Asociación Argentina
de Mecánica Computacional

Mecánica Computacional Vol XXXIV, págs. 1459-1477 (artículo completo)
Sebastián Giusti, Martín Pucheta y Mario Storti (Eds.)
Córdoba, 8-11 Noviembre 2016

RAYLEIGH QUOTIENT ALGORITHM FOR MODAL ANALYSIS OF STRUCTURAL MODELS

José A. Inaudi

Departamento de Física, Facultad de Ciencias Exactas, Físicas y Naturales, Universidad Nacional de Córdoba, Av. Velez Sársfield 1611, 5000 Córdoba, Argentina, inaudijose@gmail.com, http://www. www.efn.uncor.edu.

Keywords: vibration, natural frequency, Rayleigh quotient

Abstract. An algorithm for the computation of natural frequencies in continuum models based on classical separation of variables and the Rayleigh quotient is presented. The iterative method uses frequency-dependent shape functions derived from conventional modal analysis to assemble the boundary condition matrix resulting from the application of separation of variables for modal analysis, and the corresponding mass and stiffness matrices. An order reduction to a single generalized coordinate allows the application of the Rayleigh quotient for the estimation of a particular natural frequency. An iterative procedure is followed which starts from an initial estimate of a desired natural frequency. The evaluation of the Rayleigh quotient provides an improved estimate of the squared natural frequency and the corresponding improvement of the shape functions that estimate a mode shape. The recursive application of the proposed algorithm allows the estimation of any natural frequency and mode shape of the continuum model. The application of the proposed technique is illustrated using continuum-parameter rod and beam elements in longitudinal and flexural vibrations, respectively. Some characteristics of the speed of convergence and regions of convergence are explored. The algorithm shows excellent convergence characteristics.

1 INTRODUCTION

The Finite Element Method (FEM) is the most frequently used technique for modal analysis of linear models in structural analysis software. The main reason is that efficient computational methods have been developed for the solution of the standard eigenvalue problem. The application of conventional FEM and hp-FEM (finite element formulation using high order polynomial shape functions) to free vibration of linear structures, leads to structural models with finite number of degrees of freedom that satisfy the following equations of motion (Clough and Penzien 1993, Chopra 1995):

$$\mathbf{M}\,\ddot{\mathbf{q}}(\mathrm{t}) + \mathbf{K}\,\mathbf{q}(\mathrm{t}) = \mathbf{0} \tag{1}$$

The concepts of mesh refinement, adaptive mesh and high order finite element models have been covered in the literature and implemented in structural analysis software as strategies to improve model accuracy (Babushka and Guo, 1992). Once the discrete model mass and stiffness matrix have been computed, the following eigenvalue problem is formulated for modal analysis:

$$\mathbf{K}\,\boldsymbol{\varphi} = -\omega^2\mathbf{M}\boldsymbol{\varphi} \tag{2}$$

In Eq. (2) $\boldsymbol{\varphi}$ and ω are a mode shape and the corresponding natural frequency of the model, respectively.

Iterative methods are frequently used for natural frequency and mode computation, solution of the generalized eigenvalue problem given by Eq. (2). Starting with an initial guess vector, $\mathbf{v}_o$, the following sequence is applied:

$$\mathbf{K}\,\mathbf{v}_{n+1} = \mathbf{M}\mathbf{v}_n \tag{3}$$

The iterative technique converges to the fundamental mode of vibration of the model as the number of iterations increases. Successive rotations align any random initial vector to the mode of vibration with the smallest natural frequency (fundamental mode of vibration). The search requires the solution of the linear set of equations indicated in Eq. (3) and the normalization of the iteration vector using the Euclidean norm or the norm of the vector in the mass matrix. Convergence to higher modes requires the mass orthogonalization of the trial vector with respect to lower mode shapes. Power iteration techniques are used usually in standard finite-element analysis software packages (Wilson, 2002) usually in combination with subspace order reduction (subspace iteration method). Inverse iteration with variable shift is also a frequently used technique to accelerate convergence.

The Rayleigh Quotient Iteration (RQI) has been developed for real symmetric matrices and applied to modal analysis of discrete models (Eq. 2). In 1951 Crandall investigated the three variants: the original Rayleigh quotient iteration, inverse iteration with fixed shift and symmetric RQI (Crandall, 1951; Hodges 1997). At each step of the RQI procedure the trial vector is adapted using the updated estimate of natural frequency. RQI produces rotations of the trial vector towards the eigenvector whose associated natural frequency (eigenvalue) is close to the frequency estimation.

Another technique that can be used to accelerate convergence along with RQI is the following. Given an estimate of a particular natural frequency we can compute estimates of a mode of vibration using the dynamic stiffness matrix $\boldsymbol{D}$ of the discrete model:

$$\mathbf{D}\boldsymbol{\varphi} = (\mathbf{K} - \omega^2\mathbf{M})\boldsymbol{\varphi} = \mathbf{0} \tag{2}$$

The dynamic stiffness matrix and the mode shape can be partitioned as follows:

$$\mathbf{D}\boldsymbol{\varphi} = \begin{bmatrix} \mathbf{D}_{\mathrm{rr}} & \mathbf{D}_{\mathrm{rs}} \\ \mathbf{D}_{\mathrm{sr}} & \mathbf{D}_{\mathrm{ss}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varphi}_{\mathrm{r}} \\ \boldsymbol{\varphi}_{\mathrm{s}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{3}$$

for any partitions of the mode-shape estimate, φ_{r} and φ_{s}.

Given φ_{r}, φ_{s} can be computed from the solution of the linear set of equations

$$\mathbf{D}_{\mathrm{ss}}\boldsymbol{\varphi}_{\mathrm{s}} = -\mathbf{D}_{\mathrm{sr}}\boldsymbol{\varphi}_{\mathrm{r}} \tag{4}$$

If we set a particular coordinate in the mode shape estimate equal to one, $\varphi_{\mathrm{r}} = 1$, the rest of the mode shape vector can be computed using the linear set of equations (Eq. 4). An improved estimation of the squared natural frequency can be computed then using the Rayleigh Quotient

$$R = \frac{\boldsymbol{\varphi}^{\mathrm{T}}\mathbf{K}\,\boldsymbol{\varphi}}{\boldsymbol{\varphi}^{\mathrm{T}}\mathbf{M}\,\boldsymbol{\varphi}} \tag{5}$$

Using R as a new estimate of squared frequency, an updated dynamic stiffness matrix can be computed to proceed with the computation of an updated mode shape estimate using Eq. (3). A convergence criterion based on mode shape rotation in successive steps or natural frequency estimates in successive steps can be used to establish convergence. To compute a different natural frequency, the algorithm must start with an initial estimate of this natural frequency in the dynamic stiffness matrix (Eq. 2).

The aforementioned techniques are introduced as background because the computational method reported herein uses the Rayleigh quotient as the frequency estimation in each step of the algorithm in an analogous manner. However, instead of using a weak formulation to approximate the solution of the vibration problem with frequency-independent shape functions in a finite dimensional configuration space, the mode of vibration is searched in evolutionary subspaces generated by trial functions that satisfy the differential equation of the mode of vibration.

If continuum models are used to represent vibrations of structures, displacement fields are governed by partial differential equations instead of ordinary differential equations of discrete models (Clough and Penzien 1993, Rao 2007). In the case of constant parameter conservative models, the computation of natural frequencies using the method of separation of variables leads to the solution of a nonlinear eigenvalue problem of the type

$$\mathbf{B}_{\mathrm{c}}(\omega)\mathbf{c} = 0 \tag{6}$$

where the square matrix $\mathrm{B_c}$ is derived from the boundary conditions of the model (Clough and Penzien 1993; Rao 2007; Lee 2009; Inaudi and Matusevich 2006) and the vector c contains the participation coefficients of special displacement fields that satisfy the modal differential equation. Natural frequency computation requires the search of values of frequency that make matrix $\mathrm{B_c}$ singular.

