum_{n=0}^{N} \left(A_{mn}^{(+)}\right)^{neg} J_m\left(k_{mn}^{(+)} r\right) e^{im\theta} e^{-ik_z^{(+)}z} \tag{A2}$$

The modal amplitudes of the above equation are found as a superposition of the disturbance and the pressure radiated by the sources. Moreover, the axial particle velocity v_z can be found in terms of the pressure using the axial component of the equilibrium equation

$$v_z = \sum_m \sum_n \frac{k_z}{\rho c\left(k_o - k_z M\right)} p_{mn} \tag{A3}$$

Then, the acoustic power is found by integrating the acoustic intensity in (A1) over the cross section of the duct

$$W = \int_0^{2\pi} \int_0^{a} I_Z \, r \, dr d\theta \qquad (A4)$$