Asociación Argentina
de Mecánica Computacional

Mecánica Computacional Vol XXXIX, págs. 59-68 (artículo completo)
F.E. Dotti, M. Febbo, S.P. Machado, M. Saravia, M.A. Storti (Eds.)
Bahía Blanca, 1-4 Noviembre 2022

STOCHASTIC EVALUATION OF THE RUN-TIME OF A STICK-SLIP OSCILLATOR PROBLEM

Mariana Gomes[a], **Hector E. Goicoechea**[a,b], **Roberta Lima** [a] **and Rubens Sampaio**[a]

[a]*Mechanical Engineering Department, Pontifícia Universidade Católica do Rio de Janeiro, Rua Marquês de São Vicente 255, Gávea, Rio de Janeiro, RJ, 22451-900, Brazil*

[b]*Instituto de Física del Sur (IFISUR), Departamento de Física, Universidad Nacional del Sur, CONICET, Av. L. N. Alem 1253, Bahía Blanca, B8000CPB, Buenos Aires, Argentina*

Keywords: Analytical approximations, stick-slip phenomenon, Monte Carlo method, stochastic computation costs, random vibrations

Abstract.

In this article, the run-times associated to a Monte Carlo simulation are studied from a stochastic perspective for a problem consisting of an oscillator with uncertain dry friction that can exhibit stick-slip vibrations. Some of the novelties of the paper are that the run-times are modeled as random variables, which is not a common approach; and that a comparison of the run-times obtained with two different simulation strategies is included to assess their performance. In one of the simulations, the Monte Carlo (MC) is used in combination with a Runge-Kutta numerical integration scheme. In another, the MC method combined with an analytical approximation obtained with the Multiple Scales method is used. The hypotheses that some characteristics of the friction force and the stick-phase duration may have a direct influence on the run-time are investigated. The dependency of the run-times on the magnitude of the dynamic friction and the first stick-duration is explored, and the histograms for the run-times are constructed. The results, for $5 \cdot 10^5$ realizations, show that the run-times where an analytical approximation was used are, in mean, 3.8 times faster than those of the numerical integration, and they are also less sensitive to variations considering that the standard deviation is 2.7 times smaller. The improvement observed in the run-time obtained with the analytical approximation can be a determining factor for the feasibility of a stochastic study, given that the Monte Carlo approach requires a large number of realizations to calculate a statistical model with accuracy.

1 INTRODUCTION

A mathematical model is, in general, composed by some inputs, a transformation (e.g. an equation or an initial value problem), and some outputs, that are objects of interest. When some of the inputs are random, characterized as for example random variables, the outputs become random too. In these cases, given the probabilistic models of the inputs, one of the objectives could be to determine the probabilistic models of the outputs.

However, for a great number of stochastic problems, to determine the probabilistic models of the outputs is not a simple task. It can be a challenge, especially when the problem involves differential equations. In these situations, an option to deal with the problem is to construct approximations to the probabilistic models of the outputs using statistical models.

One of the most common tools used to construct such statistical models is the method of Monte Carlo (MC), as shown in Sobol (1994). To construct the statistical models, the method transforms the stochastic problem into many deterministic ones. In each one of them, a realization of the random input is used. First, a random sample of the input is generated. Then, each realization of the sample is transformed according to the given mathematical transformation. The obtained results, which are samples of output, are stored and used to construct a statistical model.

Even though each realization is deterministic in nature, neither the inputs, the outputs, or the computational costs are. Moreover, the elevated number of calculations required to assure an accurate statistical model makes the MC method a big data problem, especially when the transformation is given by a differential equation that is solved by numerical integration, as shown in Lima and Sampaio (2017, 2021). Given that the computational resources are limited, the computational costs, such as the total run-time, can be of the utmost importance.

