Asociación Argentina
de Mecánica Computacional

Mecánica Computacional Vol XXXVI, págs. 771-784 (artículo completo)
José G. Etse, Bibiana M. Luccioni, Martín A. Pucheta, Mario A. Storti (Eds.)
San Miguel de Tucumán, 6-9 Noviembre 2018

A BIASED PRESENTATION OF SMOOTH DECOMPOSITION AND SOME APPLICATIONS

Rubens Sampaio[a], **Damien Foiny**[a], **Gustavo Wagner**[a], **Roberta Lima**[a] **and Emmanuel Pagnacco**[b]

[a]*Department of Mechanical Engineering, PUC-Rio, Rua Marquês de São Vicente, 225, Gávea, 22451-900 RJ, Brazil*

[b]*LMN, INSA de Rouen, 685 Avenue de l'Université, Saint-Etienne-du-Rouvray, 76800, France*

Keywords: Smooth Decomposition (SD), System Identification, Operational Modal Analysis (OMA), Nonlinear Parameters Identification.

Abstract. Smooth Decomposition (SD) is a multivariate data or statistical analysis method used to identify normal modes, natural frequencies and energy partition of systems. The method is based on the knowledge of the system response (spatial data field) to a random excitation. It should be noted that only the output data of the system is needed for the identification. The excitation has to satisfy some properties, normally well met by a white noise, but doesn't need to be measured. This turns the method the ideal way to deal with the identification of systems under ambient excitations, as wind or waves for instance, which can be hard to compute or to describe. The output data of the system response is then projected into a basis and an optimization problem is created. It consists of finding the basis that gives the maximum variance of the displacement-projection and the minimum variance of the velocity-projection. This optimization problem can then be written as an eigenvalue problem with the covariance matrices of the displacement field, and of the corresponding velocity field. Solving this problem the system is identified and no further considerations and approximations are needed. From the eigenvalues, the "energy" participation of each normal mode in the response during the simulation or the experimental test can be evaluated. Since this information is crucial for non-linear systems identification, the Smooth Decomposition method can be used to identify linear and non-linear systems. The objective of the paper is to explain the Smooth Decomposition method and to present an application of it. First we present the method and show how the results of SD can be interpreted. Then, an application of SD on a simulated numerical model of a cantilever beam is performed and discussed to understand how SD can be a nice tool for modal analysis.

1 INTRODUCTION

The Smooth Decomposition (SD) is a statistical analysis technique for finding structures in an ensemble of spatially distributed data such that the vector of generalized displacement not only keeps the maximum possible variance but also the velocity field is as smooth in time as possible. Closely related with the SD are the dual smooth modes used in the framework of oblique projection to expand a random response of a system. The concept of dual mode with the associated decomposition defines a tool that transforms the SD in an efficient modal analysis tool. This method of identification can be used for linear and nonlinear systems and uses only output data provide the excitation satisfies some properties normally met by a well chosen random excitation, as a white noise, for example.

The main properties of the SD are discussed and some optimality characteristics of the expansion are deduced. The parameters of the SD (using the dual smooth modes and the smooth values) give access to modal parameters of a linear system in terms of mode shapes, natural frequencies, and modal energy partition. This part is a remarkable improvement with respect to the standard modal analysis methods. This novel modal analysis of a linear system is illustrated by examples.

In this paper we will consider a numerical model of a cantilever beam which is excited by a random force at the free edge of the beam. The first case consists of the identification and modal analysis of this beam considering that there is no modal damping in order to show the power of the SD. Then we take into account the modal damping of the beam and we will show that SD is still a good method for low levels of damping. In previous paper we have worked on discrete systems and on systems with unobserved degree(s) of freedom which was a first step to continuous systems. It is explained how the SD can be applied to identify the parameters of a continuous system discretized by finite element method.

It is interesting to stress out that this is a new method, not yet compared with the methods known in the literature as Operational Modal Analysis (OMA). So far the only association between SD and OMA is the fact that both methods use only output signals for the identification and they require random excitation. However the theories are different. SD is a type of Karhunen-Loève Decomposition, using correlations and projections in the modes whereas OMA uses the controllability matrix and correlations of the measured signals that are not necessarily the state of the system.

2 DESCRIPTION OF THE SMOOTH DECOMPOSITION

In this Section of the paper we present the basis of the smooth decomposition method based on the following works Bellizzi, S. and Sampaio, R. (2012c), Bellizzi, S. and Sampaio, R. (2013), Sampaio, R. and Bellizzi, S. (2011), Sampaio, R. and Bellizzi, S. (2014b,c). As consequence, the discussion is biased as remarked in the paper title. We also compare this method with another well known method called the "Karhunen-Loève Decomposition (KLD)" or the "Proper Orthogonal Decomposition (POD)" used to analyze random data as Bellizzi, S. and Sampaio, R. (2009b), Bellizzi, S. and Sampaio, R. (2009a), Bellizzi, S. and Sampaio, R. (2006).

The main objective of KLD, or POD, consists in finding the basis that will be, with a fixed number of elements, the best representation of the initial field. All those methods have been used for another interesting aspect which is the model order reduction as it is presented in Bellizzi, S. and Sampaio, R. (2012a) and Bellizzi, S. and Sampaio, R. (2012b) for the SD and in Ritto, T. and Buezas, F.S. and Sampaio, R. (2012) for the POD, or Karhunen-Loève Decomposition.

KLD (or POD) and SD are based on the projection of the data field such as the generalized displacement vector has the maximum variance in order to be sure that all the modes we are looking for are excited. Indeed, the bigger is the variance of the displacement vector, the higher is the probability of a mode to be excited. The SD method is a bit different because we also consider the derivative of this generalized displacement vector, the velocity field. The objective is to find the basis that gives the maximum variance for the generalized displacement vector and the minimum variance for the velocity vector (in order to keep the motion as smooth as possible in time).