An alternative for continuum model analysis is the assembly of the dynamic stiffness matrix in the frequency domain (Clough and Penzien 1975; Leung 1983; Preziemienieki 1985; Yu and Roesset 2001; Lee 2009) . The computational effort to find and compute natural frequencies as singularities of the dynamic stiffness matrix or singularities of the boundary condition matrix is the main disadvantage of these approaches when compared with standard discrete FEM.

The basic idea in this paper is to compute the Rayleigh quotient in continuum models with frequency-dependent shape functions to provide an improved-precision estimation of natural frequencies. Rather than using subspaces with fixed basis to approximate the continuum fields using a linear combination of trial functions as conventional FEM does, the proposed method uses only frequency dependent trial functions parameterized by a frequency variable. At each step, shape functions are updated and the natural frequency estimation is improved using the Rayleigh Quotient. The convergence and accuracy of the proposed method is analysed using examples of straight rods in longitudinal vibration and beams in flexural vibrations.

The paper is organized as follows. First, the proposed technique is explained using axial vibration of constant parameter rods and beams in flexure as application examples. Next, the convergence and regions of convergence of the technique are investigated for continuum models containing different number of frequency dependent functions and different boundary conditions. Finally, the main results of this piece of research are presented in a discussion and a brief conclusion.

2 RAYLEIGH QUOTIENT ITERATION IN CONTINUUM MODELS

In this section, the convergence of the proposed iterative procedure for computation of natural frequencies using frequency-dependent shape functions is analyzed using continuum models with different boundary conditions. The proposed technique is explained using application examples of axial vibration of rods and flexural vibration of beams Even though the proposed methodology can be extended to other types of continuum models, the analytical simplicity of axial vibrations and flexural vibrations of straight frame elements is considered convenient for this introductory paper.

2.1 Longitudinal vibrations of a rod.

A single bar and two connected bars in axial vibrations are considered. Regions of convergence are explored by numerical simulation of the proposed algorithm for a range of initial natural frequency estimates.

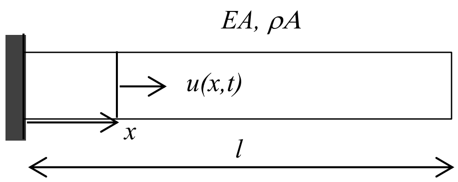

Figure 1: Continuous model of a rod in longitudinal vibration.

Using the separation of variable method, an analytical expression of the mode shape of a clamped-free rod in longitudinal vibration (see Figure 1) is obtained; this expression is used as trial function of the Rayleigh Quotient of the continuum model for frequency estimation.

The axial displacement field of a portion of a rod in free longitudinal vibration is governed by the following partial differential equation:

$$\frac{\partial}{\partial x}\left[EA\frac{\partial u(x,t)}{\partial x}\right] - \rho A\frac{\partial^2 u(x,t)}{\partial x^2} = 0 \tag{7}$$

In Eq. (6), (x, t) is the axial displacement field, E and ρ are the Young modulus and mass density of the rod material, respectively, and A is the cross-sectional area. In a domain with constant parameters, the natural mode of vibration of the rod satisfies

$$\frac{d^2\varphi(x)}{dx^2} + \frac{\omega^2\rho}{E}\varphi(x) = 0 \tag{8}$$

where $\varphi(x)$ is a natural mode shape and ω a natural frequency of the structure. The general solution of this equation can be expressed as

$$\varphi(x) = c_1 \sin\left(\sqrt{\frac{\omega^2\rho}{E}}x\right) + c_2 \cos\left(\sqrt{\frac{\omega^2\rho}{E}}x\right) \tag{9}$$

The boundary conditions for the model in Fig. 1 require

$$\varphi(0) = 0 \tag{10}$$

$$EA\frac{d\varphi}{dx}(l) = 0 \tag{11}$$

In standard modal analysis, Eqs. (10) and (1) are expressed as the boundary matrix, B_c, times the vector of coefficients, c, as follows:

$$B_c\,\mathbf{c} = \begin{bmatrix} 0 & 1 \\ EA\sqrt{\frac{\omega^2\rho}{E}}\cos\left(\sqrt{\frac{\omega^2\rho}{E}}l\right) & -EA\sqrt{\frac{\omega^2\rho}{E}}\sin\left(\sqrt{\frac{\omega^2\rho}{E}}l\right)\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \tag{12}$$

The computation of natural frequencies requires the search for singularities of B_c as a function of ω, The determinant of this matrix set equal to zero provides the characteristic equation for a root search:

$$\det(\mathbf{B}_c) = EA\sqrt{\frac{\omega^2\rho}{E}}\cos\left(\sqrt{\frac{\omega^2\rho}{E}}l\right) = 0 \tag{13}$$

In this case, an analytical expression of natural frequencies can be obtained

$$\omega_n\sqrt{\frac{\rho}{E}}l = n\frac{\pi}{2} \qquad n = 1,2,3,\ldots \tag{14}$$

The corresponding modes of vibration can be resolved from Eqs. (9) and (12). In this case, letting $c_1 = 1$,

$$\varphi_n(x) = \sin\left(\sqrt{\frac{\omega_n^2\rho}{E}}x\right) \tag{15}$$

For other boundary conditions or in the case of model including several continuum elements, the search of singularities of B_c requires a numerical iterative search because analytical expressions cannot be obtained for the singularities.

Alternatively, let us formulate the problem using frequency dependent interpolating functions. The displacement field is expressed as a function of the estimated frequency $\hat{\omega}$

$$\varphi(x) = c_1\zeta_1(x) + c_2\zeta_2(x) = c_1 \sin\left(\sqrt{\frac{\hat{\omega}^2\rho}{E}}x\right) + c_2\cos\left(\sqrt{\frac{\hat{\omega}^2\rho}{E}}x\right) \tag{16}$$

For the proposed set of frequency-dependent shape functions (Eq. 16), the mass matrix of the rod element can be computed as

$$M_{ij} = \rho A\int_0^l \zeta_i(x)\zeta_j(x)dx \tag{17}$$

Where the shape functions, $\zeta_j(x)$, are defined in Eq. (16). For this model, defining $\beta = \hat{\omega}\sqrt{\frac{\rho}{E}}$, the mass matrix can be computed as

$$\mathbf{M}_e = \rho A \begin{bmatrix} \frac{l}{2} - \frac{1}{4\beta}\sin(2\beta l) & \frac{1-\cos^2(\beta l)}{2\beta} \\ \frac{1-\cos^2(\beta l)}{2\beta} & \frac{2\beta l+\sin^2(2\beta l)}{4\beta} \end{bmatrix} \tag{18}$$

Similarly, the element stiffness matrix can be computed as

$$K_{ij} = EA\int_0^l \frac{d}{dx}\zeta_i(x)\,\frac{d}{dx}\zeta_j(x)dx \tag{19}$$

$$\mathbf{K}_e = EA \begin{bmatrix} \frac{\beta^2 l}{2} + \frac{\beta}{4}\sin(2\beta l) & -\frac{\beta}{4}(1-\cos(2\beta l)) \\ -\frac{\beta}{4}(1-\cos(2\beta l)) & \frac{\beta^2 l}{2} - \frac{\beta}{4}\sin(2\beta l) \end{bmatrix} \tag{20}$$

From the kinematic boundary condition, first row of B_c in Eq. (12), we can define $c_2 = 0$ and c_1 as an arbitrary constant. Thus, we can express the complete $\boldsymbol{c}$ vector as a linear function of a single coordinate c1

$$\mathbf{c} = \mathbf{L}\, c_1 = \begin{bmatrix}1\\0\end{bmatrix} c_1 \tag{21}$$

Therefore, for the single degree of freedom model defined by c_1, the Rayleigh quotient can be evaluated given an estimate of a particular natural frequency as

$$R(\hat{\omega}_n) = \frac{\mathbf{L}(\hat{\omega}_n)^T\mathbf{K}(\hat{\omega}_n)\mathbf{L}(\hat{\omega}_n)}{\mathbf{L}(\hat{\omega}_n)^T\mathbf{M}(\hat{\omega}_n)\mathbf{L}(\hat{\omega}_n)} \tag{22}$$

Where L (Eq. 21), $\mathbf{K} = \mathbf{K}_e$ (Eq. 20) and $\mathbf{M} = \boldsymbol{M}_e$ (Eq. 18) are evaluated as functions of the assumed or estimated natural frequency.