In this paper, an analysis of the computational cost associated to the simulation run-time of a random oscillator is tackled from a stochastic perspective. A comparison is made considering two cases where 1) the MC method is combined with a Runge-Kutta numerical integration scheme; and 2) the MC is combined with an analytical approximation based on the Multiple Scale method. Up to the authors' knowledge, this study is a novelty given that most papers concerning stochastic simulations ignore the fact that the run-times are also of stochastic nature, Wilhelm et al. (2008); Lee et al. (2009). Eventually, the run-times play a role in determining whether a stochastic analysis is feasible or not, and their behavior could have an impact regarding the efficient assignment of the available resources.

In the context of the dry friction oscillator that can exhibit stick-slip vibrations, the accuracy of the response in the numerical integrations is related to time-step used, as it influences the accuracy with which the transitions are captured. In other words, an improvement in the accuracy of the numerical integrations is usually linked to a decrease in the integration time-step, but the smaller the step, the higher the amount of data generated and the higher the run-time required. For this reason, the hypothesis that the stick-phase duration and the magnitude of the dry friction have a direct influence on the run-time is explored, especially when numerical integrations are used. Also, the MC with analytical approximations is employed as a means to improve the computation costs, and the results of both approaches are compared.

2 SYSTEM DYNAMICS AND DEFINITION OF THE STICK AND SLIP PHASES

The system analyzed in this paper is composed of a spring, a damper, and a mass that slides over a constant-speed moving belt, as shown in Fig. 1. The dry friction between the two surfaces in contact can induce stick-slip vibrations. When this occurs, the system's response is characterized by two different behaviors, called stick and slip phases. These phases alternate with an abrupt transition and have a non-zero duration, that is, they are not instantaneous.

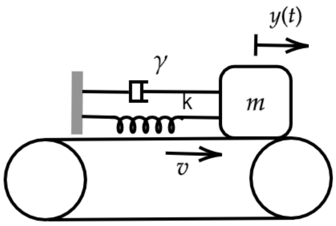

Figure 1: Mass-spring-damper system with dry friction.

The initial value problem (IVP) for the system is given by

$$m\,\ddot{y}(t) + \gamma\,\dot{y}(t) + k\,y(t) = f_{at}(V)\,, \tag{1}$$

with initial conditions $y(0)$ and $\dot{y}(0)$. In the previous equations, y is the position of the mass m, γ the damping coefficient, k the spring constant, f_{at} the friction force between the mass and the belt and V the relative velocity between them. For $V \neq 0$, the friction force is modeled as

$$f_{at}(V) = \frac{1}{3}\,a\,V\,(V^2 - 3) + f_d\,sign(V) \tag{2}$$

where $V = (v - \dot{y})$, a is constant, v is the speed of the belt, and f_d is the dynamic friction force. For $V = 0$, the dry friction force can assume values between $[-f_e, f_e]$, where f_e is the magnitude of the static friction. Fig. 2 shows how the friction force varies with V.

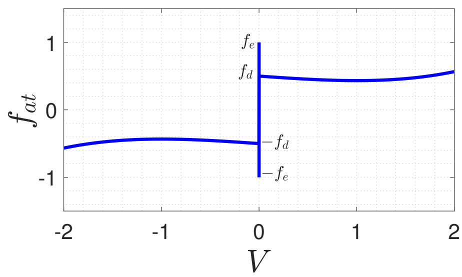

Figure 2: Friction force model with $a = 0.1$ and $f_d = 0.5$.

Due to the discontinuity in the transition between static and dynamic frictions, the dynamic is discontinuous and the solution is expressed with a piece-wise switch as

$$y(t) = \begin{cases} y_j(t) & \text{, if stick conditions are met} \\ y_p(t) & \text{, otherwise} \end{cases} \tag{3}$$

where $y_j(t)$ is the solution corresponding to the stick phases, $y_p(t)$ is the solution to the slip phases, with

$$j \in [1, ..., N_{sticks}], \quad p \in [1, ..., N_{slips}], \quad l \in [1, ..., N_{sticks} + N_{slips}] \tag{4}$$

In the previous, N_{sticks} and N_{slips} are the total number of sticks and slips phases, respectively. The conditions to predict whether the current phase is a stick or a slip will be obtained in what comes next. In all cases, the notation $t_{0,l} \leq t \leq t_{e,l}$ is used to refer to the beginning and ending instant of each phase l, whether a stick or slip.