2.1 Decomposition Principle

First, let us describe the data field used in this method. We consider the sampled scalar field $\mathbf{X}(t)$ formed of random values (in the matrix form) as a function of the time t ($t \in \mathbb{R}$). This field is such as $\mathbf{X}(t) \in \mathbb{R}^{n\times m}$ where n represents the different instants and m represents the spacial points where we measure the information. The displacement field is considered as a stationary second-order process with zero-mean value that admits a time derivative which is also a stationary zero-mean value process.

The central point of this method is to find a linear projections such as:

$$\mathbf{Y}_{\mathbf{X}}(t) = \mathrm{proj}_{\boldsymbol{\phi}}\mathbf{X}(t) = \mathbf{X}(t)\boldsymbol{\phi}, \qquad (1)$$

$$\mathbf{Y}_{\dot{\mathbf{X}}}(t) = \mathrm{proj}_{\boldsymbol{\phi}}\dot{\mathbf{X}}(t) = \dot{\mathbf{X}}(t)\boldsymbol{\phi}, \qquad (2)$$

where $\mathbf{Y}_{\mathbf{X}}(t) \in \mathbb{R}^{n\times m}$, $\mathbf{Y}_{\dot{\mathbf{X}}}(t) \in \mathbb{R}^{n\times m}$ and $\boldsymbol{\phi} \in \mathbb{R}^{m\times m}$ (representing a projection basis). Now, the objective of this method is to find this projection basis such as it keeps the maximum variance for the projection of the original field $\mathbf{X}(t)$ (the generalized displacement field) and the smallest projection of the velocity field in order to keep the variation in time as smooth as possible. The objective is to find the $\max\limits_{\boldsymbol{\phi}} ||\mathbf{Y}_{\mathbf{X}}(t)||^2$ and $\min\limits_{\boldsymbol{\phi}} ||\mathbf{Y}_{\dot{\mathbf{X}}}(t)||^2$ which is exactly as maximizing $f(\phi)$ with:

$$f(\phi) = \frac{||\mathbf{Y}_{\mathbf{X}}(t)||^2}{||\mathbf{Y}_{\dot{\mathbf{X}}}(t)||^2} = \frac{||\mathbf{X}(t)\boldsymbol{\phi}||^2}{||\dot{\mathbf{X}}(t)\boldsymbol{\phi}||^2}. \qquad (3)$$

Now we can simplify this ratio using the auto-correlation matrices de Cursi, E.S. and Sampaio, R. (2015) $\mathbf{R}_{\mathbf{XX}}$ and $\mathbf{R}_{\dot{\mathbf{X}}\dot{\mathbf{X}}}$, respectively, for the displacement field ($\mathbf{X}(t) \in \mathbb{R}^{n\times m}$) and the velocity field ($\dot{\mathbf{X}}(t) \in \mathbb{R}^{n\times m}$). Indeed, we can write:

$$||\mathbf{X}\boldsymbol{\phi}||^2 = \mathbb{E}\left((\mathbf{X}\boldsymbol{\phi})^T\mathbf{X}\boldsymbol{\phi}\right) = \boldsymbol{\phi}^T\mathbb{E}\left(\mathbf{X}^T\mathbf{X}\right)\boldsymbol{\phi} = \boldsymbol{\phi}^T\mathbf{R}_{\mathbf{XX}}\boldsymbol{\phi}, \qquad (4)$$

$$||\dot{\mathbf{X}}\boldsymbol{\phi}||^2 = \mathbb{E}\left(\left(\dot{\mathbf{X}}\boldsymbol{\phi}\right)^T\dot{\mathbf{X}}\boldsymbol{\phi}\right) = \boldsymbol{\phi}^T\mathbb{E}\left(\dot{\mathbf{X}}^T\dot{\mathbf{X}}\right)\boldsymbol{\phi} = \boldsymbol{\phi}^T\mathbf{R}_{\dot{\mathbf{X}}\dot{\mathbf{X}}}\boldsymbol{\phi}, \qquad (5)$$

where $\mathbb{E}(\square)$ is the expected value. Finally we get this new expression for $f(\boldsymbol{\phi})$ (keeping in mind that n is the number of time samples which is rather big and thus can be simplified in the ratio in the case of the derivative method used do not conserve the same number of samples as in the original field) and we want to find:

$$\max_{\boldsymbol{\phi}}\left\{f(\boldsymbol{\phi}) = \frac{\boldsymbol{\phi}^T\mathbf{R}_{\mathbf{XX}}\boldsymbol{\phi}}{\boldsymbol{\phi}^T\mathbf{R}_{\dot{\mathbf{X}}\dot{\mathbf{X}}}\boldsymbol{\phi}}\right\}. \qquad (6)$$

In order to find the maximum of $f(\boldsymbol{\phi})$ we can express its derivative with respect to $\boldsymbol{\phi}$, called $\nabla f(\boldsymbol{\phi})$, such as:

$$\nabla f(\boldsymbol{\phi}) = \frac{\partial f(\boldsymbol{\phi})}{\partial \boldsymbol{\phi}} = \frac{2\left(\boldsymbol{\phi}^T\mathbf{R}_{\dot{\mathbf{X}}\dot{\mathbf{X}}}\boldsymbol{\phi}\right)\mathbf{R}_{\mathbf{XX}}\boldsymbol{\phi} - 2\left(\boldsymbol{\phi}^T\mathbf{R}_{\mathbf{XX}}\boldsymbol{\phi}\right)\mathbf{R}_{\dot{\mathbf{X}}\dot{\mathbf{X}}}\boldsymbol{\phi}}{\left(\boldsymbol{\phi}^T\mathbf{R}_{\dot{\mathbf{X}}\dot{\mathbf{X}}}\boldsymbol{\phi}\right)^2}, \qquad (7)$$

and then find when $\nabla f(\boldsymbol{\phi})$ vanishes. We can also find this maximum using Lagrange multiples. In both cases we will find the following eigenvalue problem as the expression of the two initial propositions. The problem is equivalent to:

$$\mathbf{R}_{\mathbf{XX}}\boldsymbol{\phi}_k = \lambda_k \mathbf{R}_{\dot{\mathbf{X}}\dot{\mathbf{X}}}\boldsymbol{\phi}_k, \qquad k = 1, ..., m. \tag{8}$$

