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A NODAL AVERAGED FORMULATION FOR AXISYMMETRIC SOLIDS USING LINEAR QUADRILATERALS

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Abstract. In this work we present a finite element formulation for the stress analysis of axisymmetric solids based on the nodal recovering of stresses and deformations. This procedure is based on a truncated Taylor series expansion of the gradient matrices. For constant constitutive properties the resultant stiffness matrices have polynomial integrands and can be integrated analytically without using numerical integration. If the constitutive properties are variable over the bilinear quadrilateral element, it is shown that we can use an equivalent constant constitutive matrix made by nodal averaged properties for computing the stiffness matrices. Also, the procedure is convergent for any type of material nonlinearity, including plasticity.

1 INTRODUCTION

The isoparametric formulation of bilinear quadrilaterals leads to complex integrands of stiffness matrices, and although some authors have integrated analytically these matrices (Babu and Pinder, 1984; Rathod, 1988) numerical integration using Gauss's quadrature is the preferred choice (Zienkiewicz et al., 2013). This is due to many factors, for example, there is no appreciable gain in accuracy using exact integration and the analytical integration produces rather complex results involving the use of rational polynomials and logarithmic functions that require a great number of computations.

On the other side, the use of numerical integration can be implemented by a systematic approach (Zienkiewicz et al., 2013), independently of the complexity of the integrands. By using Gauss's quadrature the integrands need to be evaluated at certain *integration points* and a minimum number of integration points is required for each finite element to ensure convergence of the formulation. For bilinear quadrilaterals the minimum number of integration points (full quadrature) is four (Zienkiewicz and Hinton, 1976). Full quadrature is costly but the use of reduced quadrature (one integration point) can generate spurious modes or mechanisms known as hourglass modes (Irons and Ahmad, 1980; Belytschko et al., 1984).

Many authors (Flanagan and Belytschko, 1981; Jacquotte and Oden, 1984; Liu and Belytschko, 1984; Liu et al., 1985; Schulz, 1985; Hansbo, 1998) have developed effective controls of the spurious modes that results in a stiffness matrix $\mathbf{K}$ composed of two parts a consistent stiffness matrix $\mathbf{K}^C$ plus a stabilization stiffness matrix $\mathbf{K}^S$.

$$\mathbf{K} = \mathbf{K}^C + \mathbf{K}^S \tag{1}$$

The consistent stiffness matrix $\mathbf{K}^C$ is the matrix obtained with one-point quadrature that guarantees convergence. The stabilization matrix $\mathbf{K}^S$ is added to avoid the appearance of spurious modes (null energy modes) in the complete stiffness matrix $\mathbf{K}$. The resultant matrices can be integrated explicitly.

For axisymmetric quadrilaterals the same procedure can be applied to obtain a consistent and a stabilization matrix. In this work we will use the procedure developed by (Liu et al., 1985, 1994) to obtain the consistent and stabilization matrices.

For nonlinear material properties usually the constitutive matrix is evaluated in each integration point, in the case of stabilized finite elements it is possible to assume constant properties over all the element, this is equivalent to compute this properties at the centroid of each element. But this procedure does not take into account the strong variation that occurs in an advancing front of plasticity or near the boundaries. Then is preferable to compute the material properties at the nodes of the mesh by a procedure, usually employed in meshfree methods and known as *nodal integration* (Beissel and Belytschko, 1996; Puso and Solberg, 2006; Morrev and Gordon, 2018). It should be noted that in these methods special procedures are required to compute a global stiffness matrix.

For the case of a mesh of bilinear quadrilaterals we will show that if the constitutive matrices at each nodal point of the mesh are known, then the element stiffness matrices can be computed in a conventional manner using a nodal averaged constitutive matrix, and the procedure is valid for linear or nonlinear materials, even in the case of plasticity.

The paper is organized as follows, firstly we present the main equations for stress analysis of axisymmetric solids by finite elements, then we show the consistent and stabilization matrices, as deduced in (Liu et al., 1985). Finally, we show the deduction of the averaged constitutive matrix and a numerical example is presented.

