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STOCHASTIC MODELLING OF WAVE SCATTERING IN METASTRUCTURES FOR VIBRATION ATTENUATION

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Abstract. Metamaterials, or locally resonant metamaterials, are a class of structures that have been used to control and to manipulate acoustic and elastic waves with applications in vibration attenuation. A great amount of research has been done on acoustic and structural metamaterials but very little attention has been given to the effects of coupling conditions on structural assemblies, even though this is typical case on mechanical engineering applications. In this work, the wave attenuation in a metamaterial beam assembly is investigated considering uncertain connections. A beam, with attached resonators, undergoing longitudinal and flexural vibration is connected to homogeneous beams at each end. It is assumed a large enough number of identical resonators such that effective longitudinal and flexural wavenumbers are derived. Wave modes are assumed unchanged by the attachments and analytical expressions can be derived. A point connection is considered with an assembly angle such that wave mode conversion, between flexural and longitudinal waves, can happen. The reflection and transmission properties of the full assembly are then calculated and it is shown that the connection angle has significant effects on the band gap performance, which cannot be captured by a purely deterministic model of the straight assembly. Furthermore, the effects of some stochastic models, derived based on the Maximum Entropy principle, on the overall metastructure vibration attenuation performance are investigated. It is shown that the connection angle can considerably widen the metastructure band gap and that the joint uncertainties can play a major role on the vibration attenuation.

1 INTRODUCTION

Metamaterials, or locally resonant metamaterials, are a class of structures that have been used to control and to manipulate acoustic and elastic waves (Hussein et al., 2014) with several applications, including vibration attenuation, e.g. (Sugino et al., 2017; Huang and Sun, 2009). Although periodicity of the resonators positioning is not required, it is used for a cell-based description of the wave propagation. In metamaterials, the attenuation effect is created due to inclusions or attachments that work as internal resonators (Liu et al., 2000) and are able to create band gaps at sub-wavelength frequencies, unlike the phononic crystals, which rely on spatial periodicity and the Bragg scattering effect (Hussein et al., 2014).

The use of a resonator for vibration control is very common in engineering applications (Den Hartog, 1985) but its efficacy is restricted to a very narrow frequency band. Some development has been proposed to widen the frequency band of actuation using adaptive or non-linear mechanisms (Brennan, 2006). The advantage of the concept introduced by locally resonant materials is that it is possible to widen the frequency band of attenuation by simply adding several resonators while keeping the same mass ratio (Sugino et al., 2017). This concept is particularly useful for lightweight vibro-acoustic metamaterial design are being recently explored in NVH applications (Claeys et al., 2017; de Melo Filho et al., 2019).

The dynamics of joints has been investigated for many decades and several approaches have been proposed in terms of energy flow (Beshara and Keane, 1997), vibrational modes (Arruda and Santos, 1993) and wave reflection and transmission coefficients (Zhang et al., 2010). The modelling of joints in mechanical assemblies can be very challenging and usually requires the inclusion of some level of uncertainty (Ibrahim and Pettit, 2005). Typically, the mechanical properties of the joints are considered uncertain (Dohnal et al., 2009) and are handled by a parametric approach in which case a stochastic model of the parameters is used or a non-parametric approach in which case the mechanical model itself is considered to be random (Fabro and Mencik, 2018).

In this work, the wave attenuation in a metamaterial beam assembly is investigated considering uncertain connections. A beam, with attached resonators, undergoing longitudinal and flexural vibration is connected to homogeneous beams at each end. A point connection is considered with an assembly angle such that wave mode conversion, between flexural and longitudinal waves, can happen. The reflection and transmission properties of the full assembly are then calculated and it is shown that the connection angle has significant effects on the band gap performance, which cannot be captured by a purely deterministic model of the straight assembly. Furthermore, the effects of some stochastic models, derived based on the Maximum Entropy principle, on the overall metastructure vibration attenuation performance are investigated. It is shown that the connection angle can considerably widen the metastructure band gap and that the joint uncertainties can play a major role on the vibration attenuation.