Solving this eigenvalue problem we get the eigenvalues, the λ_k's, and the eigenvectors, the $\boldsymbol{\phi}_k$'s, such as the λ_k's are in ascending order ($\lambda_1 > \lambda_2 > \ldots > \lambda_m$). There is a relation between the auto-correlation of the velocity and the correlation of the acceleration and the displacement such as $\mathbf{R}_{\dot{\mathbf{X}}\dot{\mathbf{X}}} = -\mathbf{R}_{\ddot{\mathbf{X}}\mathbf{X}}$. From this relation we get the following eigenvalue problem which leads to the same results.

$$\mathbf{R}_{\mathbf{XX}}\boldsymbol{\phi}_k = -\lambda_k \mathbf{R}_{\ddot{\mathbf{X}}\mathbf{X}}\boldsymbol{\phi}_k, \qquad k = 1, ..., m. \tag{9}$$

Using $\boldsymbol{\phi}_k$, it is possible to define

$$\boldsymbol{\psi}_k = \mathbf{R}_{\mathbf{XX}}\boldsymbol{\phi}_k. \tag{10}$$

At this step we are able to find several parameters from a displacement field of a mechanical system. We can identify the λ_k (the Smooth Value - SV), the $\boldsymbol{\phi}_k$ (Smooth Mode - SM) and the $\boldsymbol{\psi}_k$ (Dual Smooth Mode - DSM). Depending on the characteristics of the system, we can interpret these parameters differently, as seen in the contributions of Bellizzi and Sampaio (Bellizzi, S. and Sampaio, R. (2012c), Bellizzi, S. and Sampaio, R. (2013), Sampaio, R. and Bellizzi, S. (2011), Sampaio, R. and Bellizzi, S. (2014b) and Sampaio, R. and Bellizzi, S. (2014c)) and also in the one of the Farooq and Feeny's works Farooq, U. and Feeny, B.F. (2008).

2.2 Expansion Principle

From this decomposition we do have two different bases, the smooth basis called $\boldsymbol{\Phi}$, formed with the $\boldsymbol{\phi}_k$'s, and the smooth dual basis, called $\boldsymbol{\Psi}$, formed with the $\boldsymbol{\psi}_k$'s (for $k = 1, ..., m$ with m as the number of measuring points). Now we propose to use these two bases to find the smooth expansion of $\mathbf{X}(t)$ and its dual smooth expansion.

2.2.1 Smooth expansion in the Φ-basis

Considering the expansion of $\mathbf{X}(t)$ in the $\boldsymbol{\Phi}$-basis we can write $\mathbf{X}(t) = \sum_{k=1}^{m} \xi_k(t)\boldsymbol{\phi}_k^T$, which can be simplified using a normalization condition $\phi_k^T \mathbf{R}_{\dot{\mathbf{X}}\dot{\mathbf{X}}}\phi_k = 1$ and using ψ_k. Then, we can find the Dual Smooth Components (DSC) $\xi_k(t)$:

$$\xi_k(t) = \frac{\mathbf{X}(t)\boldsymbol{\psi}_k}{\lambda_k}. \tag{11}$$

2.2.2 Dual smooth expansion in the Ψ-basis

Let us consider the $\boldsymbol{\Psi}$-basis to express $\mathbf{X}(t)$. The dual smooth expansion of this field into this basis can be written as $\mathbf{X}(t) = \sum_{k=1}^{m} \chi_k(t)\boldsymbol{\psi}_k^T$. Following the same procedure applied in the Smooth expansion in the Φ-basis, it is possible to obtain the the Smooth Components (SC) $\chi_k(t)$:

$$\chi_k(t) = \frac{\mathbf{X}(t)\boldsymbol{\phi}_k}{\lambda_k}, \tag{12}$$

At this step, we can notice an interesting property for the Smooth Components. Let us consider the square of it and develop to the following form:

$$\mathbb{E}(\chi_k^2(t)) = \boldsymbol{\phi}_k^T \mathbf{R}_{\mathbf{XX}} \boldsymbol{\phi}_k. \tag{13}$$

Now considering the original eigenvalue problem formulated in Eq.(8) we can write:

$$\mathbb{E}\left(\chi_k^2(t)\right) = \lambda_k. \tag{14}$$

2.3 Energetic point of view

An interesting thing with SD is the energetic study that can be made with this method. Let us call the "energy" of the field $\mathbf{X}(t)$ the expression $\mathbb{E}\left(||\mathbf{X}(t)||^2\right)$ that can be reduced (using the dual smooth expansion). From the dual smooth expansion we get:

$$\mathbf{X}(t) = \sum_{k=1}^{m} \chi_k(t) \boldsymbol{\psi}_k^T \Rightarrow ||\mathbf{X}(t)||^2 = \sum_{k=1}^{m} \left|\left|\chi_k(t)\boldsymbol{\psi}_k^T\right|\right|^2. \tag{15}$$

The expression of the "energy" can then be simplified using the previous formulation and we get:

$$\mathbb{E}\left(||\mathbf{X}(t)||^2\right) = \mathbb{E}\left(\sum_{k=1}^{m} \left|\left|\chi_k(t)\boldsymbol{\psi}_k^T\right|\right|^2\right) = \sum_{k=1}^{m} \left[\mathbb{E}\left(\chi_k^2(t)\right)\mathbb{E}\left(||\boldsymbol{\psi}_k||^2\right)\right]. \tag{16}$$

Simplifying using Eq.(14) we can find the final expression for the "energy" of $\mathbf{X}(t)$ as:

$$\mathbb{E}\left(||\mathbf{X}(t)||^2\right) = \sum_{k=1}^{m} \lambda_k \, ||\boldsymbol{\psi}_k||^2. \tag{17}$$

Note that, from this formula it is quite easy to find the energy captured in each mode (the identified mode with the SD which, sometimes, does not correspond to a physical mode) during the simulation since the expression:

$$\frac{\lambda_i \, ||\boldsymbol{\psi}_i||^2}{\sum_{k=1}^{m} \lambda_k \, ||\boldsymbol{\psi}_k||^2}, \tag{18}$$

represents the fraction of energy captured by the mode i during the simulation. This value can be a really good way to verify if a mode has been well excited during a simulation and then if the estimation of its frequency and mode shape can be validated. Also, this parameter is crucial for identification of non-linear systems since knowing the energy is essential.