2 FINITE ELEMENT STRESS ANALYSIS OF AXISYMMETRIC SOLIDS

Consider the mapping of a four node quadrilateral element from the physical space of coordinates (r, z) to the biunit square $[-1, 1] \times [-1, 1]$ in the parametric space of coordinates (ξ, η).

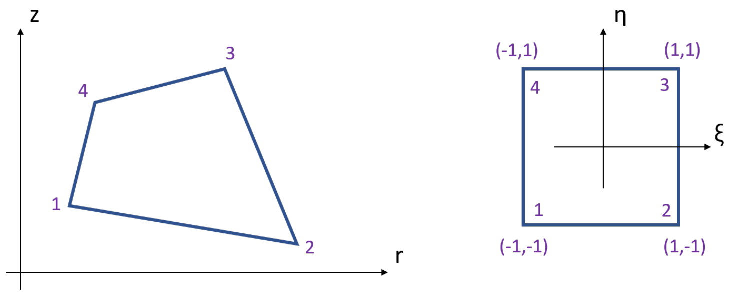


Figure 1: Four node isoparametric quadrilateral

The geometry of the finite element is define by interpolating the spatial coordinates (r, z) from the nodal values (r_i, z_i) as

$$
\begin{aligned}
r(\xi, \eta) &= \sum_{i=1}^{4} N_i(\xi, \eta) r_i = \mathbf{N}_b^T \mathbf{r} \\
z(\xi, \eta) &= \sum_{i=1}^{4} N_i(\xi, \eta) z_i = \mathbf{N}_b^T \mathbf{z}
\end{aligned}
\tag{2}
$$

where $\mathbf{r}, \mathbf{z}$ are the vectors of nodal coordinates.

$$
\begin{aligned}
\mathbf{r} &= \left\{ r_1 \quad r_2 \quad r_3 \quad r_4 \right\}^T \\
\mathbf{z} &= \left\{ z_1 \quad z_2 \quad z_3 \quad z_4 \right\}^T
\end{aligned}
\tag{3}
$$

and $\mathbf{N}_b$ is the vector of bilinear shape functions

$$
\mathbf{N}_b = \left\{ N_1 \quad N_2 \quad N_3 \quad N_4 \right\}^T \tag{4}
$$

The bilinear shape functions can be expressed as

$$
N_i(\xi, \eta) = \frac{1}{4}(1 + \xi_i \xi)(1 + \eta_i \eta) \tag{5}
$$

where (ξ_i, η_i) are the nodal values in the parametric space.

Adopting an *isoparametric* mapping the displacements are interpolated with the same bilinear shape functions:

$$
\begin{aligned}
u(\xi, \eta) &= \sum_{i=1}^{4} N_i(\xi, \eta) u_i = \mathbf{N}_b^T \mathbf{u} \\
w(\xi, \eta) &= \sum_{i=1}^{4} N_i(\xi, \eta) w_i = \mathbf{N}_b^T \mathbf{w}
\end{aligned}
\tag{6}
$$

where $\mathbf{u}, \mathbf{w}$ are the vectors of nodal displacements

$$
\begin{aligned}
\mathbf{u} &= \left\{ u_1 \quad u_2 \quad u_3 \quad u_4 \right\}^T \\
\mathbf{w} &= \left\{ w_1 \quad w_2 \quad w_3 \quad w_4 \right\}^T
\end{aligned}
\tag{7}
$$

After substitution of the approximated displacement field (6) into the strain vector (**??**) we obtain the approximated strains $\boldsymbol{\epsilon}_e$ over the element.

$$
\boldsymbol{\epsilon}_e = \mathbf{B}\mathbf{d}_e \tag{8}
$$

where $\mathbf{B}$ is the *gradient matrix* defined as

$$
\mathbf{B} = \begin{bmatrix} \dfrac{\partial \mathbf{N}_b^T}{\partial r} & 0 \\ 0 & \dfrac{\partial \mathbf{N}_b^T}{\partial z} \\ \dfrac{\partial \mathbf{N}_b^T}{\partial z} & \dfrac{\partial \mathbf{N}_b^T}{\partial r} \\ \dfrac{\mathbf{N}_b^T}{r} & 0 \end{bmatrix} \tag{9}
$$

and $\mathbf{d}_e^T = \left\{ \mathbf{u}^T \ \mathbf{w}^T \right\}$ is the vector of nodal displacements of the element.