The stick phase exhibits the following properties: a null relative velocity, $V = 0$, and a friction force that lies within the interval $-f_e \leq f_{at} \leq f_e$. During a stick phase, the velocity of the mass equals the belt velocity, $\dot{y}_j(t) = v$, therefore the mass acceleration is zero, $\ddot{y}_j(t) = 0$. Using these observations in combination with Eq. (1), one obtains

$$\gamma\, v + k\, y_j = f_{at}\,, \tag{5}$$

that, together with $-f_e \leq f_{at} \leq f_e$ give an expression limiting the values that y can take for stick phases to occur. These limits are given by

$$\frac{-f_e - \gamma\, v}{k} \leq y_j \leq \frac{f_e - \gamma\, v,}{k}\,. \tag{6}$$

During this phase the mass exhibits uniform motion with an analytical solution given by

$$y_j(t) = y_j(t = t_{0,l}) + v(t - t_{0,l}) \text{ for } t_{0,l} \leq t \leq t_{e,l}, \tag{7}$$

where $t_{0,l}$ and $t_{e,l}$ are the instants associated to the start and the end of the j-th phase, in this case, a stick, respectively. Thus, $y_j(t = t_{0,j})$ represents the position of the mass at the instants of initiation of the stick phases. These values also coincide with the position of the mass at the end of the immediately previous slip phase.

If the system is not in a stick phase, then it is in a slip phase. In this case, the relative velocity between the mass and the belt is different from zero and the friction forces are defined by Eq. (2), so Eq. (1) becomes

$$m\, \ddot{y}_p(t) + \gamma\, \dot{y}_p(t) + k\, y_p(t) = \frac{1}{3}\, a\, V\, (V^2 - 3) + f_d\, sign(V)\,. \tag{8}$$

As we can observe from the equation above, the slip phase is governed by a nonlinear IVP with an unknown analytical solution. For each slip phase we will compute an analytical approximation using the Multiple Scale method. A detailed derivation of the previous equations can be found in Gomes et al. (2021).

Summarizing the results obtained so far, the solution of this problem can be written as a piece-wise response in the form of

$$y(t) = \begin{cases} y_j(t) = y_j(t = t_{0,l}) + v(t - t_{0,l}) & \text{, if } -f_e \leq f_{at} \leq f_e \text{ and } \frac{-f_e - \gamma\, v}{k} \leq y \leq \frac{f_e - \gamma\, v,}{k} \\ y_p(t) & \text{, otherwise} \end{cases} \tag{9}$$

where the condition to check if the current state corresponds to a stick or a slip phase have been is now stated explicitly.

3 THE ANALYTICAL APPROXIMATION

The perturbation method used in this paper is the Multiple Scale method. The central idea of this technique is to transform the often complex IVP with an unknown analytical solution into a family of linear IVPs with known analytical solution, by introducing a perturbation parameter Kevorkian and Cole (1996). To do this, a solution in the form of a power series of the perturbation parameter is proposed. When the proposal is introduced into the equation, by application of the Fundamental Theorem of Perturbation Theory, a new set of IVP is obtained Simmonds and Mann (1988). To compute a solution all these linear IVPs should be solved hierarchically, but this is an impossible task, as it would involve solving an infinite number of IVPs. For this reason, we produce an N-order approximation by truncating the series. The number of terms is related to the quality of the approximation, i.e. the domain of validity . The details to computing an analytical approximation to the solution of Eq. (8) are explained in Gomes et al. (2021). A first-order approximation is given by

$$y_p(t) \approx e^{z\,(t-t_{0,l})}\left[C_{1,p}\,\cos(\sqrt{k}\,(t-t_{0,l})) \,+\, C_{2,p}\,\sin(\sqrt{k}\,(t-t_{0,l}))\right] + \frac{f_d}{k}\,, \tag{10}$$