2 WAVE MODEL

The governing equation of motion of a general one-dimensional undamped system of distributed parameter can be given by $L(x)w(x,t) + \mu\ddot{w}(x,t) = p(x,t)$, where $L(x)$ is a linear homogeneous self-adjoint stiffness differential operator of order $2q$, where $q \geq 1$ is an integer defining the order of the system, μ is the mass density per unity length, $p(x,t)$ is the force per unity length and $w(x,t)$ is the displacement. For rods undergoing longitudinal vibration $L(x) = -d/dx\,[EA(x)d/dx]$, where $EA(x)$ is the longitudinal stiffness, and for beams undergoing flexural vibration, $L(x) = d^2/dx^2\,[EI(x)d^2/dx^2]$, where $EI(x)$ is the flexural stiffness. Assuming harmonic motion such that $w(x,t) = e^{i(\omega t - kx)}$, homogeneous properties and free vibrations, where ω is the angular frequency and k is the wavenumber, then it is possible to define the dispersion relation $L(-ik) - \mu\omega^2 = 0$. Using the stiffness operators for rods and beams leads to $k_l = (\rho/E)^{1/2}\sqrt{\omega}$ and $k_b = (\rho A/EI)^{1/4}\omega^{1/2}$, which are the longitudinal and flexural wavenumbers, respectively. The displacement field in the rod $u(x,t)$ and in the beam $w(x,t)$ is then given by $u(x,t) = \left(a_l^+ e^{-ik_l x} + a_l^- e^{ik_l x}\right)e^{i\omega t}$, and $w(x,t) = \left(a_b^+ e^{-ik_b x} + a_{bN}^+ e^{-k_b x} + a^- e^{ik_b x} + a_{bN}^- e^{k_b x}\right)e^{i\omega t}$, where $a_l^\pm$, $a_b^\pm$ and $a_{bN}^\pm$ are the wave amplitudes of the positive and negative going propagating and non-propagating longitudinal and flexural waves, respectively. A linear transformation from the wave domain to the physical domain can be given for a generalized displacement and generalized force, respectively, by

$$\mathbf{q} = \mathbf{\Phi}_q^+\mathbf{a}^+ + \mathbf{\Phi}_q^-\mathbf{a}^-, \quad \mathbf{f} = \mathbf{\Phi}_f^+\mathbf{a}^+ + \mathbf{\Phi}_f^-\mathbf{a}^-, \tag{1}$$

where $\mathbf{\Phi}_q^\pm$ and $\mathbf{\Phi}_f^\pm$ are, respectively, displacement and internal forces matrices. For a waveguide undergoing both longitudinal and flexural waves, then $\mathbf{q} = [u \quad w \quad \theta]^T$ and $\mathbf{f} = [P \quad V \quad M]^T$, where $\theta = dw/dx$, P is the rod axial force, V is the beam shear force and M is the bending moment, and the wave amplitude vectors are given by $\mathbf{a}^\pm = \left[a_l^\pm \quad a_b^\pm \quad a_{bN}^\pm\right]^T$.

The equation of motion of a continuous system with S periodically attached resonators can be given in the general form by (Sugino et al., 2017)

$$L(x)w(x,t) + \mu\ddot{w}(x,t) - \sum_{p=1}^{S} k_p u_p(t)\delta(x - x_p) = p(x,t), \tag{2}$$

and one additional equation for each resonator $m_p\ddot{u}_p(t) + k_p u_p(t) + m_p\ddot{w}(x_p,t) = 0$, where $u_p(t)$ is the displacement of each resonator attached at x_p, with mass m_p and stiffness k_p and $\delta(x)$ is the Dirac delta function. This expression was originally proposed for a modal analysis in metastructures and allows the derivation of closed form expression for the band gap frequency edges. In this work, it will be used for finding the dispersion equation. Also, assuming that the wave modes are unchanged due to the resonators attachments, it provides a analytical framework for calculating reflection and transmission coefficients. Note that it is similar to Eq. **??** and thus a similar procedure can be applied for finding the dispersion equation. Assuming identical resonators and a large enough number of attachments, it can be shown that