3 CONSISTENT AND STABILIZATION MATRICES

Using the procedure described in (Liu et al., 1985) we approximate the gradient matrix $\mathbf{B}$ by its Taylor's series around the element centroid and retaining up to linear terms we have

$$
\bar{\mathbf{B}}(\xi, \eta) = \frac{1}{J_{00}} \left(\bar{\mathbf{B}}_{00} + \bar{\mathbf{B}}_{10}\xi + \bar{\mathbf{B}}_{01}\eta \right) \tag{10}
$$

The B-bar gradient matrix $\bar{\mathbf{B}}_{00}$ is

$$
\bar{\mathbf{B}}_{00} = \frac{1}{2} \begin{bmatrix} z_{24} & -z_{13} & -z_{24} & z_{13} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -r_{24} & -r_{13} & r_{24} & r_{13} \\ -r_{24} & -r_{13} & r_{24} & r_{13} & z_{24} & -z_{13} & -z_{24} & z_{13} \\ \dfrac{J_{00}}{2r_m} & \dfrac{J_{00}}{2r_m} & \dfrac{J_{00}}{2r_m} & \dfrac{J_{00}}{2r_m} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{11}
$$

where

$$
\begin{aligned}
r_{ij} &= r_i - r_j \\
z_{ij} &= z_i - z_j
\end{aligned}
\tag{12}
$$

and r_m is the mean radius, that is the value of the radius at the centroid

$$
r_m = (r_1 + r_2 + r_3 + r_4)/4 \tag{13}
$$

Also J_{00} is the *jacobian* evaluated at the centroid (which is equal to one quarter of the area A of the element

$$
J_{00} = A/4 = a_1 b_2 - a_2 b_1 \tag{14}
$$

and the coefficients a_1, b_1, a_2, b_2 are

$$\begin{aligned} a_1 &= (-r_1 + r_2 + r_3 - r_4)/4, \quad & b_1 &= (-z_1 + z_2 + z_3 - z_4)/4 \\ a_2 &= (-r_1 - r_2 + r_3 + r_4)/4, \quad & b_2 &= (-z_1 - z_2 + z_3 + z_4)/4 \end{aligned} \tag{15}$$

The B-bar gradient matrices $\bar{\mathbf{B}}_{10}, \bar{\mathbf{B}}_{01}$ are

$$\bar{\mathbf{B}}_{10} = \frac{1}{4} \begin{bmatrix} \gamma_1 b_1 & \gamma_2 b_1 & \gamma_3 b_1 & \gamma_4 b_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\gamma_1 a_1 & -\gamma_2 a_1 & -\gamma_3 a_1 & -\gamma_4 a_1 \\ -\gamma_1 a_1 & -\gamma_2 a_1 & -\gamma_3 a_1 & -\gamma_4 a_1 & \gamma_1 b_1 & \gamma_2 b_1 & \gamma_3 b_1 & \gamma_4 b_1 \\ -\dfrac{J_{10}}{r_m} & -\dfrac{J_{10}}{r_m} & -\dfrac{J_{10}}{r_m} & -\dfrac{J_{10}}{r_m} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{16}$$

and

$$\bar{\mathbf{B}}_{01} = \frac{1}{4} \begin{bmatrix} \gamma_1 b_2 & \gamma_2 b_2 & \gamma_3 b_2 & \gamma_4 b_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\gamma_1 a_2 & -\gamma_2 a_2 & -\gamma_3 a_2 & -\gamma_4 a_2 \\ -\gamma_1 a_2 & -\gamma_2 a_2 & -\gamma_3 a_2 & -\gamma_4 a_2 & \gamma_1 b_2 & \gamma_2 b_2 & \gamma_3 b_2 & \gamma_4 b_2 \\ -\dfrac{J_{01}}{r_m} & -\dfrac{J_{01}}{r_m} & -\dfrac{J_{01}}{r_m} & -\dfrac{J_{01}}{r_m} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{17}$$