An improved estimate of the natural frequency is obtained as

$$\hat{\omega}_{n+1} = \sqrt{R(\hat{\omega}_n)} \tag{23}$$

The recursive application of Eqs.(12), (21), (22) and (23) to estimate a specific natural frequency and convergence regions of the initial frequency estimate are investigated numerically. The following parameters are assumed: $E = 1$, , $\rho = 1$, $A = 1$. The tolerance criterion to define convergence in natural frequency is $|\hat{\omega}_{n+1} - \hat{\omega}_n| < 0.001$. Figure 2 shows the convergence of the algorithm for several initial estimates of natural frequencies. The x-axis represents the n-th iteration step and y-axis represents natural frequency estimate in dot symbol. The initial natural frequency estimate is presented for $n = 0$. The exact natural frequencies, to which the iteration converges, are presented in continuum horizontal lines. The figure shows that for all values of initial estimates, the algorithm converges to a natural frequency of the model. The speed of convergence is dependent on the absolute value of the difference between the initial estimate and the natural frequency to which the algorithm converges. The speed of convergence reduces significantly in the case of initial frequency selected close to the boundary of the interval of convergence of a particular natural frequency.

Figure 3 shows the behavior of the algorithm in one step. The x-axis shows the value of frequency estimate at step n and the y-axis shows the value of the natural frequency estimate of the algorithm at step n+1 (solid line). The dashed line corresponds to $\hat{\omega}_{n+1} = \hat{\omega}_n$. The o symbols represent the natural frequencies of the model. The regions of convergence of initial estimates to each natural frequency of the model of the algorithm can be identified in Fig 3; the values of frequency for which the solid line intersects the dashed line with positive slope define the boundary values for each convergence regions. As the figure shows, there exist regions (intervals) of convergence for each natural frequency. Provided the initial estimate of the natural frequency belongs to this convergence interval, the recursive application of the algorithm will converge to the corresponding natural frequency with any desired accuracy level (fixed point of the algorithm).

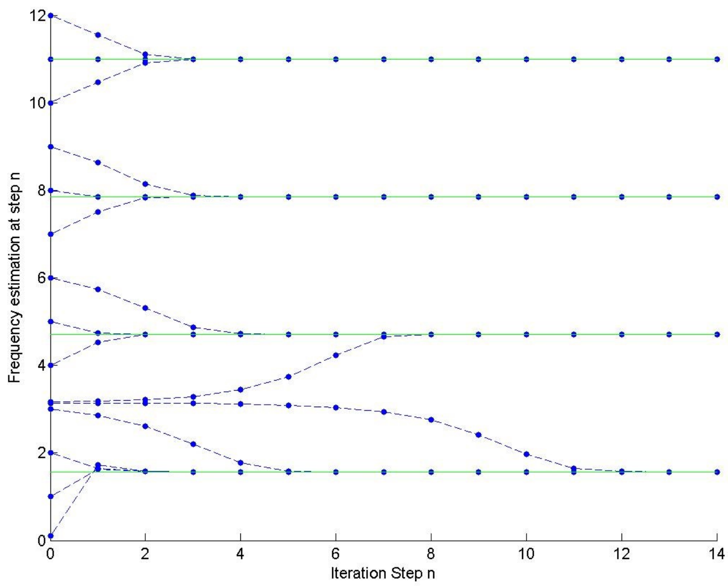

Figure 2: Convergence of Rayleigh quotient iteration method. Clamped-free rod of parameters: $E = 1$, $A = 1$, $\rho = 1$, and $l = 1$.

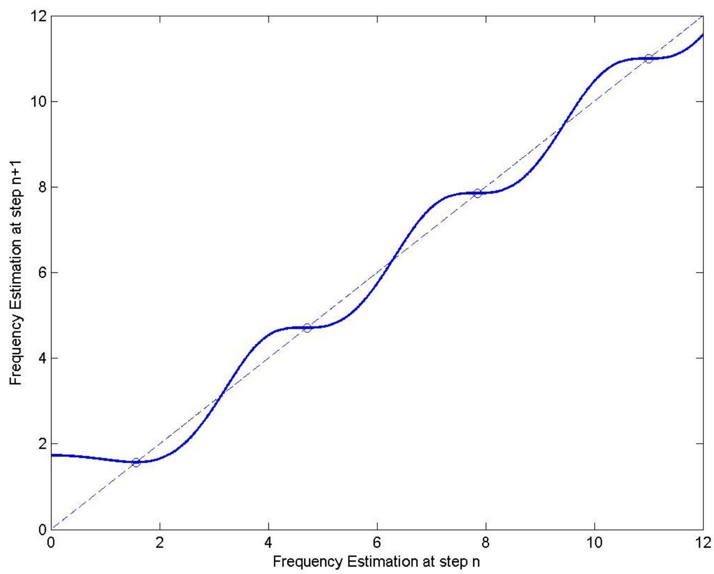

Figure 3: One step frequency estimate. Clamped-free rod of parameters: $E = 1$, $A = 1$, $\rho = 1$, and $l = 1$.

Figure 4 presents, in solid line, the quotient $\frac{\hat{\omega}_{n+1}}{\hat{\omega}_n}$ as a function of $\hat{\omega}_n$. The "o" symbols represent the natural frequencies of the model, attraction points or fixed points of the algorithm. These are stable points of the algorithm because a small perturbation around this value of $\hat{\omega}_n = \omega_j + \varepsilon$ for $j = 1,2,3, \ldots$, produces a value of $\hat{\omega}_{n+1}$ closer to a natural frequency, ω_j. On the other hand, the values of frequency $\hat{\omega}_n$ of intersection with positive slope of the solid line with the dashed line $\frac{\widehat{\omega}_{n+1}}{\widehat{\omega}_n} = 1$ define the boundaries of convergence regions of initial estimates. These are unstable points of the algorithm because positive perturbations around these values of $\widehat{\omega}_n$ produce values of $\widehat{\omega}_{n+1}$ larger than $\widehat{\omega}_n$, and a negative perturbation around these values of $\widehat{\omega}_n$ produce a value of $\widehat{\omega}_{n+1}$ even smaller than $\widehat{\omega}_n$, indicating divergence.