where ϵ is a perturbation parameter that was included in Eq. (8) as a factor multiplying $\dot{y}_p$, $z = \epsilon(a-\gamma)/2$, $C_{1,p} = y_p(t = t_{0,l}) - f_d/k$ e $C_{2,p} = [\dot{y}_p(t = t_{0,l})) - zd]/\sqrt{k}$. The value of the expressions $y_p(t = t_{0,l})$ and $\dot{y}_p(t = t_{0,l})$ vary. At the beginning of the simulation, they match the initial conditions $y(0)$ and $\dot{y}(0)$, respectively. After that, they coincide with the value at the ending of the previous phase, that is, if the current phase is a slip: $y_p(t = t_{0,l}) = y_j(t = t_{e,l-1})$ and $\dot{y}_p(t = t_{0,l}) = \dot{y}_j(t = t_{e,l-1})$. Otherwise, $y_j(t = t_{0,l}) = y_p(t = t_{e,l-1})$ and $\dot{y}_j(t = t_{0,l}) = \dot{y}_p(t = t_{e,l-1})$.

One of the features of the Perturbation Theory is that the perturbation parameter should assume small values, i.e., $\epsilon << 1$ Simmonds and Mann (1988). Thus, once the value of z depends on ϵ, the value of z is also small, i.e, $z \to 0$ and consequently $e^{z\,t} \to 1$. With these assumptions and using trigonometric transformations, Eq. (10) can be rewritten as

$$y_p(t) \approx C_{3,p}\,cos(\sqrt{k}\,(t-t_{0,l}) \,-\, C_{4,p}) + \frac{f_d}{k} \tag{11}$$

where $C_{3,p} = \sqrt{C_{1,p}^2 + C_{2,p}^2}$ and $C_{4,p} = atan(C_{2,p}/C_{1,p})$. This rewriting of Eq. (11) is used as an aid to calculating an expression for the transition instants between phases.

4 TRANSITION INSTANTS BETWEEN STICK-SLIP PHASES

As already described, this system exhibits a dynamics characterized by two highly distinctive phases. The mass during stick phases has a uniform motion given by Eq. (5). While during slip phases, the movement is given by a nonlinear expression as in Eq. (8). The principal information about this dynamic is the transition instants between phases.

When numerical integrations are used, a small time-step is often needed to resolve each phase as well as to accurately capture the transition instant between phases. But the smaller the time-step, the more calculations are required, and the more costly the problem is. This can be a decisive factor for the feasibility of the problem. In addition, the conditions of the stick and slip phases must be tested at every single time-step. All these details influence the run-time of the numerical integrations. In contrast, an analytical approximation is valid for the whole duration of the current phase, and from it the transition instants can be calculated directly, reducing the computation cost.

To calculate the transition instants analytically, the parameter t is isolated in the right side of the Eq. (7) and (11) accordingly to the condition of each phase. First, we calculate the transition instants from of the stick-to-slip phases using Eq. (7). The stick phase is defined by the relative velocity between the mass and the belt being equal to zero and the mass position being in the range of $(y_{min}\,,\quad y_{max})$. The sign of the belt velocity determines if the mass position is either y_{min} or y_{max} at the stick-to-slip transition instant. So, we have

$$y_j(t = t_{0,l}) \; + \; v\,(t - t_{0,l}) \; = \begin{cases} y_{max}, & \text{if } v > 0 \\ y_{min}, & \text{if } v < 0 \end{cases} \tag{12}$$

and depending on the sign of the belt velocity, the transition instants for the stick-to-slip phases are given by

$$t \; = \begin{cases} \dfrac{y_{max} \; - \; y_j(t = t_{0,l})}{v} + t_{0,l}, & \text{if } v > 0 \\[2em] \dfrac{y_{min} \; - \; y_j(t = t_{0,l})}{v} + t_{0,l}. & \text{if } v < 0 \end{cases} \tag{13}$$