where

$$\begin{aligned} \gamma_1 &= 1 - \gamma_{01} - \gamma_{10} \\ \gamma_2 &= -1 + \gamma_{01} - \gamma_{10} \\ \gamma_3 &= 1 + \gamma_{01} + \gamma_{10} \\ \gamma_4 &= -1 - \gamma_{01} + \gamma_{10} \end{aligned} \tag{18}$$

and

$$\gamma_{10} = \frac{J_{10}}{J_{00}}, \qquad \gamma_{01} = \frac{J_{01}}{J_{00}} \tag{19}$$

with

$$\begin{aligned} J_{10} &= a_1 b_3 - a_3 b_1 \\ J_{01} &= a_3 b_2 - a_2 b_3 \end{aligned} \tag{20}$$

where coefficients a_1, b_1, a_2, b_2 are given in (15) and coefficients a_3, b_3 are

$$a_3 = (r_1 - r_2 + r_3 - r_4)/4, \quad b_3 = (z_1 - z_2 + z_3 - z_4)/4 \tag{21}$$

3.1 Approximated stiffness matrix

Then the element stiffness matrix $\mathbf{K}_e$ can be approximated as

$$\mathbf{K}_e = \int_{A_e} \mathbf{B}^T \mathbf{C} \mathbf{B} \, r dA \approx \int_{A_e} \bar{\mathbf{B}}^T \mathbf{C} \bar{\mathbf{B}} \, r dA \approx \int_{-1}^{1} \int_{-1}^{1} \bar{\mathbf{B}}^T \mathbf{C} \bar{\mathbf{B}} \, r_m / J_{00} \, d\xi d\eta \tag{22}$$

After substituting the approximate gradient matrix (10) we have

$$
\begin{aligned}
\mathbf{K}_e \approx & \int_{-1}^{1}\int_{-1}^{1} \bar{\mathbf{B}}_{00}^T \mathbf{C} \bar{\mathbf{B}}_{00}\, r_m/J_{00}\, d\xi d\eta \\
& + \int_{-1}^{1}\int_{-1}^{1} \bar{\mathbf{B}}_{00}^T \mathbf{C} \left(\bar{\mathbf{B}}_{10}\xi + \bar{\mathbf{B}}_{01}\eta\right)\, r_m/J_{00}\, d\xi d\eta \\
& + \int_{-1}^{1}\int_{-1}^{1} \left(\bar{\mathbf{B}}_{10}^T\xi + \bar{\mathbf{B}}_{01}^T\eta\right) \mathbf{C}\, \bar{\mathbf{B}}_{00}\, r_m/J_{00}\, d\xi d\eta \\
& + \int_{-1}^{1}\int_{-1}^{1} \left(\bar{\mathbf{B}}_{10}^T\xi + \bar{\mathbf{B}}_{01}^T\eta\right) \mathbf{C} \left(\bar{\mathbf{B}}_{10}\xi + \bar{\mathbf{B}}_{01}\eta\right)\, r_m/J_{00}\, d\xi d\eta
\end{aligned}
\tag{23}
$$

Taking into account the properties of the integrals:

$$
\begin{array}{lll}
\int_{-1}^{1}\int_{-1}^{1} d\xi d\eta = 4 & , \int_{-1}^{1}\int_{-1}^{1} \xi d\xi d\eta = 0 & \int_{-1}^{1}\int_{-1}^{1} \eta d\xi d\eta = 0 \\
\int_{-1}^{1}\int_{-1}^{1} \xi\eta d\xi d\eta = 0 & \int_{-1}^{1}\int_{-1}^{1} \xi^2 d\xi d\eta = \frac{4}{3} & \int_{-1}^{1}\int_{-1}^{1} \eta^2 d\xi d\eta = \frac{4}{3}
\end{array}
\tag{24}
$$