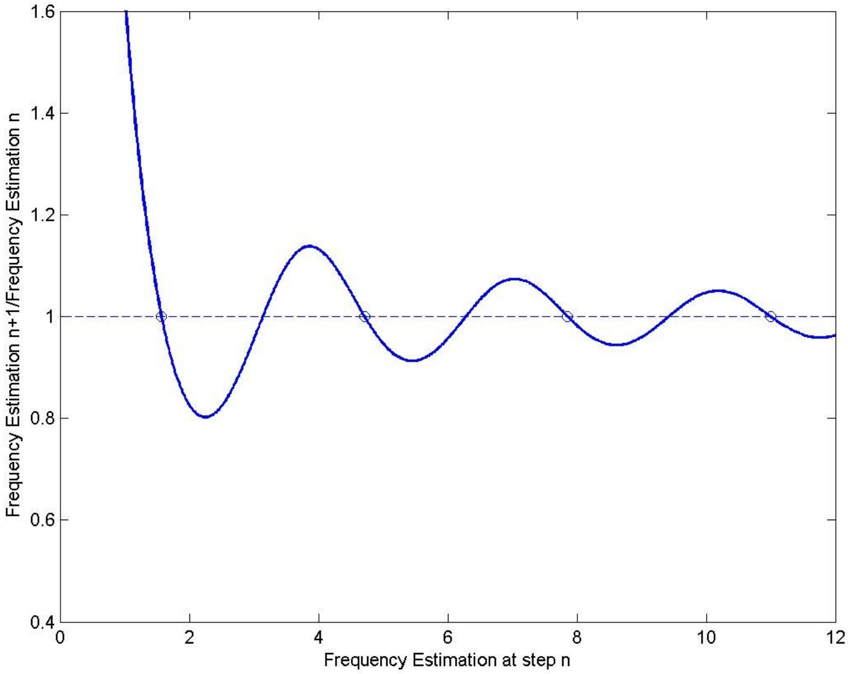

Figure 4: One step frequency estimate ratio. Clamped-free rod of parameters: $E = 1$, $A = 1$, $\rho = 1$, and $l = 1$. x-coordinates of circles ('o' symbols) indicate natural frequencies of the model.

The application of a method to a bar in longitudinal vibrations with a fixed support at the left end and a linear spring at the other end ($x = l$) is developed. The assumed parameters in the numerical example are: $E = 1$, $A = 1$, $\rho = 1$, $l = 1$ and $k = 1$. For comparison, exact natural frequencies are computed from the singularities of the corresponding $\mathrm{B_c}$ matrix:

$$\mathbf{B}_c = \begin{bmatrix} 0 & 1 \\ \mathrm{EA}\sqrt{\frac{\omega^2\rho}{\mathrm{E}}}\cos\left(\sqrt{\frac{\omega^2\rho}{\mathrm{E}}}\mathrm{l}\right) + \mathrm{k}\sin(\sqrt{\frac{\omega^2\rho}{\mathrm{E}}}\mathrm{l}) & -\mathrm{EA}\sqrt{\frac{\omega^2\rho}{\mathrm{E}}}\sin(\sqrt{\frac{\omega^2\rho}{\mathrm{E}}}\mathrm{l}) + \mathrm{k}\cos(\sqrt{\frac{\omega^2\rho}{\mathrm{E}}}\mathrm{l}) \end{bmatrix} \quad (24)$$

From the first boundary condition

$$\mathbf{c} = \mathbf{L}\,\mathrm{c}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\mathrm{c}_1 \quad (25)$$

The mass of the model is defined in Eq. (18). The stiffness matrix associated to coordinates c, can be expressed as

$$\mathbf{K}_e = \mathrm{EA}\begin{bmatrix} \frac{\beta^2 \mathrm{l}}{2} + \frac{\beta}{4}\sin(2\beta\mathrm{l}) & -\frac{\beta}{4}(1-\cos(2\beta\mathrm{l})) \\ -\frac{\beta}{4}(1-\cos(2\beta\mathrm{l})) & \frac{\beta^2 \mathrm{l}}{2} - \frac{\beta}{4}\sin(2\beta\mathrm{l}) \end{bmatrix} + \mathrm{k}\begin{bmatrix} \sin^2(\beta\mathrm{l}) & \sin(\beta\mathrm{l})\cos(\beta\mathrm{l}) \\ \sin(\beta\mathrm{l})\cos(\beta\mathrm{l}) & \cos^2(\beta\mathrm{l}) \end{bmatrix} \quad (26)$$

Therefore, the Rayleigh quotient is

$$R = \frac{\mathbf{L}^T\mathbf{K}\mathbf{L}}{\mathbf{L}^T\mathbf{M}\mathbf{L}} = \frac{EA\left(\frac{\beta^2 l}{2}+\frac{\beta}{4}\sin(2\beta l)\right)+k\sin(\beta l)^2}{\rho A(\frac{l}{2}-\frac{1}{4\beta}\sin(2\beta l))} \tag{27}$$

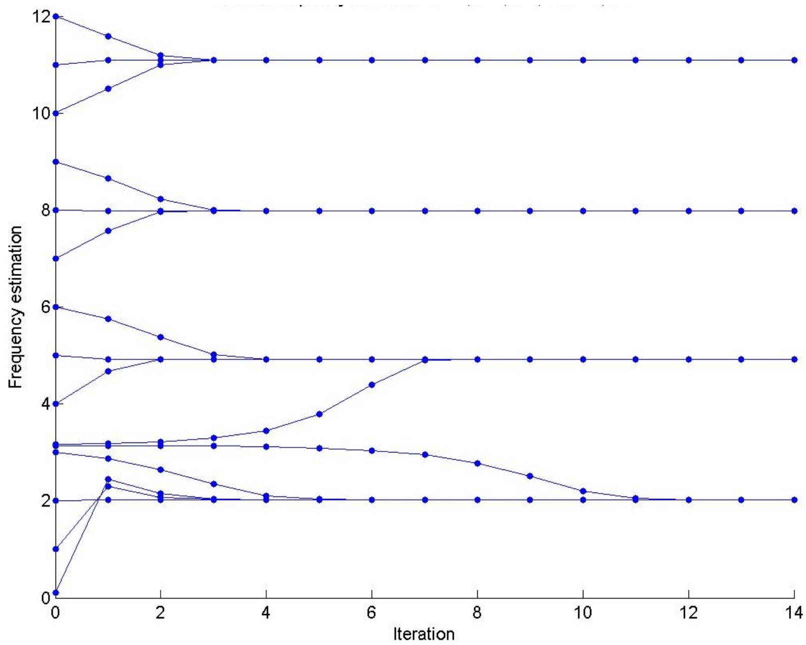

Figure 5: Convergence of Rayleigh quotient iteration method for bar with fixed left end and axial spring at right end. Parameters: $E = 1$, $A = 1$, $\rho = 1$, $l = 1$ and $k = 1$.

Figure 5 shows the convergence to the natural frequencies, of the proposed algorithm for several initial frequency estimates. Unless the initial estimate is selected very close to the boundary of a particular convergence region, the application of four or five steps of the proposed algorithm provides a precise estimation of natural frequency and mode shape. As the figure shows, the selection of an initial estimate close to the boundary of a convergence region may require twelve or more steps to achieve a satisfactory precision.

In the following another structural model is analyzed, applying the algorithm to a clamped-clamped rod. The model in Figure 1 is considered but with both ends fixed. The boundary conditions in this case are

$$\varphi(0) = 0 \tag{28}$$

$$\varphi(l) = 0 \tag{29}$$

Therefore,

$$\mathbf{B}_c = \begin{bmatrix} 0 & 1 \\ \sin\left(\sqrt{\frac{\omega^2\rho}{E}}l\right) & \cos\left(\sqrt{\frac{\omega^2\rho}{E}}l\right) \end{bmatrix} \tag{30}$$

The proposed algorithm requires the use of shape functions that satisfy all geometric boundary conditions. Because in this model all boundary conditions are geometric in nature (restrained displacement), the selection of either c_1 or c_2 does not allow the satisfaction of all geometric boundary conditions. To circumvent this limitation, the model can be conceived as two rods of identical properties connected as shown in Figure 6, assuming $E_1 = E_2$, $A_1 = A_2$, , $\rho_1 = \rho_2$, and $l_1 + l_2 = 1$. The order of the model augments but the boundary conditions now include both geometric and natural boundary conditions, allowing the application of the algorithm.