2.4 Modal Assurance Criterion - MAC

In order to evaluate the mode basis $\boldsymbol{\Psi}$ found from SD ($\boldsymbol{\Psi}_{SD}$) with respect to the expected ones we will use the Modal Assurance Criterion, called MAC representation. According to Allemang Allemang, R.J. (2003), this tool is a good way to verify if modes found from one method (SD in our case) correspond to modes found by another method (from the initial eigenvalue problem for us, defined as the modes shapes base $\boldsymbol{\Psi}_{EIG}$). The formulation of this criteria is:

$$MAC(\boldsymbol{\Psi}_{SD}, \boldsymbol{\Psi}_{EIG}) = \frac{\left|\boldsymbol{\Psi}_{SD}^H \boldsymbol{\Psi}_{EIG}\right|^2}{\left(\boldsymbol{\Psi}_{SD}^H \boldsymbol{\Psi}_{SD}\right)\left(\boldsymbol{\Psi}_{EIG}^H \boldsymbol{\Psi}_{EIG}\right)}, \tag{19}$$

where the notation $\square^H$ denotes the complex conjugate transpose of the quantity. Let us note that for identical modes from two different methods the MAC should give one. However, since the orthogonality is in relation to the stiffness matrix metric, the MAC value is not null for two different modes.

3 SMOOTH DECOMPOSITION AS A MODAL ANALYSIS TOOL FOR DISCRETIZED UNDAMPED SYSTEMS

Any continuous undamped system can be written (thanks to the discretization by finite element for instance) in a matrix form of the classical dynamic equation, where $\mathbf{M}$ is the mass matrix, $\mathbf{K}$ is the stiffness, $\mathbf{x}(t)$ and $\ddot{\mathbf{x}}(t)$ represent respectively the generalized displacement and the acceleration fields of the system. The excitation of the system is represented by $\mathbf{f}(t)$.

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{f}(t). \tag{20}$$

From Eq.(20) we can write the acceleration as

$$\ddot{\mathbf{x}}(t) = -\mathbf{M}^{-1}\mathbf{K}\mathbf{x}(t) + \mathbf{M}^{-1}\mathbf{f}(t). \tag{21}$$

The modal parameters are found solving the classical eigenvalue problem:

$$\mathbf{M}^{-1}\mathbf{K}\boldsymbol{\psi}_k = \omega_k^2\boldsymbol{\psi}_k. \tag{22}$$

where, ω_k^2's are the squares of the natural frequencies of the mechanical system with associated normal modes $\boldsymbol{\psi}_k$ (results used as reference). First let us remind the initial eigenvalue problem, given by Eq.(9), and adapt it to our case:

$$\mathbf{R}_{\mathbf{xx}}\boldsymbol{\phi}_k = -\lambda_k\mathbf{R}_{\ddot{\mathbf{x}}\mathbf{x}}\boldsymbol{\phi}_k, \qquad k = 1, ..., m. \tag{23}$$

Then, using Eq.(21) and the cross-covariance definition, we can write:

$$\begin{aligned}
\mathbf{R}_{\mathbf{xx}}\boldsymbol{\phi}_k &= -\lambda_k\mathbb{E}\left[\ddot{\mathbf{x}}(t)\mathbf{x}^T(t)\right]\boldsymbol{\phi}_k \\
&= -\lambda_k\mathbb{E}\left[\left(-\mathbf{M}^{-1}\mathbf{K}\mathbf{x}^T(t) + \mathbf{M}^{-1}\mathbf{f}(t)\right)\mathbf{x}^T(t)\right]\boldsymbol{\phi}_k \\
&= \lambda_k\mathbf{M}^{-1}\mathbf{K}\mathbb{E}\left[\mathbf{x}(t)\mathbf{x}^T(t)\right]\boldsymbol{\phi}_k - \lambda_k\mathbf{M}^{-1}\mathbb{E}\left[\mathbf{f}(t)\mathbf{x}^T(t)\right]\boldsymbol{\phi}_k \\
&= \lambda_k\mathbf{M}^{-1}\mathbf{K}\mathbf{R}_{\mathbf{xx}}\boldsymbol{\phi}_k - \lambda_k\mathbf{M}^{-1}\cancelto{0}{\mathbf{R}_{\mathbf{fx}}}\boldsymbol{\phi}_k.
\end{aligned} \tag{24}$$

Then, as the response and the forcing are not correlated at the same time instant t (see later in this paper) and zero-mean value signals, also using the definition of the auto-covariance, we can write (reorganizing):

$$\mathbf{M}^{-1}\mathbf{K}\mathbf{R}_{\mathbf{xx}}\boldsymbol{\phi}_k = \frac{1}{\lambda_k}\mathbf{R}_{\mathbf{xx}}\boldsymbol{\phi}_k. \tag{25}$$

The eigenvalue problems given by Eq. (22) and (25) correspond to same mechanical system, thus we get the following equivalence:

$$\mathbf{M}^{-1}\mathbf{K}\boldsymbol{\psi}_k = \omega_k^2\boldsymbol{\psi}_k \quad \Longleftrightarrow \quad \mathbf{M}^{-1}\mathbf{K}\mathbf{R}_{\mathbf{xx}}\boldsymbol{\phi}_k = \frac{1}{\lambda_k}\mathbf{R}_{\mathbf{xx}}\boldsymbol{\phi}_k \tag{26}$$