then we have

$$
\boxed{\mathbf{K}_e \approx \mathbf{K}^C + \mathbf{K}^S}
\tag{25}
$$

where $\mathbf{K}^C$ is the *consistent* stiffness matrix

$$
\mathbf{K}^C = \int_{-1}^{1}\int_{-1}^{1} \bar{\mathbf{B}}_{00}^T \mathbf{C} \bar{\mathbf{B}}_{00}\, r_m/J_{00}\, d\xi d\eta = \frac{4 r_m}{J_{00}} \left[\bar{\mathbf{B}}_{00}^T \mathbf{C} \bar{\mathbf{B}}_{00}\right]
\tag{26}
$$

and $\mathbf{K}^S$ is the *stabilization* matrix

$$
\begin{aligned}
\mathbf{K}^S = & \int_{-1}^{1}\int_{-1}^{1} \bar{\mathbf{B}}_{10}^T \mathbf{C} \bar{\mathbf{B}}_{10}\xi^2\, r_m J_{00} d\xi d\eta + \int_{-1}^{1}\int_{-1}^{1} \bar{\mathbf{B}}_{01}^T \mathbf{C} \bar{\mathbf{B}}_{01}\eta^2\, r_m/J_{00}\, d\xi d\eta \\
= & \frac{4 r_m}{3 J_{00}} \left[\bar{\mathbf{B}}_{10}^T \mathbf{C} \bar{\mathbf{B}}_{10} + \bar{\mathbf{B}}_{01}^T \mathbf{C} \bar{\mathbf{B}}_{01}\right]
\end{aligned}
\tag{27}
$$

Matrix $\mathbf{K}^C$ is the one-point quadrature matrix. This matrix provides the exact internal forces for any state of constant strain (at least in the limit for infinitesimal size). It is well known(Liu et al., 1985; Liu and Belytschko, 1984) that this matrix is rank-deficient, that is, it has two improper modes in addition to the modes associated with rigid body displacements.

Matrix $\mathbf{K}^S$ is a stabilization matrix that eliminates these spurious modes of the complete stiffness matrix.

4 NODAL AVERAGED CONSTITUTIVE MATRICES

Suppose that constitutive properties are nonlinear and are known at each node i of the element by a constitutive matrix C_i. We are not restricted to any material law or phenomena, only that at nodal point i the relation between nodal stresses $\boldsymbol{\sigma}_i$ must be related to nodal strains $\boldsymbol{\epsilon}_i$ as

$$
\boldsymbol{\sigma}_i = \mathbf{C}_i \boldsymbol{\epsilon}_i
\tag{28}
$$

Note that all these quantities need to be univaluated at the node, so in general some kind of postprocessing must be necessary to unify values at the nodes, see for example (Jia et al., 2020).

Consider that the region of influence of each node on the element is the quarter area containing that node, as shown in figure 2.

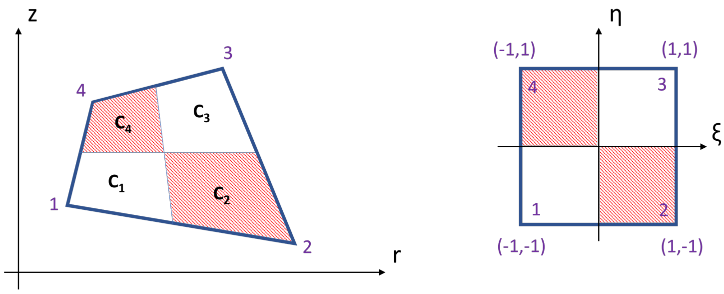


Figure 2: Influence regions of each node

If we assume constant constitutive properties C_i on the quarter area containing node i then the approximated stiffness matrix (22) can be computed as