$$L(-ik) - \mu\omega^2\left(1 + \epsilon\frac{1}{1 - \Omega_r^2}\right) = 0, \tag{3}$$

where $\Omega_r = \omega/\omega_r$ and $\omega_r^2 = k_p/m_p$ and $\epsilon = m_p/\mu\Delta l$ is the mass ratio for resonators spaced by Δl. Derivation details are shown in the Appendix. The suitable stiffness operators can be applied to find the effective wavenumbers for longitudinal

$$k_{rl} = \sqrt{\frac{\rho}{E}\left(1 + \epsilon\frac{1}{1 - \Omega_r^2}\right)}\omega, \tag{4}$$

and flexural waves

$$k_{rb} = \sqrt[4]{\frac{\rho A}{EI}\left(1 + \epsilon\frac{1}{1 - \Omega_r^2}\right)}\sqrt{\omega}. \tag{5}$$

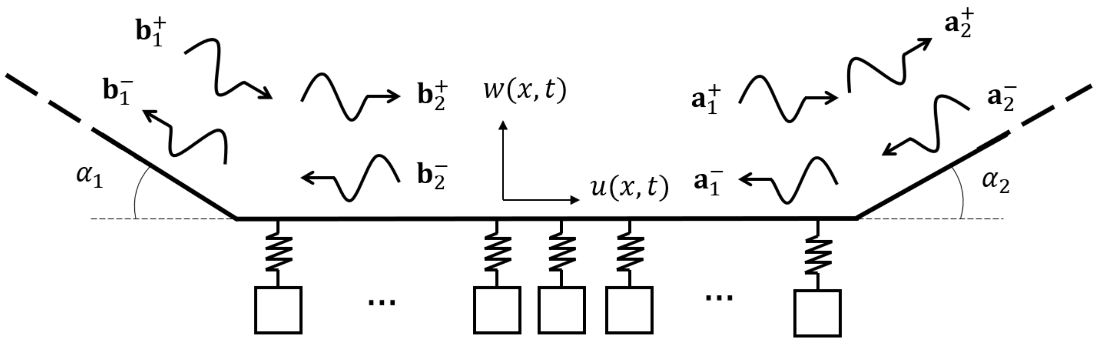

Figure 1: Metamaterial beam assembly with one semi-infinite homogeneous beam at each end undergoing longitudinal and flexural vibration.

This result is equivalent to (Gao et al., 2011) for a continuous neutralizer attached to the beam, in which the mass ratio is given in terms of wave length. Assuming that the attached resonators do not change the wave types, these wavenumbers can then be used to describe the displacement field and the matrices $\mathbf{\Phi}_q^{\pm}$, $\mathbf{\Phi}_f^{\pm}$ are the same as for the simple beam.

2.1 Metamaterial assembly

A metamaterial beam undergoing longitudinal and flexural waves is connected to two other homogeneous beam, one at each end, as shown in Fig. 1. At the left end, the connection angle is α_1, $\mathbf{b}_1^{\pm}$ are the amplitude of the incoming and outgoing waves. At the right end, the connection angle is α_2, $\mathbf{a}_1^{\pm}$ are the amplitude of the outgoing and incoming waves. A scattering matrix can be defined relating the incoming and outgoing waves of the assembly by (Harland et al., 2001; Fabro et al., 2015)

$$\begin{bmatrix} \mathbf{a}_2^+ \\ \mathbf{b}_1^- \end{bmatrix} = \begin{bmatrix} \mathbf{r}^+ & \mathbf{t}^+ \\ \mathbf{t}^- & \mathbf{r}^- \end{bmatrix} \begin{bmatrix} \mathbf{a}_2^- \\ \mathbf{b}_1^+ \end{bmatrix}, \tag{6}$$