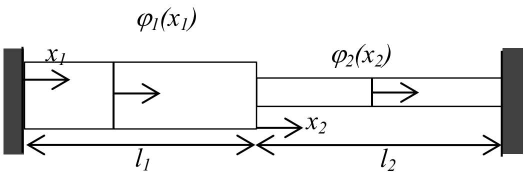

Figure 6: Continuous model of a rod in longitudinal vibration.

As an additional example, this section focuses in the application of proposed algorithm to a structural model consisting of more than one element: two straight bars connected to each other and fixed at left and right ends, respectively. The properties of each bar can be identical or different. In this example, the model requires the consideration of two shape functions for each element and four coefficients in vector $\boldsymbol{c}$.

The solution of this equation in each subdomain j of the rod with constant parameters can be expressed as:

$$\varphi_j(x_j) = c_{j1} \sin\left(\sqrt{\frac{\omega^2 \rho_j}{E_j}} x_j\right) + c_{j2} \cos\left(\sqrt{\frac{\omega^2 \rho_j}{E_j}} x_j\right) \tag{31}$$

The boundary conditions for the model in Figure 6 are

$$\varphi_1(0) = 0 \tag{32}$$

$$\varphi_2(l_2) = 0 \tag{33}$$

$$\varphi_1(l_1) - \varphi_2(0) = 0 \tag{34}$$

$$E_1 A_1 \frac{d}{dx} \varphi_1(l_1) - E_2 A_2 \frac{d}{dx} \varphi_2(0) = 0 \tag{35}$$

The assembly of the corresponding boundary condition matrix for this model leads to

$$\mathbf{B}_c\, \mathbf{c} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \sin\left(\sqrt{\frac{\omega^2\rho_2}{E_2}} l_2\right) & \cos\left(\sqrt{\frac{\omega^2\rho_2}{E_2}} l_2\right) \\ \sin\left(\sqrt{\frac{\omega^2\rho_1}{E_1}} l_1\right) & \cos\left(\sqrt{\frac{\omega^2\rho_1}{E_1}} l_1\right) & 0 & 1 \\ A_1\sqrt{\frac{\omega^2\rho_1}{E_1}} \cos\left(\sqrt{\frac{\omega^2\rho_1}{E_1}} l_1\right) & -E_1 A_1 \sqrt{\frac{\omega^2\rho_1}{E_1}} \sin\left(\sqrt{\frac{\omega^2\rho_1}{E_1}} l_1\right) E_1 & E_2 A_2 \sqrt{\frac{\omega^2\rho_2}{E_2}} & 0 \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{12} \\ c_{21} \\ c_{22} \end{bmatrix} = \mathbf{0} \tag{36}$$

The values of ω that make singular this boundary condition matrix Bc are the natural frequencies of the model. To apply the Rayleigh Quotient iteration method to search for a particular natural frequency, we select c11 as degree of freedom,

$$\mathbf{c} = \mathbf{L}\, c_{11} \tag{37}$$

and express vector $\boldsymbol{c}$, using the geometric boundary conditions (rows 1 through 3 in Eq. 36),

$$\mathbf{L} = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\left(\sqrt{\frac{\hat{\omega}^2\rho_2}{E_2}}l_2\right) & \cos\left(\sqrt{\frac{\hat{\omega}^2\rho_2}{E_2}}l_2\right) \\ \cos\left(\sqrt{\frac{\hat{\omega}^2\rho_1}{E_1}}l_1\right) & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \sin\left(\sqrt{\frac{\hat{\omega}^2\rho_1}{E_1}}l_1\right) \end{bmatrix} \tag{38}$$

The Rayleigh quotient for the model can be computed using Eqs. (22) and (23), with matrix L given by Eq. (38) and the assembled mass and stiffness matrices for the model

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_1 & \mathbf{O} \\ \mathbf{O} & \mathbf{M}_2 \end{bmatrix} \qquad \mathbf{K} = \begin{bmatrix} \mathbf{K}_1 & \mathbf{O} \\ \mathbf{O} & \mathbf{K}_2 \end{bmatrix} \tag{39}$$

In Eq. (39), $\mathbf{M}_1$ and $\mathbf{M}_2$ are the mass matrices of elements 1 and 2, respectively, that can be computed with Eq. (18) using the length, cross sectional area and density of each element. $\mathbf{K}_1$ and $\mathbf{K}_2$ are the stiffness matrices of elements 1 and 2, respectively, that can be computed using Eq. (20) using the corresponding element properties.

Once convergence has been achieved to a particular natural frequency and taking the selected degree of freedom with a unit value (Eq. 37), the corresponding mode of vibration is determined by the coefficients that represent the displacement field (Eq. 9 with parameters E, A, ρ and l of the corresponding element), elements of vector L.

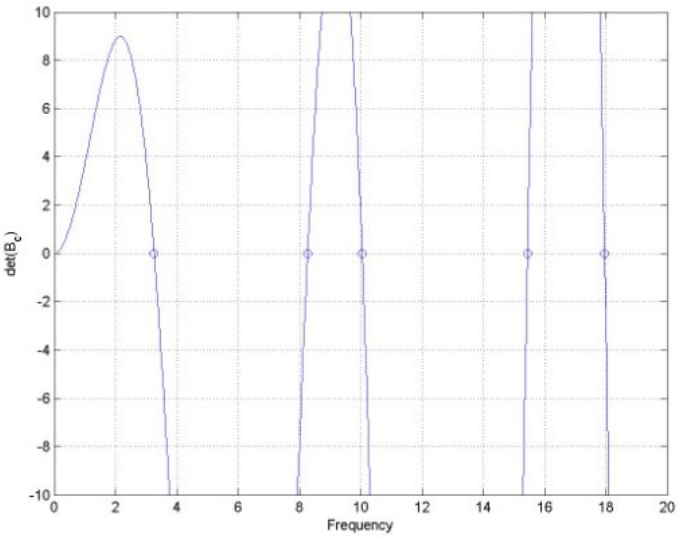

Figure 7: Continuous model of a two connected rods in longitudinal vibration. Model fixed in left and right ends. Parameters: $E_1 = 2,\ E_2 = 1, A_1 = 1,\ A_2 = 10,$, $\rho_1 = \rho_2 = 1$ $l_1 = l_2 = 0.5$.

To analyze the constant parameter fixed-fixed bar, the following parameters are assumed: $E_1 = E_2 = 1$, $A_1 = A_2 = 1$, , $\rho_1 = \rho_2 = 1$, and $l_1 = l_2 = 0.5$. The algorithm converges to natural frequencies, but not to all natural frequencies. Only odd numbered frequencies are estimated (ω_1, ω_3,…), even though random initial estimates close to even numbered frequencies are considered. This is the first case in which the proposed algorithm shows convergence problems in this investigation. This problem was not solved changing the c coefficient defined as master degree of freedom for the computation of L; other selections provided the same results: convergence to symmetric modes of the model was obtained (ω_1, ω_3,…),, but anti-symmetric modes were not detected by the algorithm (ω_2, ω_4,…),. The condition of equal length bar elements was modified to analyze the behavior of the proposed algorithm. Using $l_1 = 0.3l$ and $l_2 = 0.7l$ or any other pair of lengths summing the total length l, both symmetric and anti-symmetric modes were computed by the algorithm.