$$\begin{aligned}\mathbf{K}_e \approx &\int_{-1}^{0}\int_{-1}^{0} \bar{\mathbf{B}}^T \mathbf{C}_1 \bar{\mathbf{B}}\, r_m/J_{00}\, d\xi d\eta + \int_{0}^{1}\int_{-1}^{0} \bar{\mathbf{B}}^T \mathbf{C}_2 \bar{\mathbf{B}}\, r_m/J_{00}\, d\xi d\eta + \\ &\int_{0}^{1}\int_{0}^{1} \bar{\mathbf{B}}^T \mathbf{C}_3 \bar{\mathbf{B}}\, r_m/J_{00}\, d\xi d\eta + \int_{-1}^{0}\int_{0}^{1} \bar{\mathbf{B}}^T \mathbf{C}_4 \bar{\mathbf{B}}\, r_m/J_{00}\, d\xi d\eta\end{aligned} \tag{29}$$

And taking into account the values of the integrals on each quarter:

$$\begin{aligned}
&\int_{-1}^{0}\int_{-1}^{0} d\xi d\eta = \int_{0}^{1}\int_{-1}^{0} d\xi d\eta = \int_{0}^{1}\int_{0}^{1} d\xi d\eta = \int_{-1}^{0}\int_{0}^{1} d\xi d\eta = 1 \\
&-\int_{-1}^{0}\int_{-1}^{0} \xi d\xi d\eta = \int_{0}^{1}\int_{-1}^{0} \xi d\xi d\eta = \int_{0}^{1}\int_{0}^{1} \xi d\xi d\eta = -\int_{-1}^{0}\int_{0}^{1} \xi d\xi d\eta = \frac{1}{2} \\
&-\int_{-1}^{0}\int_{-1}^{0} \eta d\xi d\eta = -\int_{0}^{1}\int_{-1}^{0} \eta d\xi d\eta = \int_{0}^{1}\int_{0}^{1} \eta d\xi d\eta = \int_{-1}^{0}\int_{0}^{1} \eta d\xi d\eta = \frac{1}{2} \\
&\int_{-1}^{0}\int_{-1}^{0} \xi\eta d\xi d\eta = -\int_{0}^{1}\int_{-1}^{0} \xi\eta d\xi d\eta = \int_{0}^{1}\int_{0}^{1} \xi\eta d\xi d\eta = -\int_{-1}^{0}\int_{0}^{1} \xi\eta d\xi d\eta = \frac{1}{4} \\
&\int_{-1}^{0}\int_{-1}^{0} \xi^2 d\xi d\eta = \int_{0}^{1}\int_{-1}^{0} \xi^2 d\xi d\eta = \int_{0}^{1}\int_{0}^{1} \xi^2 d\xi d\eta = \int_{-1}^{0}\int_{0}^{1} \xi^2 d\xi d\eta = \frac{1}{3} \\
&\int_{-1}^{0}\int_{-1}^{0} \eta^2 d\xi d\eta = \int_{0}^{1}\int_{-1}^{0} \eta^2 d\xi d\eta = \int_{0}^{1}\int_{0}^{1} \eta^2 d\xi d\eta = \int_{-1}^{0}\int_{0}^{1} \eta^2 d\xi d\eta = \frac{1}{3}
\end{aligned} \tag{30}$$

Then after substituting these results in (29) we have that the approximated stiffness is