where $\mathbf{r}^{\pm}$ are reflection matrices and $\mathbf{t}^{\pm}$ are transmission matrices. They can be obtained from the equilibrium and continuity conditions at the beams connections and the wave propagation along the metamaterial beam. The full derivation is presented in the Appendix. Assuming $\mathbf{a}_2^- = \mathbf{0}$, i.e., a incident wave on the left end only, then the scattering simplifies to $\mathbf{a}_2^+ = \mathbf{t}^+\mathbf{b}_1^+$ and $\mathbf{b}_1^- = \mathbf{r}^-\mathbf{b}_1^+$. Therefore, the transmission coefficient $\mathbf{t}^+$ can be used as to access the vibration attenuation of the metamaterial beam in the assembly.

In this case, the reflection and transmission matrices are size 3×3 and relate the longitudinal, propagating flexural and non-propagating flexural (near field) wave amplitudes at the both sides of the assembly. For $\alpha_1 = \alpha_2 = 0$, i.e., a straight assembly, no wave mode conversion is expected and these matrices are diagonal. However, for $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$, they are full matrices and wave mode conversion plays a role on the metamaterial vibration attenuation performance. Moreover, asymmetries in the assembly can be given by differences in the connection angle, i.e. $\alpha_1 \neq \alpha_2$, and also play a role on the reflection coefficients $\mathbf{r}^{\pm}$, while $\mathbf{t}^+ = \mathbf{t}^-$ due to reciprocity.

3 PROBABILISTIC MODELLING

Two cases are considered in the analysis. In the first, it defined that the first connection angle α_1 is fixed while $\alpha_2 = \alpha_1 + \theta$, where θ is a sample of the random variable Θ. In the second case, it is considered that both connection angles α_1 and α_2 can be modelled by the random variables $\mathtt{A}_1$ and $\mathtt{A}_2$, respectively. For each analysis case, some probabilistic models are defined in the following section such that the probability distribution of the random variables take in to account typical physical constrains of the problem.

3.1 Maximum Entropy based probabilistic models

Typically, manufacturing processes can only guarantee minimum θ_1 and maximum θ_2 values from the tolerances in the assembly process. It is also reasonable to assume that the angles in both connections are not correlated. Given the lack of any experimental information, the Maximum Entropy principle (J.N. Kapur and H. K. Kesavan, 1992; Cursi and Sampaio, 2015) for modelling a random variable namely is applicable. In this sense, two possible probability density functions (PDF) are derived from this principle for increasing level of available information, considering only the upper and lower bounds, the mean value and the standard-deviation.

The first model is given by the Uniform distribution, for the case when only the lower bound θ_1 and the lower bound θ_2 of the random variable are know. The PDF of the random variables Θ, $\mathtt{A}_1$ and $\mathtt{A}_2$ is defined by

$$f^{(1)}_{\Theta,\mathtt{A}_1,\mathtt{A}_2}(x) = \frac{1}{\theta_2 - \theta_1}\,, \quad \theta_1 \leq x \leq \theta_2, \tag{7}$$

The second model assumes, the Beta distribution of the first kind, is found from the Maximum Entropy principle when the statistical moments $E[\ln x]$ and $E[\ln(1-x)]$ are known, where $E[\cdot]$ is mathematical expectation. They allow the description of the problem in terms of a mean value $\bar{\mu}$ and a standard-deviation $\bar{\sigma}$, which are more practical for engineering applications. The Beta distribution of the first kind is given by

$$f^{(2)}_{\Theta,\mathtt{A}_1,\mathtt{A}_2}(x) = \frac{(x-a)^{\alpha-1}(b-x)^{\beta-1}}{B(\alpha,\beta)(b-a)^{\alpha+\beta-1}}\,, \quad a \leq x \leq b, \tag{8}$$

where $B(\alpha,\beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$ is the Beta function, $\Gamma(z) = \int_0^\infty x^{z-1}\exp(-z)dx$ is the Gamma function and α and β are real positive shape parameters. Typical functions for Beta distributed sampling usually generate random samples for $a = 0$ and $b = 1$, given the shape parameters α and β. In this case, a rescaling is necessary such that $\alpha = ((1-\mu)/\sigma^2 - 1/\mu)\mu^2$ and $\beta = \alpha(1/\mu - 1)$, where $\mu = (\bar{\mu} - a)/(b-a)$ and $\sigma = \bar{\sigma}/(b-a)$. The generated samples are then added of a after being multiplied by $(b-a)$.