Thanks to Eq.(26) we get the following interpretation of the parameters of a linear undamped system:

$$\begin{cases} \omega_k^2 = \frac{1}{\lambda_k}, \\ \boldsymbol{\psi}_k = \mathbf{R}_{\mathbf{xx}}\boldsymbol{\phi}_k. \end{cases} \tag{27}$$

3.1 Correlation between random force and displacement

In this section we will rapidly show why the term with the correlation between the force and the displacement is equal to zero in Eq.(24). First, let us consider a force $f(t)$ and the associated response of a system $x(t)$. The correlation between the force f at the instant u and the response at the instant s is:

$$\begin{aligned}
\mathbf{R}_{fx}(u,s) &= \mathbb{E}\left[f(u)x(s)\right] \\
&= \mathbb{E}\left[f(u)\int_{-\infty}^{s} h(v)f(s-v)dv\right] \\
&= \mathbb{E}\left[\int_{-\infty}^{s} h(v)f(u)f(s-v)dv\right] \\
&= \int_{-\infty}^{s} h(v)\mathbf{R}_{ff}(u,s-v)dv.
\end{aligned} \tag{28}$$

In Eq.(28), the function $h(v)$ is the response of the system to an impulse. Then, considering the force is stationary we can proceed with:

$$\mathbf{R}_{fx}(u,s) = \int_{-\infty}^{s} h(v)\mathbf{R}_{ff}(s-v-u)dv. \tag{29}$$

Even though the integrand depends on $s-u$, for small s, $\mathbf{R}_{fx}(u,s)$ is a function of s as it enters in the limit of the integral. Now, considering $h(t)$ does not diverge for $t \longrightarrow \infty$, which means that the system is stable, and, considering that $u, s \longrightarrow \infty$, one can show that $\mathbf{R}_{fx}(u,s)$ approaches a simple dependence on $s-u$. Now, an interesting point consists in considering the delay the response needs to become stationary. The delay directly depends on the damping of the system (indeed, the higher is the damping, the faster the signal reaches stationarity). For undamped cases, the number of samples has to be big enough to ensure stationarity. Then, the response of the system approaches stationarity when $s \longrightarrow \infty$, we can define the variable $\tau = s-u$ and get:

$$\begin{aligned}
\mathbf{R}_{fx}(u,s) &= \mathbf{R}_{fx}(s-u) \\
&= \mathbf{R}_{fx}(\tau) \\
&= \int_{-\infty}^{\infty} h(v)\mathbf{R}_{ff}(\tau-v)dv \\
&= \int_{-\infty}^{\infty} h(\tau-v)\mathbf{R}_{ff}(v)dv.
\end{aligned} \tag{30}$$

As we have said earlier in this paper, the force is random and assumed to be a white noise with a frequency range defined in the frequency band $\left[-\frac{f_s}{2};\frac{f_s}{2}\right]$ where f_s is the acquisition frequency (known from the simulation or the experiment), the auto-correlation of the force is:

$$\mathbf{R}_{ff}(v) = \frac{\sigma_f^2 \sin\left(f_s\, v\right)}{f_s\, v}. \tag{31}$$

In this equation, σ_f^2 represents the variance of the force (white noise). From the previous equations we get:

$$\begin{aligned}
\mathbf{R}_{fx}(\tau) &= \frac{\sigma_f^2 \sin\left(f_s\, v\right)}{f_s\, v}\int_{-\infty}^{\infty} h(\tau-v)\delta(v)dv \\
&= \frac{\sigma_f^2}{f_s} h(-\tau).
\end{aligned} \tag{32}$$

As it is well known, the response function $h(t)$ to an impulse (applied at $t = 0$) is zero when $t \leq 0$ and oscillating when $t > 0$. In Eq.(32) we consider $h(-\tau)$ then we switch the oscillating part of the response with the other one (the zero-one). It is important to say that in our cases, we consider causal mechanical systems then the correlation for $\tau < 0$ is not relevant (the oscillating part of the function) but the other one ($\tau \geq 0$) is crucial and for this part the function is zero. Finally, for $\tau \geq 0$, $\mathbf{R}_{fx}(\tau) \to 0$ for a time of simulation or experiment long enough to get stationarity (the stationarity can be reached faster for damped systems).

In order to show the phenomena let us consider a simple damped mechanical system with only one degree of freedom. The system consists of a mass $M = 10$ kg, a spring $K = 40$ kN and a damper C which will be responsible for two different values of modal damping called ζ such as $\zeta_1 = 0.95\%$ and $\zeta_2 = 1.9\%$. We use a Gaussian white noise for the excitation $F(t)$ such as its statistics parameters are $\mu_F = 0N$ and $\sigma_F = 100$ N. The acquisition frequency $f_s = 500$ Hz. For each case of damping, we will show results of the correlation between the force and the displacement for different numbers of samples which means different time duration of simulation.