$$
\begin{aligned}
\mathbf{K}_e \approx\ & r_m/J_{00}\left[\bar{\mathbf{B}}_{00}^T\left(\mathbf{C}_1+\mathbf{C}_2+\mathbf{C}_3+\mathbf{C}_4\right)\bar{\mathbf{B}}_{00}\right] \\
& + r_m/2J_{00}\left[\bar{\mathbf{B}}_{00}^T\left(-\mathbf{C}_1+\mathbf{C}_2+\mathbf{C}_3-\mathbf{C}_4\right)\bar{\mathbf{B}}_{10}+\bar{\mathbf{B}}_{00}^T\left(-\mathbf{C}_1-\mathbf{C}_2+\mathbf{C}_3+\mathbf{C}_4\right)\bar{\mathbf{B}}_{01}\right] \\
& + r_m/2J_{00}\left[\bar{\mathbf{B}}_{10}^T\left(-\mathbf{C}_1+\mathbf{C}_2+\mathbf{C}_3-\mathbf{C}_4\right)\bar{\mathbf{B}}_{00}+\bar{\mathbf{B}}_{01}^T\left(-\mathbf{C}_1-\mathbf{C}_2+\mathbf{C}_3+\mathbf{C}_4\right)\bar{\mathbf{B}}_{00}\right] \\
& + r_m/3J_{00}\left[\bar{\mathbf{B}}_{10}^T\left(\mathbf{C}_1+\mathbf{C}_2+\mathbf{C}_3+\mathbf{C}_4\right)\bar{\mathbf{B}}_{10}+\bar{\mathbf{B}}_{01}^T\left(\mathbf{C}_1+\mathbf{C}_2+\mathbf{C}_3+\mathbf{C}_4\right)\bar{\mathbf{B}}_{01}\right] \\
& + r_m/4J_{00}\left[\bar{\mathbf{B}}_{10}^T\left(\mathbf{C}_1-\mathbf{C}_2+\mathbf{C}_3-\mathbf{C}_4\right)\bar{\mathbf{B}}_{01}+\bar{\mathbf{B}}_{01}^T\left(\mathbf{C}_1-\mathbf{C}_2+\mathbf{C}_3-\mathbf{C}_4\right)\bar{\mathbf{B}}_{10}\right]
\end{aligned} \tag{31}
$$

Assuming that in the limit, as the size h of the element tends to zero, the four nodal constitutive matrices $\mathbf{C}_i$ tends to the mean value $\mathbf{C}_m$

$$
\lim_{h\to 0}\mathbf{C}_i=\mathbf{C}_m=\left(\mathbf{C}_1+\mathbf{C}_2+\mathbf{C}_3+\mathbf{C}_4\right)/4 \tag{32}
$$

then we can assume that

$$
\begin{aligned}
\lim_{h\to 0}\left(\mathbf{C}_1+\mathbf{C}_2+\mathbf{C}_3+\mathbf{C}_4\right)&=4\mathbf{C}_m \\
\lim_{h\to 0}\left(-\mathbf{C}_1+\mathbf{C}_2+\mathbf{C}_3-\mathbf{C}_4\right)&=0 \\
\lim_{h\to 0}\left(-\mathbf{C}_1-\mathbf{C}_2+\mathbf{C}_3+\mathbf{C}_4\right)&=0 \\
\lim_{h\to 0}\left(\mathbf{C}_1-\mathbf{C}_2+\mathbf{C}_3-\mathbf{C}_4\right)&=0
\end{aligned} \tag{33}
$$

and the consistent and the stabilization matrices can be approximated as

$$
\begin{aligned}
\mathbf{K}^C &= \frac{4r_m}{J_{00}}\left[\bar{\mathbf{B}}_{00}^T\mathbf{C}_m\bar{\mathbf{B}}_{00}\right] \\
\mathbf{K}^S &= \frac{4r_m}{3J_{00}}\left[\bar{\mathbf{B}}_{10}^T\mathbf{C}_m\bar{\mathbf{B}}_{10}+\bar{\mathbf{B}}_{01}^T\mathbf{C}_m\bar{\mathbf{B}}_{01}\right]
\end{aligned} \tag{34}
$$

Note that this procedure is equivalent to assume a constant constitutive matrix $\mathbf{C}_m$, obtained by nodal averaging, to compute the approximate stiffness matrix