4 NUMERICAL RESULTS

In this section, numerical results are presented considering the metamaterial beam assembly. Both bare beams and metamaterial beam are assumed to be composed of polyamide. It is assumed that the presented metastructure has similar design and proprieties of the metamaterial beams proposed by Beli et al. (2019).

Figure 2 presents the real and imaginary part of the longitudinal and flexural wavenumber for the metamaterial beam and the absolute value of the transmission coefficient, considering $\alpha_1 = \alpha_2 = 0$, i.e., a straight assembly. For a lossless waveguide, the wavenumber can be real, leading to a propagating wave, imaginary, giving a decaying or evanescent wave, or complex, which has both behaviours, i.e. propagating and decaying. The imaginary part of the dispersion curve (negative values) shows the frequency band in which there is vibration attenuation for each wave mode, i.e. the band gap for longitudinal and flexural waves. Note that the wave types do not interact because the axial and flexural vibration are considered uncoupled at the metamaterial beam. This is also noticed in the absolute value of the transmission coefficient, which shows a very low transmission at the band gap frequencies for each individual wave mode. Additionally, from the dispersion curve it can be seen that the group velocity $c_g = \partial\omega/\partial k$ is zero at the resonator frequency and it is negative at the band gap, meaning that the velocity of energy transport is in the negative direction and therefore can be interpreted as a negative-going wave (Mace, 2014). A similar behaviour is found from the analysis of the propagation constant of the equivalent periodic structure (Beli et al., 2019) and it is usually observed in homogeneous structures (Graff, 1991).

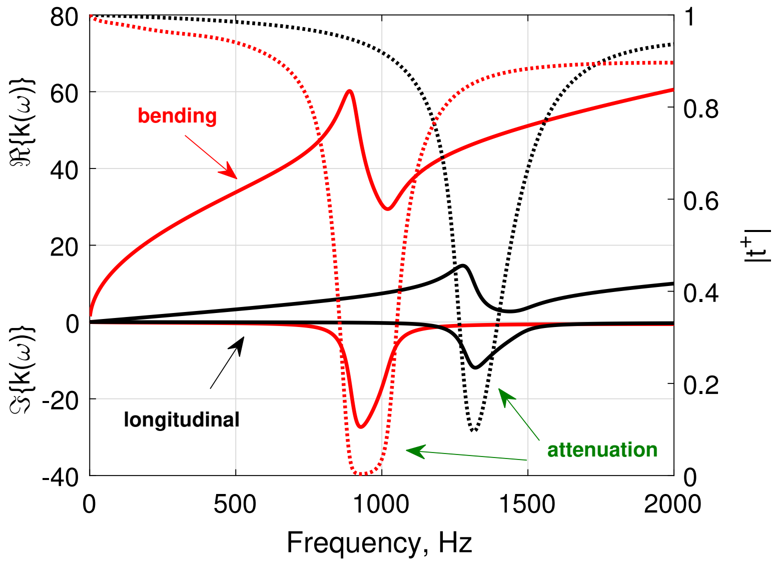

Figure 2: (left axis) Real and imaginary parts of the longitudinal (red) and flexural (black) wavenumbers for the bare beam (solid line) and metamaterial beam (dashed line) and (right axis) absolute value of the transmission coefficient considering $\alpha_1 = \alpha_2 = 0$.