To calculate the transition instants from slip-to-stick phases, Eq. (11) is derivated

$$\dot{y}_p(t) \approx -C_{3,p} \sin\left(\sqrt{k}(t - t_{0,l}) - C_{4,p}\right) \sqrt{k}\,, \tag{14}$$

and the condition that the mass velocity should be equal to the velocity of the belt is checked

$$-C_{3,p} \sin(\sqrt{k}(t - t_{0,l}) - C_{4,p})\sqrt{k} = v. \tag{15}$$

Thereby, the parameter t is isolated as

$$t = \frac{\left[asin\left(\dfrac{-v}{C_{3,p}\sqrt{k}}\right) + C_{4,p}\right]}{\sqrt{k}} + t_{0,l}. \tag{16}$$

It should be noted that the condition that leads to Eqs. (15) and (16) (the mass velocity equals the velocity of the belt) is a necessary but not sufficient condition to determine the transition instants for the slip-to-stick phases. This condition is completed with the verification that the position of the mass must be within interval $[y_{min}\,, \quad y_{max}]$.

5 STOCHASTIC APPROACH - MONTE CARLO METHOD

The Monte Carlo method permits to construct statistical models for random objects. The principal idea of the MC is the tackle the stochastic problem by solving several deterministic ones for different realizations of the random input variables. The first step is to obtain a sample of the input random object with known distribution. Then, each realization is transformed according to the deterministic problem. Lastly, the outputs of interest are saved to construct their statistical models. The resolution procedure is stopped after some convergence criterion is met, otherwise more realizations are simulated. The elevated amount of data required to construct an accurate statistical model falls under the classification of a big data problem.

In this paper, the dynamic friction force was modeled as a random variable F_d. The authors expect an influence of this parameter in the response, affecting the transition instants and the start and end position between phases, the number of sticks, the duration of each phase as well as the computational run-time. The force F_d was modeled as a random variable with uniform distribution and its probability density function (PDF) is

$$p_{F_d}(x) = \begin{cases} \frac{1}{l_s - l_i}, & l_i \leq x \leq l_s \\ 0, & \text{to any other value}\,. \end{cases} \tag{17}$$

Parameter	Value	Unit	Parameter	Value	Unit
m	1	Kg	k	0.1	N/m
v	-2	m/s	$y(0)$	1	m
γ	1	(N s)/m	$\dot{y}(0)$	4	m/s
a	0.1	(Kg s)/(m^2)	f_e	2	N
ϵ	0.0001	-	g	9.81	m/s^2

Table 1: Parameters used in the analytical and numerical simulations.

where l_i is the inferior limit of the PDF range and l_s is the upper limit. The limits of this distribution were chosen to assure two features: the range must be positive and the upper limit must not exceed the static friction force, f_e. Thereby, the support was defined as $[0.8N, \quad 1.8N]$. The relation between the friction forces and friction coefficients is given by $f_e = \mu_e \, m \, g$ e $f_d = \mu_d \, m \, g$.

6 RESULTS AND DISCUSSIONS

With the distribution for the random variable representing the dry friction F_d defined, the stochastic problem is tackled with two different strategies: 1) a 'numerical' case that combines the Monte Carlo method with a Runge-Kutta numerical integration and 2) an 'analytical' case that combines Monte Carlo with an analytical approximation based on the Multiple Scale method. The numerical and analytical approximations were simulated using the same parameters values shown in Tab. 1 and time interval of $[0\,\mathrm{s}\,, 2000\,\mathrm{s}]$. A total number of $5 \cdot 10^5$ realizations was used.