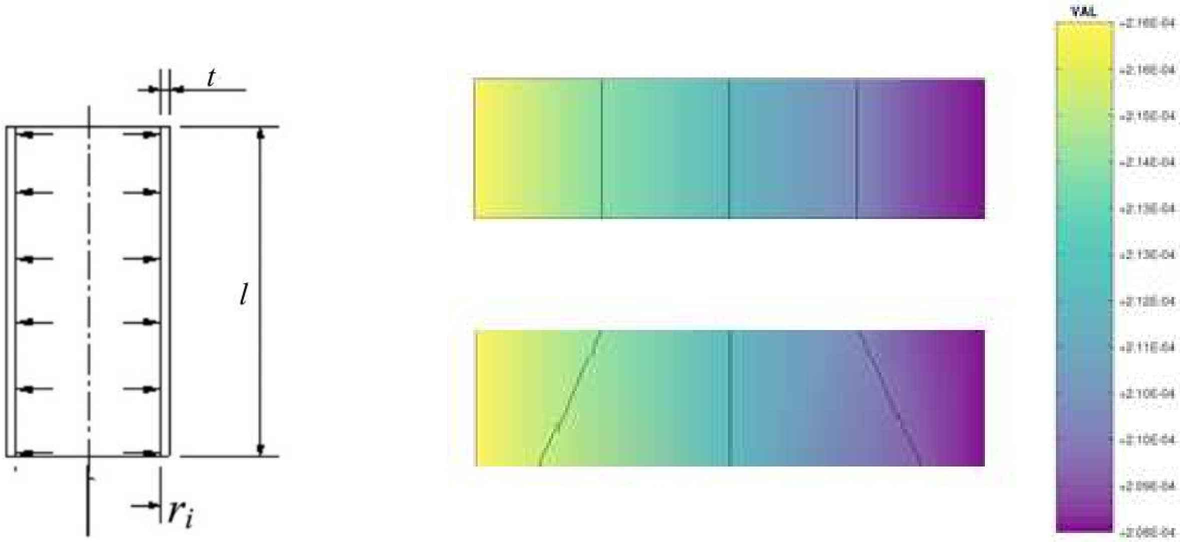
$$
\boxed{\mathbf{K}_e \approx \int_{-1}^{1}\int_{-1}^{1}\bar{\mathbf{B}}^T\mathbf{C}_m\bar{\mathbf{B}}\, r_m/J_{00}\, d\xi d\eta} \tag{35}
$$

Then the assembling procedure can be done in the same manner as for a constant constitutive matrix without making any distinction for material nonlinearities.

5 NUMERICAL EXAMPLES

To test the formulation we analyze a long pressurized linear elastic pipe as shown in figure 3a

The pipe geometry is defined by its inner radius ($r_i = 4.5mm$) and by the thickness/inner radius ratio ($t/r_i = 0.1$). The pipe is considered long enough to assume plane strain conditions. An isotropic, linear elastic material was adopted, characterized by a Young's modulus $E = 210GPa$ and a Poisson ratio $\nu = 0.27$. A unitary ($1MPa$) inner pressure was applied.



(a) Long pressurized pipe.

(b) Finite element model. Regular and distorted meshes.

Figure 3: Ejemplo y resultados

We have modeled this example with four squared finite elements in the thickness and for comparisons we have used another distorted mesh. In figure 3b both meshes are shown with the radial displacement results.

In table 1 the results for the radial displacement at the inner surface are shown. The column labeled CAX4 corresponds to the standard four node axisymmetric quadrilateral (Bathe, 2014) integrated with 2 Gauss points in each direction and the exact solution for this example can be found in (Di Puccio and Celi, 2012). Also, in this table we can see the results for the distorted mesh.

Table 1: Internal radial displacements.

	CAX4	error %	current	error %	exact
regular	0.000216399	0.009	0.000216399	0.009	0.000216418
distorted	0.000216393	0.009	0.000211631	0.012	0.000216418

Analyzing the results we can see that the current and the conventional formulations give similar results.

6 CONCLUSIONS

A nodal averaged formulation has been presented for axisymmetrical problems solved with quadrilateral finite elements. In this formulation the constitutive matrices only need to be computed at nodes and this implies a substantial reduction reduction in the number of integration points over the conventional formulation. Also, an element averaged procedure has been presented to compute the finite element stiffness matrices where the element constitutive matrix is calculated as an average from the nodal constitutive matrices. The procedure can be applied to any type of nonlinear material, including plasticity, without need of using Gauss numerical integration.

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