To analyze the convergence of the Rayleigh quotient algorithm in a new study case with different material properties and sections of the connected bars (see Figure 6), we consider the following parameters, $E_1 = 2,\ E_2 = 1,\ A_1 = 1,\ A_2 = 10,\ ,\ \rho_1 = \rho_2 = 1\quad l_1 = l_2 = 0.5$. The tolerance criterion selected to define convergence to natural frequency is $|\widehat{\omega}_{n+1} - \widehat{\omega}_n| < 0.001$. As Figure 7 shows, the application of the algorithm converges to estimates of natural frequencies; the computed natural frequencies showed in 'o' symbol are precisely the roots of the determinant of the boundary matrix presented in solid line.

The algorithm shows convergence to all natural frequencies, provided the initial frequency guess belongs to the region of convergence that includes the fixed point of the sequence (natural frequency of interest).

2.2 Beam element in flexural vibrations.

In this section, the application of the proposed technique to beam vibrations is considered briefly. The idea is to prove that the proposed algorithm can be applied to other types of continuum models. The transverse displacement field for a mode of vibration of a piece of beam element can be estimated as

$$\varphi_j(x) = c_{j1}\, sin(a\, x) + c_{j2}\, cos(ax) + c_{j3}\, sinh(ax) + c_{j4}\, cosh(ax) \quad = L_j(x) r_j \tag{40}$$

$$a = \sqrt[4]{\frac{\widehat{\omega}_i{}^2 \rho_j A_j}{E_j I_j}} \tag{41}$$

In Eq. (41), $\widehat{\omega}_i$ is the estimate of the i-th natural frequency of the model included in the proposed formulation. The assumed frequency-dependent functions correspond to the exact mode shapes of constant parameter Bernoulli beams in flexure (Clough and Penzien, 1975).

The computation of mass and stiffness matrices associated to vector c can be obtained analytically using the corresponding integral expressions. Appendix I shows the stiffness and mass matrices of a beam element associated to the generalized coordinates $\boldsymbol{c}$ (interpolation defined in Eq. (40)).

In the application of the proposed technique to frame structures, the continuity of transverse displacements and rotations are enforced imposing adequate conditions on the displacement field and rotation fields (first derivative of displacement field) at the boundaries of adjacent elements. Additional constraints are imposed by structural boundary conditions in terms of displacements, rotations, and or moment and shear equilibrium at both ends, to form the corresponding rows to the boundary condition matrix $\mathbf{B}_c$.

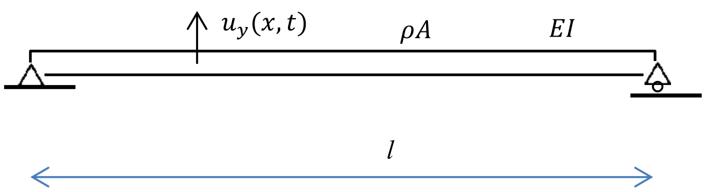

Figure 8: Simply supported beam in flexural vibration.

In an analogous way, as in the case of bars in longitudinal vibrations, the iterative computation of a natural frequency from an initial estimate can be performed using the

Rayleigh quotient algorithm presented in Eqs. (21) and (22) computing **L** vector from the boundary condition matrix evaluated at the frequency estimate at each algorithm step.

Consider as an application example, the simple supported beam shown in Figure 8. The parameters are E = Young modulus, A = cross sectional area, ρ = mass density, I = second moment of inertia of the cross section, and l = length of the beam.

In this model, the boundary conditions are

$$\varphi(0) = 0 \tag{42}$$

$$\varphi(l) = 0 \tag{43}$$

$$EI\varphi''(0) = 0 \tag{44}$$

$$EI\varphi''(l) = 0 \tag{45}$$

For brevity, the expression of the corresponding four by four boundary condition matrix is omitted. The parameters were selected as $E = 1$, $A = 1$, $\rho = 1$, $l = 1$ and $I = 1$. Selecting c_1 as degree of freedom and using the first three boundary conditions for the computation of $\boldsymbol{L}$ vector, the proposed method is applied starting in a range of random initial estimates of natural frequency in the interval [0 40]. For all initial estimates the convergence is correct to the first two natural frequencies included in the interval, as shown in Figure 9 (frequencies estimated by the application of the proposed algorithm are shown in 'o' symbols). The solid line is the value of the determinant of the boundary condition matrix as a function of frequency.

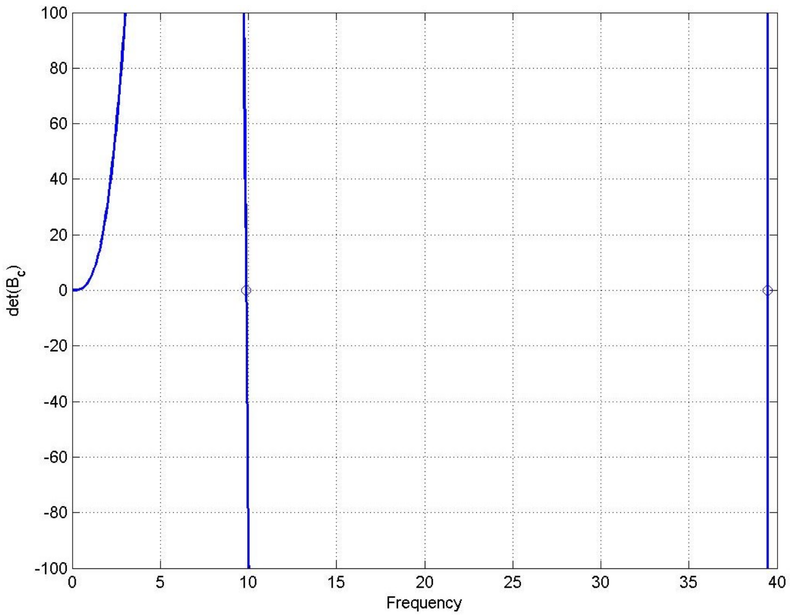

Figure 9: Convergence to natural frequencies of a pinned-pinned beam model.
Parameters: E=1, A=1, ρ=1, l=1 and I=1.

Consider now a clamped-clamped beam. Because in this case all four boundary conditions are kinematic if a single beam element is used, the proposed method cannot be applied because there are no natural boundary conditions to neglect to solve for **L** vector. To circumvent this problem a two-element model is created for the frequency search as shown in Figure 10 in an analogous way as that followed in the case of a bar in longitudinal vibration. The mode shape is expressed by two functions, $\varphi_1(x_1)$ and $\varphi_2(x_2)$, according to the four coordinate model given in Eq. (40) for each subdomain with lengths $l_1 = l_2 = l/2$.

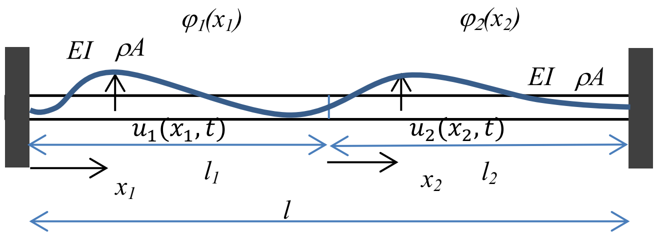

Figure 10: Clamped-clamped beam.