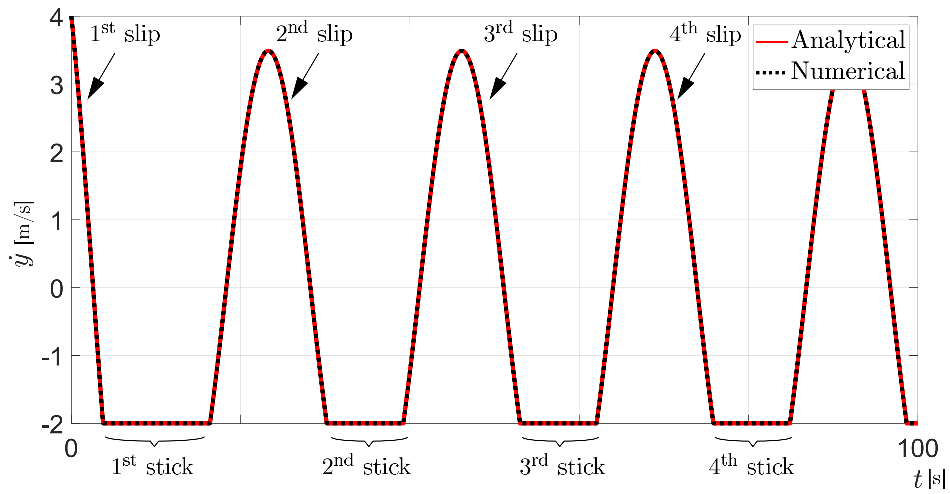

Figure 3: The speed $\dot{y}(t)$ for one of the realizations until the 4$^{\text{th}}$ stick and slip phase.

In Fig. 3 one of the simulated realizations is shown. The first four stick phases of the simulation computed using the numerical and analytical approximations are displayed. Fig. 4 shows three scatter plots. Fig. 4 shows three scatter plots. Fig. 4 (a) relates the magnitude of the dynamic friction with the first stick duration for each realization; Fig. 4 (b) relates the first stick duration with the second one; and Fig. 4 (c) relates the second stick duration with the third one. In this graph the hypothesis that friction affects the duration of the sticks is confirmed. On top of that, a dependency between the first and second sticks is observed, while all subsequent stick durations are invariant, as depicted in Fig. 4 (c) for the second and third stick durations. This is expected since the first stick duration depends on the initial conditions and the duration of the next stick phases depends on the first stick duration.

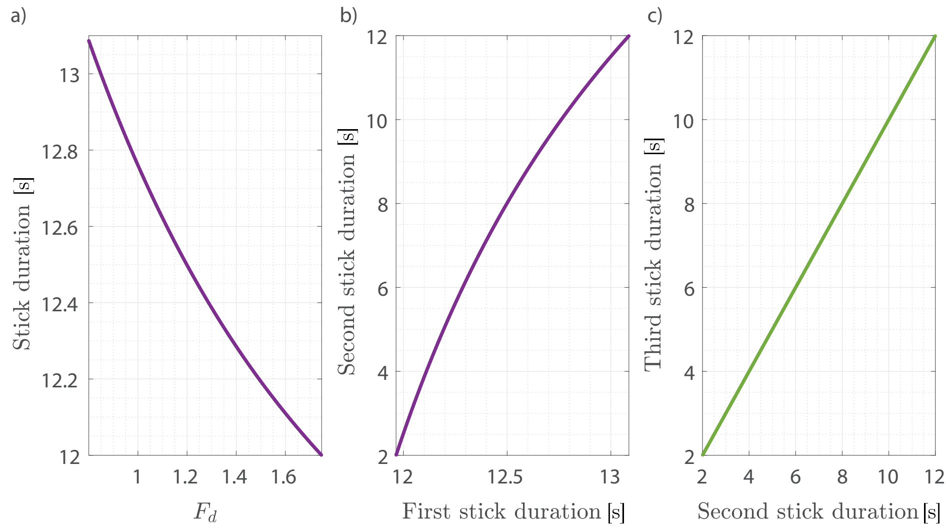

Figure 4: A scatter plot shows the relation between: a) the duration of the first stick and the magnitude of the friction force Fd; b) the duration of the first and the second stick; c) the duration of the second and the third stick.

The results for the computation run-time taken by the numerical integration and the analytical approximation, considering $5 \cdot 10^4$ realizations, are shown in Fig. 5. The support of the histogram of the numerical integration run-time is about $[30.3, \quad 49.8]$ while the support of the histogram of the analytical approximation run-time is about $[7.8, \quad 15.1]$. The mean values are 41.71 s and 10.94 s, and the standard deviations are 3.11 s and 1.17 s, for the numerical and the analytical methods, respectively. These results show that the computation run-time with the analytical approximation are, in mean, 3.81 faster than the numerical ones, and also that they are less sensitive to variations, considering that the standard deviation is 2.66 smaller than that of the numerical integrations.