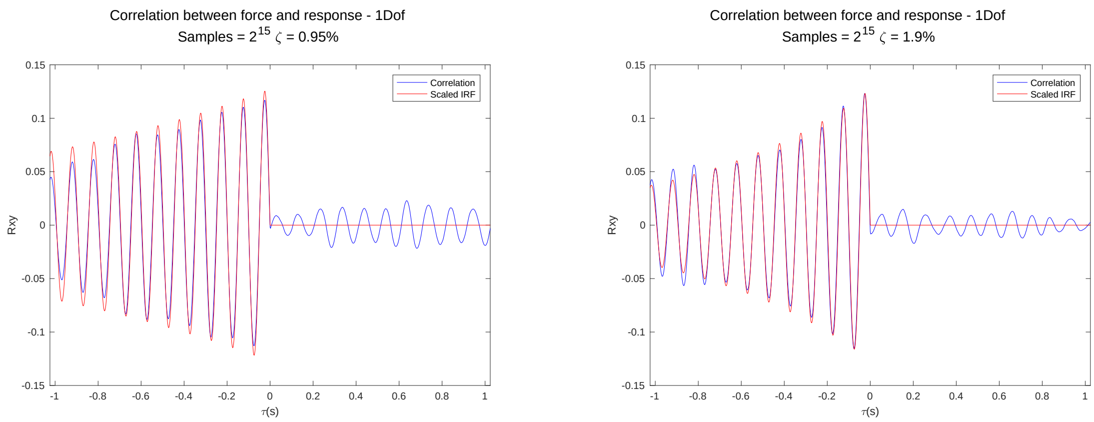

Figure 1: Comparison between the calculated correlation $\mathbf{R}_{fx}$ (blue) and the estimated one through the Scaled Impact Response Function (red) for different values of damping

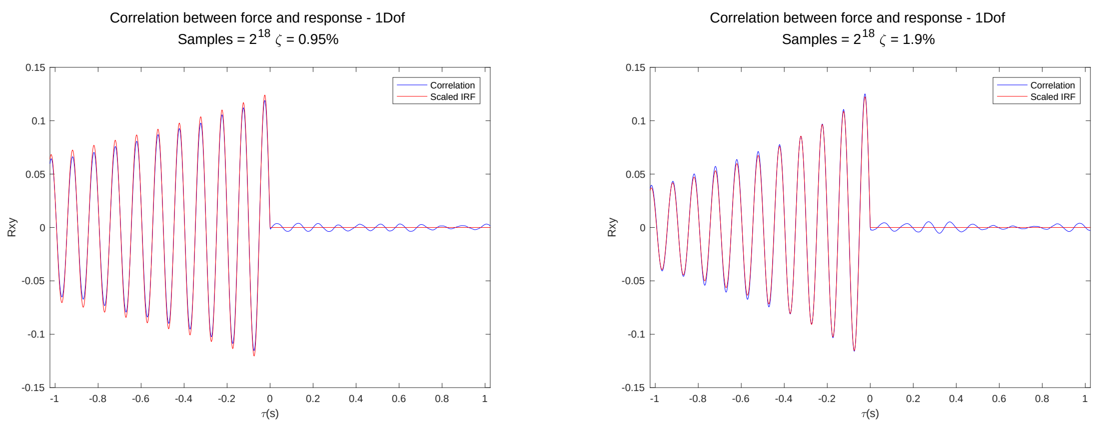

Figure 2: Comparison (bis) between the calculated correlation $\mathbf{R}_{fx}$ (blue) and the estimated one through the Scaled Impact Response Function (red) for different values of damping

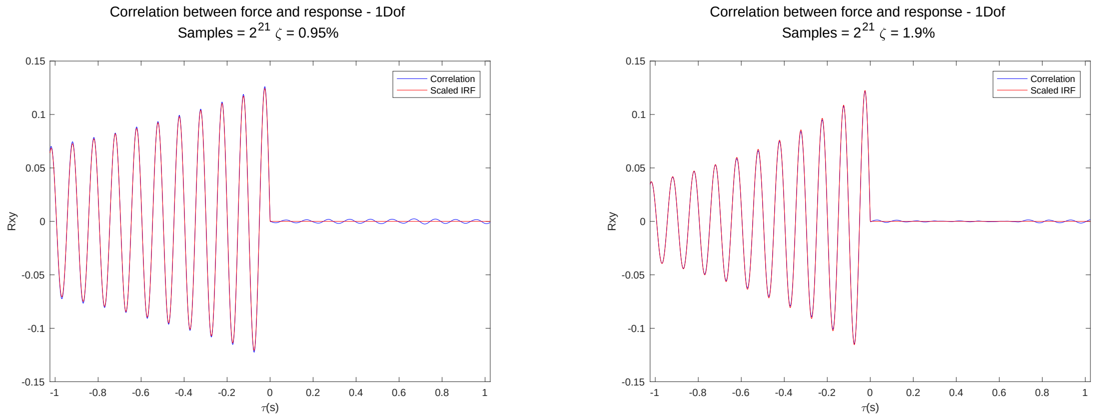

Figure 3: Comparison (ter) between the calculated correlation $\mathbf{R}_{fx}$ (blue) and the estimated one through the Scaled Impact Response Function (red) for different values of damping

Thanks to the results presented by Figs. 1, 2 and 3, we can first validate the Eq.(32) and we can also show that for a number of samples big enough (which means a simulation time also sufficiently big) the correlation between the force and the response is equal zero if the force is random for $\tau \geq 0$ which is the case in Eq.(32). The same conclusion can be reached faster if we consider the damping of the system as we have said before.

3.2 Application of the method

Let us apply this method to a problem. To illustrate this theory we can observe the cantilever beam presented in Fig. 4. This system has got the following mechanical properties: $L = 1$ m, $E = 200GPa = 200 \times 10^9$ N.m^{-2}, $I = 8.33 \times 10^{-10}$ m^4 and $\rho = 7850$ kg.m^{-3}. The concentrated force applied at the free end of the beam is assumed to be random with $\mu_F = 0$ N and $\sigma_F = 10$ N.

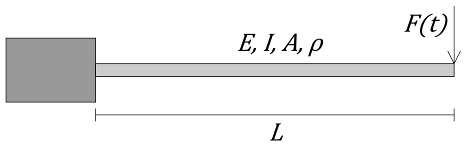

Figure 4: Cantilever beam submitted to random force at the free edge

In order to solve this continuous problem we discretize it by finite elements. For this discretization we consider the Euler-Bernoulli theory thus we get two degree of freedom for the finite element, the deflection called $\mathbf{w}(x,t)$ and the rotation of the transversal section called $\boldsymbol{\theta}(x,t)$.