The corresponding eight boundary conditions are:

$$\varphi_1(0) = 0 \tag{46}$$

$$\varphi_1'(0) = 0 \tag{47}$$

$$\varphi_2(l/2) = 0 \tag{48}$$

$$\varphi_2'(l/2) = 0 \tag{49}$$

$$\varphi_1(l/2) - \varphi_2(0) = 0 \tag{50}$$

$$\varphi_1'(l/2) - \varphi_2'(l/2) = 0 \tag{51}$$

$$EI\varphi_1''(l/1) - EI\varphi_2''(0) = 0 \tag{52}$$

$$EI\varphi_1'''(l/2) - EI\varphi_2'''(0) = 0 \tag{53}$$

In these equations, the prime indicates differentiation with respect to x. Selecting c_3 of the left element as degree of freedom, and neglecting the shear continuity boundary condition Eq. (53) from the boundary matrix to compute **L**, the natural frequencies are estimated from random initial estimates in the interval [0 120] and parameters $E = 1$, $A = 1$, $\rho = 1$; $l = 2$ ($l_1 = 1$, l_2=1) and $I = 1$. The algorithm converges, but not to every natural frequency. Only odd numbered frequencies are estimated (ω_1, ω_3 …), even though random initial estimates close to even-numbered frequencies are considered. This is the same situation observed in the case of a fixed-fixed bar in longitudinal vibrations. Again, this problem was not solved changing the selected c coefficient for the computation of **L**. Other selections provided the same results: convergence to symmetric modes of the model was obtained (ω_1, ω_3 …), but anti-symmetric modes were not detected by the algorithm ω_2, ω_4 …). The condition of equal length beam elements was modified to analyze the behavior of the proposed algorithm. Using $l_1 = 0.3l$, $l_2 = 0.7l$, and the same parameters of the beam elements, both symmetric and anti-symmetric modes were computed by the algorithm

As the previous example shows, the proposed algorithm may show lack of convergence to certain frequencies in special parameter cases. The symmetry of the model and mode shapes seems to be the cause of this lack of convergence. This conjecture requires in deep investigation to be proved true. Although special cases may be found, the use of non-symmetric models with random subdomains lengths may guarantee convergence of the algorithm to all natural frequencies. This concept and the convenience in terms of convergence properties of refined mesh models of the continuum models require further research to be pursued in the future.

3 IMPLEMENTATION OF THE ALGORITHM

As all application examples illustrate, the RQ algorithm converges for any initial natural frequency estimate. The implementation of the proposed method requires (at each iteration):

- An estimate of natural frequency, $\widehat{\omega}_n$
- Assembly of mass and stiffness matrices of each continuous element
- The assembly of $\mathbf{B}_c$, $N_{B_c} \times N_{B_c}$ matrix, where N_{B_c} is the number of coefficients used in the continuum-model formulation.
- The selection of $N_{B_c} - 1$ rows of $\mathbf{B}_c$ that include all geometric boundary conditions
- The selection of one degree of freedom of the discrete model (a coefficient c_j that is selected equal to 1)
- The computation of the **L** vector that relates the selected degree of freedom with all the generalized coordinates of the model (vector $\boldsymbol{c}$) computed from the linear relation imposed by matrix $\mathbf{B}_c$.
- The assembly of mass and stiffness matrices for all generalized *c*-coordinates at the structure level
- The computation of $\mathbf{L^T K L}$ and $\mathbf{L^T M L}$ to compute the updated Rayleigh quotient, $R(\widehat{\omega}_n) = \frac{\mathbf{L}(\widehat{\omega}_n)^\mathbf{T}\mathbf{K}(\widehat{\omega}_n)\mathbf{L}(\widehat{\omega}_n)}{\mathbf{L}(\widehat{\omega}_n)^\mathbf{T}\mathbf{M}(\widehat{\omega}_n)\mathbf{L}(\widehat{\omega}_n)}$
- The computation of the updated frequency estimate $\widehat{\omega}_{n+1}$ as the square root of the Rayleigh quotient, $\widehat{\omega}_{n+1} = \sqrt{R(\widehat{\omega}_n)}$.
- The verification of convergence to a natural frequency and mode shape or the need of applying an additional iteration. The convergence of the recurrent application of this procedure to a natural frequency can be detected by using an adequate tolerance in the norm of the difference between two successive frequency estimates or other convergence criterion using the estimated mode shape at two successive iteration steps; for example, the angle between mode estimates at two consecutive steps.

An initial estimate of natural frequency is needed to evaluate frequency-dependent shape functions at start. This estimate can be obtained with a conventional FEM model or other methods. It is worth noting that the computation of mass and stiffness matrices must be updated at each iteration step, to estimate the natural frequency from the Rayleigh quotient. Matrix $\mathbf{B}_c$ is also updated at each iteration step, and the computation of vector **L** (representing the estimate of natural mode of vibration) requires the solution of a linear set of equations at each iteration step.

The linear set of equations to be solved is of order $N_{B_c} - 1$. For example, in a constant parameter beam element in planar flexural vibration, 4 coefficients are required. If the model includes N_e frame elements in planar vibration, $N_{B_c} = 4N_e$. A beam element with flexural vibrations in two orthogonal planes, axial vibration and torsional vibration, includes $4 + 4 + 2 + 2 = 12$ generalized coordinates per element, therefore a structure model with N_e three-dimensional beam elements, would require a boundary condition matrix of size $12N_e \times 12N_e$.

In the case of frame or truss structures with arbitrary geometry, additional parameters (such as nodal displacements) can be added to the $\mathbf{c}$ vector with its corresponding kinematic condition (Matusevich et al., 2008), leading to a larger B_c. This matrix will allow the computation of any number of natural frequencies and mode shapes and would provide exact values of these frequencies of the continuum model. The assembly of mass and stiffness matrices associated to the frequency-dependent shape functions for each class of element is required to compute the Rayleigh quotient at each iteration. This procedure can be programmed to automate the proposed algorithm for the modal analysis of any structure composed by continuum parameter model elements.

The proposed method is an efficient alternative for exact natural frequency computation of continuum models suitable for parallel processing because each frequency search can be started simultaneously and dos not require any information of other natural frequencies or mode shapes. The method can estimated any number of natural frequencies and mode shapes without mesh refinement and is a stable convergent algorithm for any frequency.

4 CONCLUSIONS

The application of the Rayleigh quotient for the iterative computation of natural frequencies and mode shapes of continuum models has been presented. The method uses frequency-dependent shape functions resulting from classical modal analysis obtained from separation of variables method applied to the partial differential equations governing each part of the model structure. The use of all but one linear boundary conditions of the continuum model that include all geometric boundary conditions, allows the expression of all modal components coordinates as a function of a single degree of freedom (one of the generalized coordinates used in the continuum model). Given a frequency estimate, an improved estimation of a natural frequency is computed by the square root of the Rayleigh quotient. The method computes a single frequency at a time and does not require mesh adaptation although it does require an iterative computation of mass and stiffness matrices and the mode update given by the computation of vector L, because shape functions are adjusted as a function of natural frequency estimations done in the previous step. The initial estimations can be provided by a conventional low order FEM model with a regular mesh or other approximate methods.

The proposed method estimates natural frequencies efficiently up to any desired degree of accuracy in low and high frequencies, using a small mesh of continuum elements.

To illustrate the proposed method the methodology has been applied to rod elements in axial vibration and beam elements in flexural vibration. Using these examples, convergence to exact values has been demonstrated. The results show that the proposed method is an efficient alternative for exact natural frequency computation of continuum models suitable for parallel processing. One of the most convenient features of the proposed method is the possibility of estimating any number of natural frequencies and mode shapes without mesh refinement and using a stable convergent algorithm suitable for parallel processing.

Although the method has been presented for the analysis of structures with a few rod and/or frame elements, the method can be applied in an analogous manner to modal analysis of other types of elements and structures consisting of a large number of continuum elements. This extension of the method, the analysis of the speed of convergence and accuracy of the proposed adaptive method in the estimation of natural frequencies, and the conditions of lack of convergence to particular frequencies in certain models are aspects that require additional investigation that will be pursued by the author in future research work.