The standard deviation of the run-time with the analytical approximation is smaller than that of the numerical one. This indicates that the numerical integration is more affected by the value of the dynamic friction, which is the only input random variable taken into consideration in this study. This result agrees with the observations depicted in Fig. 6, that shows how the computational run-times are influenced by the value of the dynamic forces, F_d. A tendency is observed in Fig 6, showing that the numerical run-time increases appreciably with an increase of F_d, while the tendency of the analytical ones is be less prone to those variations with F_d, remaining the realizations confined to a band region that is closer to being horizontal. This validates the second part of the hypothesis that friction not only affects the duration of the sticks, but that it also has an effect over the computation run-times.

Finally, Fig. 7 depicts the tendency when the stick duration is plotted against the computational run-time. In this graph it is observed that the longer the stick duration, the faster the computation run-time is.

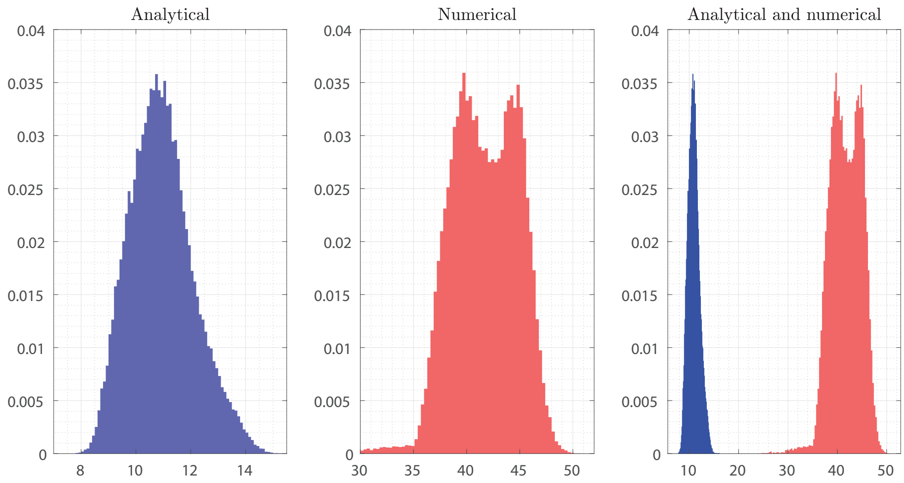

Figure 5: Normalized histograms of the computing time to simulate the Monte Carlo method using the numerical integration (red) and analytical approximation (blue).

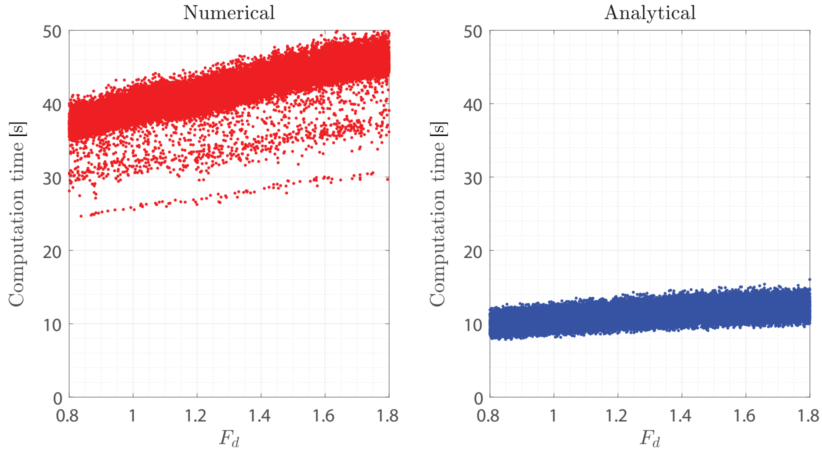

Figure 6: Scatter plot of the computing run-time to simulate the Monte Carlo method using the numerical integration (red) and analytical approximation (blue) for each value of F_d.