4.1 Stochastic analysis

The effects of uncertainties on α_1 and α_2 at the wave mode conversion and the band gap performance are also investigated. For both considered cases, i.e. models with random variables Θ and $\mathtt{A}_1$ and $\mathtt{A}_2$, each probabilistic model considered that $\alpha_1 = \alpha_2 = 0$, i.e. a straight assembly, with $\theta_1 = -\pi/10$, $\theta_2 = \pi/10$ for the uniform PDF $f^{(1)}_{\Theta,\mathtt{A}_1,\mathtt{A}_2}(x)$ and $\theta_1 = -\pi/10$, $\theta_2 = \pi/10$, $\bar{\mu} = 0$, $\bar{\sigma} = 0.05\pi$ for the Beta of the 1^{st} kind PDF $f^{(2)}_{\Theta,\mathtt{A}_1,\mathtt{A}_2}(x)$. Moreover, for the stochastic analysis, 5,000 MC samples are used which is enough for the mean-square convergence.

For the first considered case, i.e. random variable Θ, Figures 3 to 4 present the estimated PDFs of the absolute value of the longitudinal and flexural transmission coefficient as a function of the frequency assuming $f^{(1)}_{\Theta}(x)$ and $f^{(2)}_{\Theta}(x)$, respectively. These results were obtained using the Matlab function *ksdensity*. The light-grey colours represent the most probable values for the coefficients while the dark-grey colours indicates the least probable values. It can be noticed that the nominal response is not representative of the most probable values of both coefficients in all of the frequency band but at the band gap. Therefore, the deterministic analysis is not representative of the typical behaviour of the transmission coefficient outside of this regions. In fact, the results show that the nominal response gives the upper bounds of the MC samples outside the band gap regions in both cases, while it is representative of the mean response in the band gap regions. The nominal model cannot capture the wave mode conversion occurring due to the random variation of the connection angles and it cannot predict the improved attenuation features observed in these cases.

For the second case, i.e. assuming random variables $\mathtt{A}_1$ and $\mathtt{A}_2$, Figures 5 and 6 show the estimated PDFs of the absolute value of the transmission coefficients.

Moreover, the choice of sets of random variable played a much more important role in the results than the probabilistic models for the random variables. Note that changing from uniform to Beta of the first kind affects the mean values and the tails of the distribution of the results. The model considering both connection angles and uncertainty introduced qualitative changes on the response, with frequency bands with increased attenuation performance. This is because the wave mode conversion between longitudinal and flexural waves could occur at both connections.

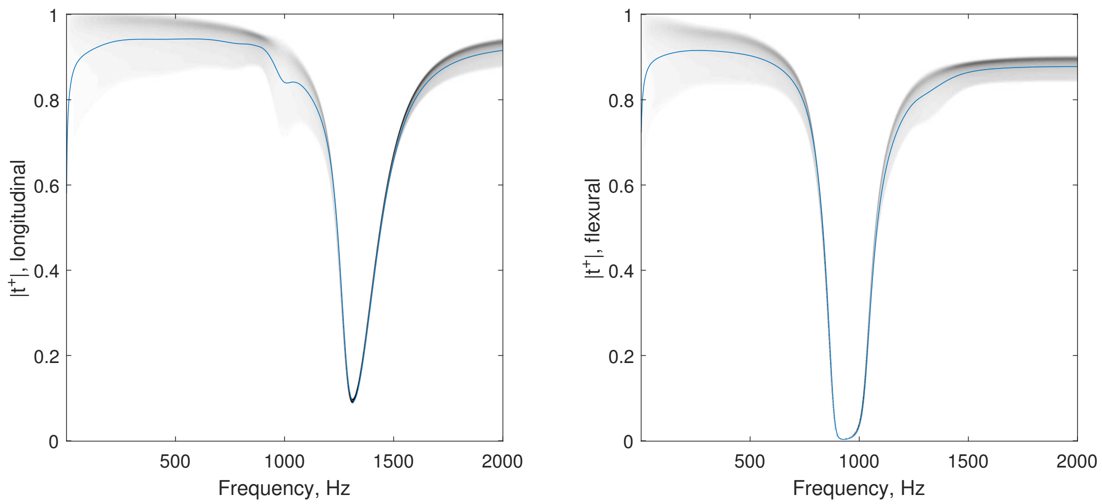

Figure 3: PDF and mean value (blue line) of the absolute value of the longitudinal (left) and flexural (right) transmission coefficients considering $\alpha_1 = \alpha_2 = 0$ and Θ. Uniform PDF $f_{\Theta}^{(1)}(x)$.