ACKNOWLEDGEMENTS

I specially thank Ariel Matusevich for his contribution in the edition of the paper and his valuable comments for improving the presentation of this paper.

APPENDIX I. Stiffness and mass matrix of beam elements in ***c*** coordinates.

The analytical expressions of the stiffness $\mathbf{K}_e$ and mass $\mathbf{M}_e$ matrices for a continuum constant parameter Bernoulli beam element model with **c** coordinates are included in this appendix.

$$\mathbf{K}_e = \begin{bmatrix} k_{ss} & k_{cs} & k_{ssh} & k_{sch} \\ k_{cs} & k_{cc} & k_{csh} & k_{cch} \\ k_{ssh} & k_{csh} & k_{shsh} & k_{chsh} \\ k_{sch} & k_{cch} & k_{schsh} & k_{chch} \end{bmatrix} \tag{I.1}$$

$$a = \sqrt[4]{\frac{\hat{\omega}_i{}^2 \rho_j A_j}{E_j I_j}} \tag{I.2}$$

$$k_{ss} = EIa^4\left(\frac{x}{2} - \sin(2ax)/(4a)\right)\Big|_0^l \tag{I.3}$$

$$k_{cc} = EIa^4(2\mathrm{a}x + \sin(2ax)/(4a))\Big|_0^l \tag{I.4}$$

$$k_{chch} = EIa^4\frac{1}{4\mathrm{a}}(\sinh(2ax) + 2\mathrm{a}x)\Big|_0^l \tag{I.5}$$

$$k_{chsh} = EIa^4\frac{1}{2\mathrm{a}}\cosh(2ax)^2\Big|_0^l \tag{I.6}$$

$$k_{shsh} = EIa^4\frac{1}{4\mathrm{a}}(\sinh(2ax) - 2\mathrm{a}x)\Big|_0^l \tag{I.7}$$

$$k_{cs} = EIa^4\left(-\frac{1}{2a}\right)\cos(ax)^2\Big|_0^l \tag{I.8}$$

$$k_{cch} = EIa^4\frac{1}{4\mathrm{a}}\frac{1}{2\mathrm{a}}(\cos(ax)\sinh(\mathrm{ax}) + \sin(\mathrm{ax})\cosh(\mathrm{ax}))\Big|_0^l \tag{I.9}$$

$$k_{csh} = EIa^4\frac{1}{4\mathrm{a}}\frac{1}{2\mathrm{a}}(\sin(ax)\sinh(\mathrm{ax}) + \cos(\mathrm{ax})\cosh(\mathrm{ax}))\Big|_0^l \tag{I.10}$$

$$k_{sch} = EIa^4\frac{1}{4\mathrm{a}}\frac{1}{2\mathrm{a}}(\sin(ax)\sinh(\mathrm{ax}) - \cos(\mathrm{ax})\cosh(\mathrm{ax}))\Big|_0^l \tag{I.11}$$

$$k_{ssh} = EIa^4\frac{1}{2\mathrm{a}}(\sin(ax)\cosh(\mathrm{ax}) - \cos(\mathrm{ax})\sinh(\mathrm{ax}))\Big|_0^l \tag{I.12}$$

$$\mathbf{M}_e = \begin{bmatrix} m_{ss} & m_{cs} & m_{ssh} & m_{sch} \\ m_{cs} & m_{cc} & m_{csh} & m_{cch} \\ m_{ssh} & m_{csh} & m_{shsh} & m_{chsh} \\ m_{sch} & m_{cch} & m_{schsh} & m_{chch} \end{bmatrix} \quad \text{(I.13)}$$

$$m_{cc} = \frac{\rho A}{4a}\,(2ax + \sin(2ax)/(4a))|_0^l \quad \text{(I.14)}$$

$$m_{ss} = \rho A\,(\frac{x}{2} - \sin(2ax)/(4a))|_0^l \quad \text{(I.15)}$$

$$m_{chch} = \rho A \frac{1}{4a}(\sin(2ax) + 2ax)|_0^l \quad \text{(I.16)}$$

$$m_{shsh} = \rho A \frac{1}{4a}(\sinh(2ax) - 2ax)|_0^l \quad \text{(I.16)}$$

$$m_{cch} = \rho A \frac{1}{2a}(\cos(ax)\sinh(\mathrm{ax}) + \sin(\mathrm{ax})\cosh(ax))|_0^l \quad \text{(I.17)}$$

$$m_{sch} = \rho A \frac{1}{2a}(\sin(ax)\sinh(\mathrm{ax}) - \cos(\mathrm{ax})\cosh(ax))|_0^l \quad \text{(I.18)}$$

$$m_{cs} = \rho A \frac{-1}{2\mathrm{a}}\cos(ax)^2|_0^l \quad \text{(I.19)}$$

$$m_{chsh} = \rho A \frac{1}{2\mathrm{a}}\cosh(ax)^2|_0^l \quad \text{(I.20)}$$

$$m_{ssh} = \rho A \frac{1}{2a}(\sin(ax)\cosh(\mathrm{ax}) - \cos(\mathrm{ax})\sinh(ax))|_0^l \quad \text{(I.21)}$$

$$m_{csh} = \rho A \frac{1}{2a}(\sin(ax)\sinh(\mathrm{ax}) + \cos(\mathrm{ax})\cosh(ax))|_0^l \quad \text{(I.22)}$$

REFERENCES

Babuska I. and Guo B.Q.: The h, p and h-p version of the finite element method: basis theory and applications, *Advances in Engineering Software*, Volume 15, Issue 3-4, 1992.

Clough R.W. and Penzien J., *Dynamics of Structures*, MacGraw-Hill, New York, 1975.

Clough R.W. and Penzien J., *Dynamics of Structures*, MacGraw-Hill, New York, 1993.

Chopra A., *Dynamics of Structures*, Prentice Hall. 1995.

Crandall, S. H., Iterative procedures related to relaxation methods for eigenvalue problems. *Proceedings of the Royal Society*. London. Series A. Mathematical, Physical and Engineering Sciences, 207:416–423, 1951.

Hodges, D.H, Improved approximations via Rayleigh's quotient, *Journal of Sound and Vibration*, 199(1), 155-164, 1997.

Inaudi J.A., Adaptive Frequency-Dependent Shape Functions for Accurate Estimation of Modal Frequencies of Frame Structures, *Journal of Engineering Mechanics*, Vol. 139, Issue 12 , pp. 1844-1855, 2013.

Lee, U., *Spectral Element Method in Structural Dynamics*, John Wiley & Sons, 2009.

Matusevich, A., and Inaudi J.A., A computational implementation of modal analysis of continuous dynamical systems, *International Journal of Mechanical Engineering Education*, 22(2), 215-234, 2005

Leung, A., Fast Convergence Modal Analysis for Continuous Systems, *Journal of Sound and vibration*, 87(3), 449-467,1983.

Matusevich, A., and Inaudi J.A. and Massa C. J., Computational formulation of boundary conditions in continuous dynamical systems (in Spanish), *Mecánica Computational Vol XXVI*, pp. 2693-2711, Asociación Argentina de Mecánica Computacional, 2007.

Przemieniecki, J.S., *Theory of Matrix Structural Analysis*, Dover Publications, New York, 1985.

Rao S.S., *Vibration of Continuous Systems*, John Wiley, 2007.

Wilson, E., Three dimensional static and dynamic analysis of structures, *SAP User Manual*, Computer & Structures, Berkeley, California, 2002.

Yu C.P. and Roesset J. M., Dynamic Stiffness Matrices for Linear Members with Distributed Mass, *Tamkang Journal of Science and Engineering*, Vol. 4, No. 4, pp. 253-264, 2001.