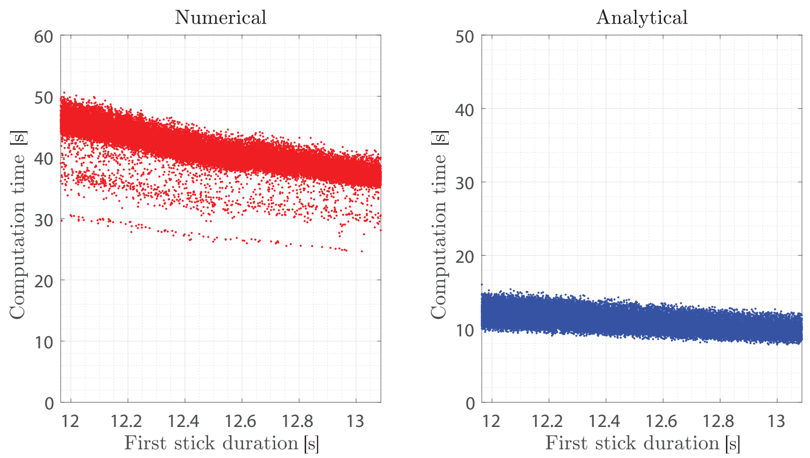

Figure 7: Scatter plot of the computing run-time to simulate the Monte Carlo method using the numerical integration (red) and analytical approximation (blue) for first stick duration.

7 CONCLUSIONS

The results confirm that the run-times are affected by the magnitude of the dynamic friction force and the first stick duration, in all cases. On top of that, MC combined with an analytical approximation was found to be 3.81 times faster in the mean, what can be a decisive factor to the feasibility of a stochastic study. These results are consistent with the fact that numerical integration schemes require relatively small integration time-steps to resolve each phase and capture the abrupt transitions between stick-slip phases, making the problem computationally costly. In contrast, with analytical strategies an approximation for the entire phase is obtained at once, and the transition instant is calculated from this approximation, resulting in a cheaper computation. Given that the computational resources are limited, this computational advantage of using analytical approximations is an important factor to consider when trying to obtain accurate statistic models that need a large amount of realizations.

REFERENCES

Gomes M., Lima R., and Sampaio R. Multiscale method: A powerful tool to reduce the computational cost of big data problems involving stick-slip oscillations. In *Proceedings of the 5th International Symposium on Uncertainty Quantification and Stochastic Modelling*, pages 69–79. Springer, 2021.

Kevorkian J. and Cole J.D. *Multiple Scale and Singular Perturbation methods*. Springer, 1 edition, 1996.

Lee M., Jeon J.h., Bae J., and Jang H.S. Parallel implementation of a financial application on a gpu. In *Proceedings of the 2nd International Conference on Interaction Sciences: Information Technology, Culture and Human*, page 1136–1141. Association for Computing Machinery, New York, NY, USA, 2009.

Lima R. and Sampaio R. Parametric analysis of the statistical model of the stick-slip process. *Journal of Sound and Vibration*, 397:141–151, 2017.

Lima R. and Sampaio R. Random stick-slip oscillations in a multiphysics system. *European Physical Journal Plus*, 136(8), 2021.

Simmonds J.G. and Mann J.E.E.. *A First Look at Perturbation Theory*. American Society of Mechanical Engineers, Mineola, NY, 2 edition, 1988. ISBN 0486675513.

Sobol I.M. *A Primer for the Monte Carlo Method*. CRC Press, 1 edition, 1994. ISBN 9780849386732; 084938673X.

Wilhelm R., Engblom J., Ermedahl A., Holsti N., Thesing S., Whalley D., Bernat G., Ferdinand C., Heckmann R., Mitra T., Mueller F., Puaut I., Puschner P., Staschulat J., and Stenström P. The worst-case execution-time problem—overview of methods and survey of tools. *ACM Trans. Embed. Comput. Syst.*, 7(3), 2008.