From the finite element discretization we get the mass and stiffness matrices of the beam (respectively $\mathbf{M}$ and $\mathbf{K}$) and from them we can easily find the natural frequencies and mode shapes of the system solving the eigenvalue problem defined in Eq. (22). These results are the reference for the comparison with results we get from SD. In this case we do not consider modal damping.

Let us now simulate the dynamic response of this system. The simulation is made (by mode superposition) such as we get a number of samples $N_s = 409600$ with an acquisition frequency $f_s = 1.3$ kHz (equivalent to a $315s$-long simulation). A convergence analysis was made on the first natural frequency calculated from the mass and the stiffness matrices. It was verified that a discretization with five linear elements, $N_{ele} = 5$ is good enough to get good approximation to the first natural frequency.

From this simulation we can plot the Power Spectral Density - PSD ($S(f)$) of the response using the following number of samples in a block $N_b = 8192$. Figure 5 shows the one-sided spectral density in Hertz $G(f)$ of the vertical displacement at the free edge. Thanks to this plot we can see that the response of the system is composed by five modes (since there are five peaks on the one-sided spectral density of the response). The frequencies were the peaks take place are then written in the following table.

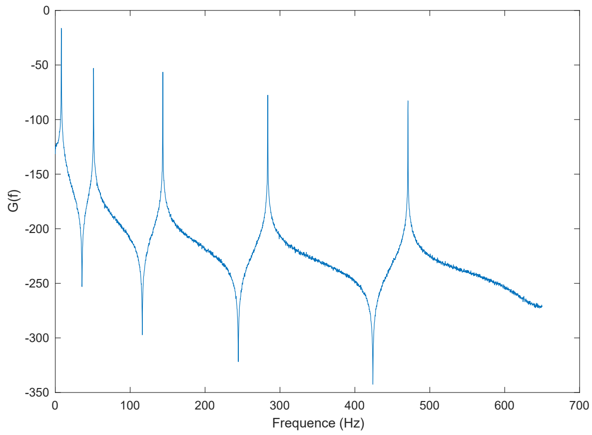

Figure 5: One-sided Power Spectral Density $G(f)$

Nº	Frequences (Hz)
f_1	8.093
f_2	51.10
f_3	143.6
f_4	283.7
f_5	470.8

Table 1: Frequencies of the peaks in the one-sided Power Spectral Density of the response.

From the response (only vertical displacement called $\mathbf{w}(x,t)$) of the system excited with the random force $F(t)$ we can apply the Smooth Decomposition and identify natural frequencies and mode shapes and compare them to the expected ones calculated from the mass and stiffness matrices generated from the finite element discretization. First let us have a look on the natural frequencies presented in the Table 2.

Nº	Freq. EIG (Hz)	Freq. SD (Hz)	Rel. Error (%)
1	8.1539	8.1537	0.0025
2	51.125	51.124	0.0015
3	143.59	143.59	0.0024
4	283.67	283.66	0.0015
5	470.81	470.82	0.0013

Table 2: Comparison of natural frequencies from SD and the ones calculated from the mass and stiffness matrices.

We can also compare the mode shapes of the five first modes of the system and their correspondence through the MAC representation to get a better idea of our results (see Fig.6).

This example shows how SD can be used as a modal analysis tool for linear undamped systems.

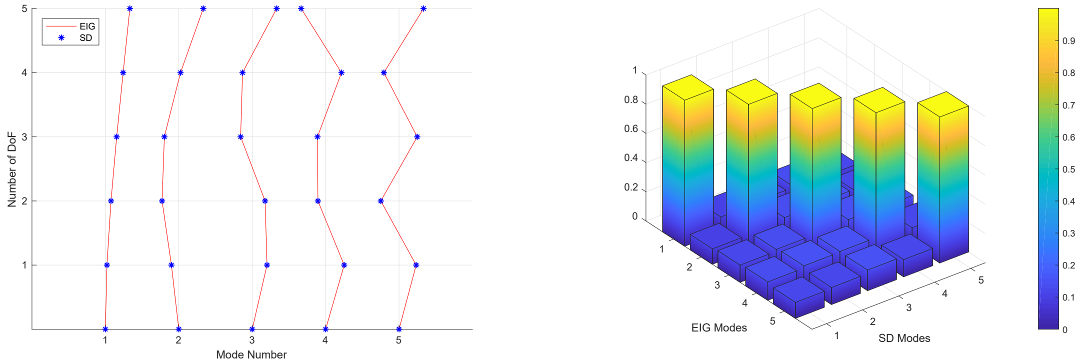

Figure 6: Left: Mode shapes comparison / Right: MAC between the EIG-resolution modes and the SD ones

4 SMOOTH DECOMPOSITION FOR GENERAL SYSTEMS

4.1 Interpretation of the method

In this part we will consider damped systems. This consideration is closer to real mechanical systems which can be formulated thanks to the dynamics equation:

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{C}\dot{\mathbf{X}} + \mathbf{K}\mathbf{X} + \mathbf{A}(\mathbf{X}) = \mathbf{F}, \tag{33}$$

where $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{C}$ are the mass, the stiffness and the damping matrices of our system. $\mathbf{F}$ is the forcing vector which, in our specific case is not monitored (unknown excitation, characteristic of the output only methods). The term called $\mathbf{A}$ represents the nonlinearity of our mechanical system.

As it was shown in the literature (namely Bellizzi and Sampaio Bellizzi, S. and Sampaio, R. (2015) and Sampaio, R. and Bellizzi, S. (2014a)), the interpretation for those cases is not as simple as for the linear ones. Indeed, we cannot find the simple equivalence shown with Eq.(27). These considerations provide from the statistical linearization method thus they give results for a linear system. If we apply these equivalences, given by Eq. (27), to non-linear systems we actually get the modal parameters for the linear equivalent system but not for the non-linear one.