5 CONCLUDING REMARKS

The wave attenuation performance of a metamaterial beam assembly is investigated considering uncertain connections. It is assumed a large enough number of identical resonators such that and effective longitudinal and flexural wavenumbers are derived. Wave modes are assumed unchanged by the attachments and then analytical expressions can be derived. The reflection and transmission properties of the assembly are be calculated and it is shown that the angle of the assembly has a significant effect on the band gap performance.

The uncertainty analysis focus on the variability of the connection angles and ensemble statistics are investigated. Monte Carlo sampling is used as the stochastic solver. It is shown that the deterministic analysis is not representative of the typical behaviour of the transmission coefficient outside the band gap region. In this case, the nominal response gives the upper bounds of the MC samples outside the band gap regions in both cases, while it is representative of the mean response in the band gap regions.

Most importantly, it is shown that the nominal model, which does not include variability in the connections, cannot capture the wave mode conversion occurring due to the randomness of the connection angles and it cannot predict the improved attenuation features observed in these

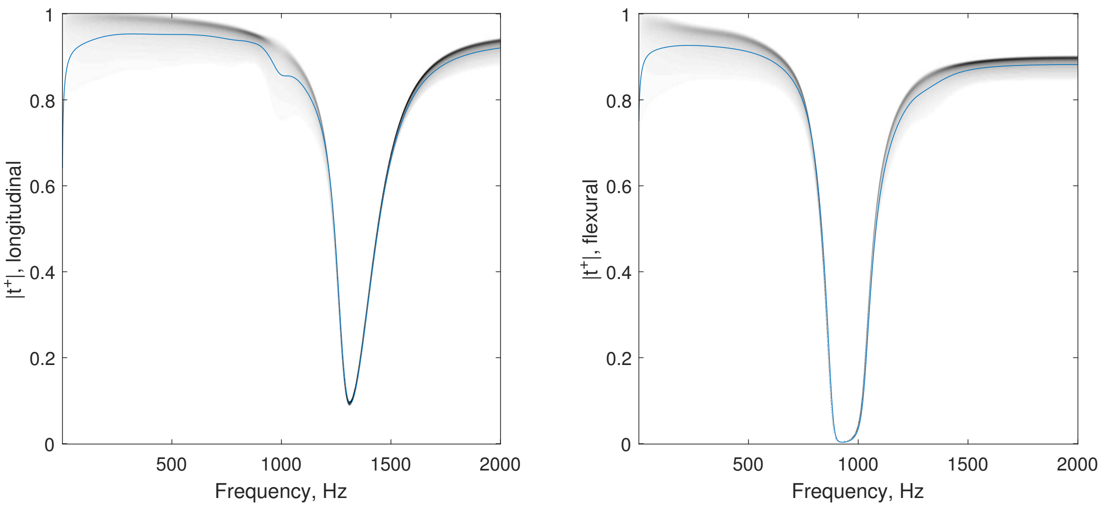

Figure 4: PDF and mean value (blue line) of the absolute value of the longitudinal (left) and flexural (right) transmission coefficients considering $\alpha_1 = \alpha_2 = 0$ and Θ. Beta of the first kind PDF $f_{\Theta}^{(2)}(x)$.