If we consider a damped system with the $\mathbf{C}$-matrix as a linear combination of the $\mathbf{M}$ and $\mathbf{K}$ ones (i.e. $\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}$) we can reach a similar interpretation as it was done for undamped systems. From this method we do have access to the normal modes of the systems.

4.2 Influence of the modal damping on the identification quality

In this part we will consider the same mechanical system but this time we consider modal damping through the ζ-coefficients. In order to observe the influence of the damping on the identification made with SD we will investigate the evolution of the relative error in the identified natural frequencies with increasing modal damping. Note that here the modal damping is considered constant i.e. the same for all modes.

This example will investigate the influence of the modal damping factor on the evaluated frequencies from SD. To discuss this we will observe the relative error in each natural frequency for different values of the modal factor $0\% \leq \zeta \leq 10\%$. Keep in mind that the modal damping is constant. On the Fig. 7 we can see that the relative error in the natural frequencies is increasing linearly.

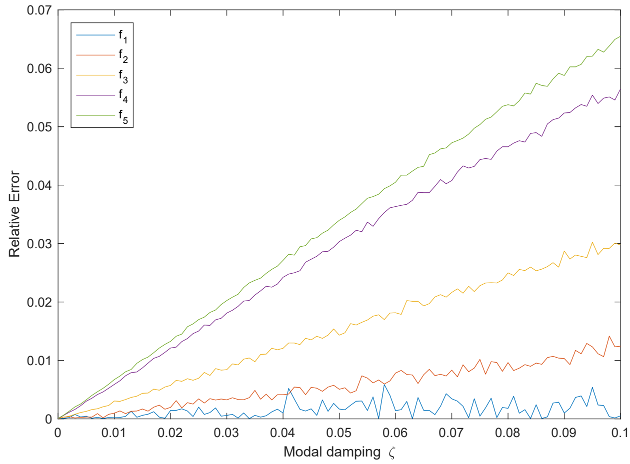

Figure 7: Evolution of the relative error in the five natural frequencies with respect to the modal damping

Now let us have a closer look on the identification made with $\zeta = 10\%$. In the Table 3 we can see that the relative error for the estimated natural frequencies is quite important.

Nº	Freq. EIG (Hz)	Freq. SD (Hz)	Rel. Error (%)
1	8.1539	8.1497	0.0517
2	51.125	50.496	1.2441
3	143.59	139.44	2.9750
4	283.67	268.51	5.6458
5	470.81	441.88	6.5472

Table 3: Comparison of natural frequencies from SD and the ones calculated from the mass and stiffness matrices.

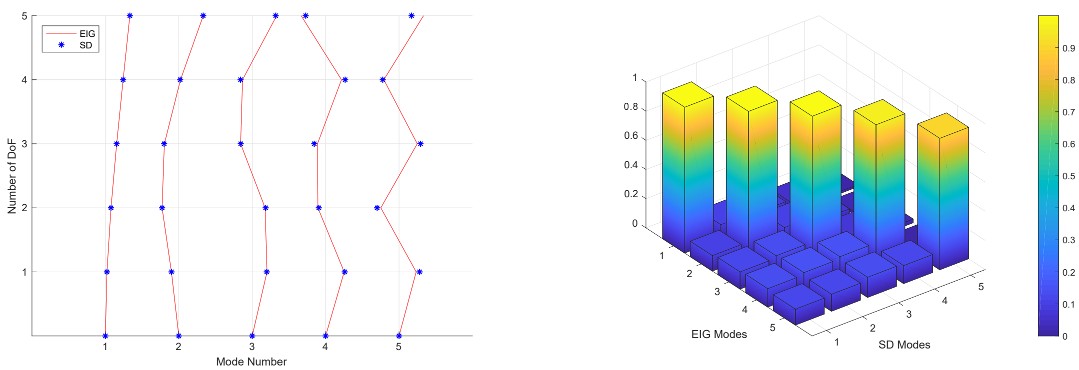

Figure 8: Results for the cantilever beam considering modal damping of 10%. Left: Mode shapes comparison / Right: MAC between the EIG-resolution modes and the SD ones

But, in the same time, we can see that the estimation of the mode shapes was not that bad for the first modes. This proves that for high modal damping factors this method is not adequate. As SD gives the evaluation of the normal modes, it was expected for high damped systems an inadequate evaluation of them and their frequencies. Concerning modes, the conclusions are similar. For small damping factors the approximations are quite good but for bigger ones we observe some mode shapes that do not correspond to the expected ones.

5 CONCLUSIONS

In this article we have discussed the concept of Smooth Decomposition and explained the method. Its properties were exposed and the concept of Dual Smooth Modes were presented. We showed simple examples where SD was used as a modal tool for the identification of mechanical systems.

As we have seen in the first part, this method can be applied for undamped linear systems which makes sense since this method was developed exactly for this kind of systems. We have seen in the second part that SD also works for low-damped systems. However, for high-damped systems SD is not really adapted and should be improved. Indeed, as SD gives the normal modes associated to a given field, this method may not work for finding modes with an imaginary part which may be the case for damped systems.

In previous articles Foiny, D. and Wagner, G.B. and Sampaio, R. and Lima, R. (2017) and Wagner, G.B. and Foiny, D. and Sampaio, R. and Lima, R. (2017) we have shown that SD is a powerful for discrete system with several degrees of freedom. In this article we have shown an application of SD to continuous systems discretized by finite element method.

A crucial point that was not developed in this article is the importance of the excitation quality. The excitation of the system has to respect some conditions to get a good approximation and estimation of the modal parameters. If the excitation does not satisfy the properties, the results are affected and this has to be taken into account for the interpretation.

Finally, it is possible to say that SD is a nice tool for modal analysis and can be applied to continuous systems. We would like to highlight that the method can also be applied to identification of non-linear systems since it is possible to define an energy indicator.

ACKNOWLEDGEMENTS

The authors acknowledge the support given by FAPERJ, CNPq and CAPES.

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