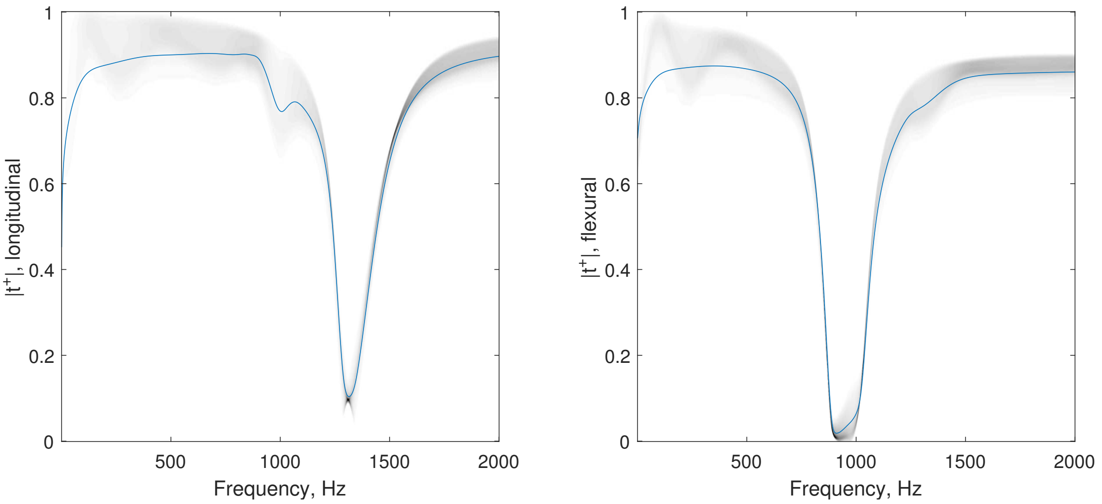

Figure 5: PDF and mean value (blue line) of the absolute value of the longitudinal (left) and flexural (right) transmission coefficients considering the random variable of both connection angles, $\mathtt{A}_1$ and $\mathtt{A}_2$. Uniform PDF $f^{(1)}_{\mathtt{A}_1,\mathtt{A}_2}(x)$.

cases. Moreover, the choice of sets of random variable played a much more important role in the results than the probabilistic models for the random variables.

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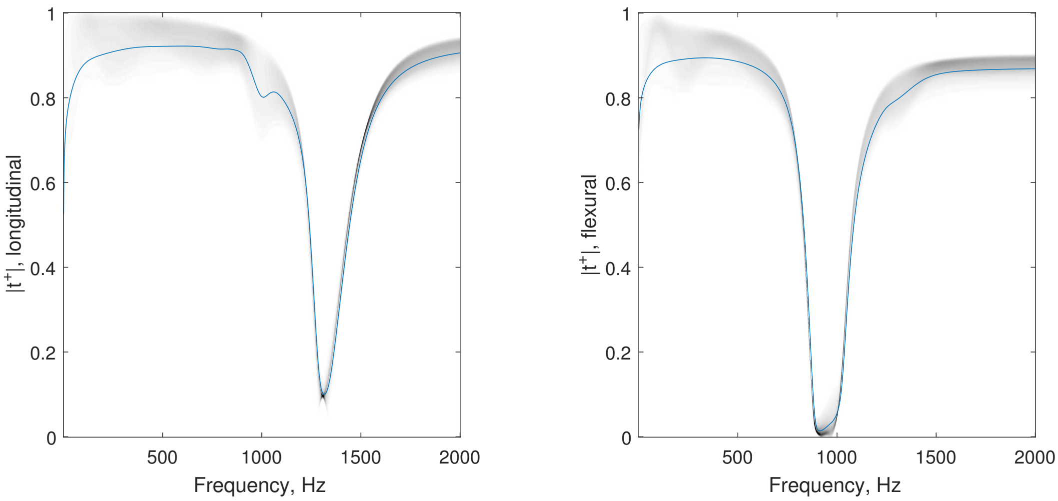

Figure 6: PDF and mean value (blue line) of the absolute value of the longitudinal (left) and flexural (right) transmission coefficients considering the random variable of both connection angles, $\mathtt{A}_1$ and $\mathtt{A}_2$. Beta of the first kind PDF $f^{(2)}_{\mathtt{A}_1,\mathtt{A}_2}(x)$.

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