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ANALYSIS OF VIBRATION AND BUCKLING PROBLEMS IN NON-LOCAL BEAM THEORIES ACCOUNTING FOR PARAMETRIC UNCERTAINTIES

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Abstract. The main concern of this work is the study of the uncertainty quantification and its propagation in the static/dynamic response of nano-beams modeled according to first-order shear theories and whose constitutive equations follow the concepts of non-local elasticity. The non-local constitutive relationships in terms of strains and stresses have parameters that could be uncertain depending on the type of material and the constructive procedure of the micro-beam or nano-beam. The structure is conceived as a thin-walled beam with motion in three axes, i.e. bending in two directions and the axial motion and twisting. A deterministic model is developed for isotropic graded beams, and it is employed to calculate the mean measure of buckling loads and vibration patterns in the frequency domain. In order to calculate the propagation of the uncertainty in the response of the beam, the geometrical parameters, such as internal and external characteristic lengths, and/or the material parameter, such as Young modulus, among other, are considered uncertain and random variables are associated to them. The probability density functions of the random variables are obtained by means of the Maximum Entropy Principle according to the available (or assumed) information. The Monte Carlo method is employed to simulate a couple of buckling and vibration problems, whose equations are proposed to be solved in the context of the finite element method. A number of examples are presented in order to show the propagation of the uncertainty associated to the non-local parameters in the buckling response and the vibratory patters of the nano/microbeams.

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1 INTRODUCTION

The exponential growing of the miniaturization technology, its present uses and potential applications in a wide variety of high competitive industries, unleashed a broad new field of research in structural mechanics at a very small scale such as nano-tubes, nano-beams, nano-composites, among others (Murmu et al., 2011; Thai, 2012; Hemmatnezhad and Ansari, 2013). These nano-sized structures possess mechanical, chemical and electrical properties among others, that make them a good choice for sensing/actuating at quite small scales. The most renowned example of these structures is the carbon nano-tube (CNT). In structures at the nano-scale there are effects of intramolecular forces as well as the spatial size and distribution of the molecular microstructure, as it was explained in the works of Di Paola et al. (2011) or Peddison et al. (2003) that cannot be successfully tackled with classical continuum models (Murmu et al., 2011; Murmu and Adhikari, 2012). A detailed and accurate dynamic analysis of nano-structures can be performed with the so-called molecular dynamics approach (Murmu and Adhikari, 2012), however it has a high computational cost that makes it prohibitive for computationally demanding problems connected with optimization or uncertainty quantification among others. Taking into account these circumstances the conventional continuum mechanics models should be enhanced and empowered for analyzing such nano-structures but avoiding the scale or size effects. The non-local theory of an elastic continuum introduced in the seventies by Eringen (Reddy, 2007-2010; Peddieson et al. 2003) is an useful tool to overcome the problems of the prominent size effects of the nano-scale that conventional continuum theories cannot handle as shown by Peddison et al. (2003) and in the work of Reddy (2007) and Reddy (2010). In the non-local elasticity approach of Eringen it is proposed that the size-effects can be captured by assuming that the stresses at a given point are function of the strains of the whole domain (Eringen, 1972). The non-local theory of Eringen contains as a limit case the conventional local elasticity employed in classical structural mechanics (Peddison et al., 2003; Reddy, 2007).

Since the beginning of the 21^{th} century many authors have been introducing and/or developing models for beams, rods, bars, plates, etc. appealing to the non-local constitutive formulation of Eringen. In fact, Peddison et al. (2003) developed a simple non-local Bernoulli-Euler type beam model analyzing the dynamics of a nano-beam with emphasis in evaluating the influence of the non-local parameters. Reddy (2007), Reddy (2010), Lu et al. (2007), Wang et al. (2007) and Simsek and Yurtcu (2013) among others developed different studies for non-local Timoshenko beams. In these studies a broad range of free vibrations, static analysis and elastic stability problems has been covered; however, in all the aforementioned papers, only simple models of bending in one plane or twisting or extension were employed.

Apparently there is a lack of studies concerning the quantification of uncertainty in this type of structures, in spite of the difficulty evidenced in gathering experimental data for characterization of material properties and non-local elasticity parameters of real nano-beam. This motivates the present research. Thus, the aim of the paper is focused in the evaluation of the uncertainty propagation in a nano-beam composed by a isotropic single cell thin-walled structure. The beam model is derived employing the non-local theory of elasticity to cope with the effects of scale. The model incorporates extensional, twisting and bending equations accounting for shear deformation. Besides, a state of normal initial stresses is incorporated. This initial stress state is useful to evaluate buckling loads of typical cases.

There are many approaches for the study of the vibratory patterns of structures subjected to several aspects of uncertainty. One of them is associating random variables to given parameters considered uncertain, what is called parametric probabilistic approach (Soize, 2001; Ritto and

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R, 2012). The deduction of the probability density functions of the associated random variables is a crucial task. It needs some "a priori" information about the statistics of the parameters (e.g. expected value or bounds and/or coefficient of variation, etc.). The Maximum Entropy Principle (Jaynes, 2003; Shannon, 1948a) is employed to construct the stochastic model of the structure. Within this context, the probability density functions of random variables are deduced in order to guarantee that they achieve the maximum uncertainty which is measured in terms of entropy defined according to Shannon (1948b). This approach is the one employed in the present paper. The material properties of the nano-beam are considered random. The non-local parameter and a geometric parameter for the location of the instability load are also considered uncertain. The deterministic and probabilistic approaches of the nano-beam are formulated in the continuum and then discretized by means of the finite element method. The Monte Carlo Method is used to obtain the statistics of the dynamic response associated to a number of independent simulations.

The article is arranged in the following form: as a first task the deterministic (or mean model) is developed in the context of linear elasticity. Then, employing the non-local constitutive formulation the non-local equations of motion are deduced. Then, the corresponding variational form is retrieved to develop the finite element formulation of the non-local beam. As the last step of the theoretical handwork the probabilistic approach is developed in the context of the previously introduced finite element formulation. Finally a couple of examples are presented in order to evaluate the propagation of uncertainty in the mechanics od nano-beams.

2 DERIVATION OF THE MEAN MODEL

2.1 Basic hypotheses and kinematic relations

In the Fig. 1 a basic sketch of a single walled nano-tube with circular configuration is shown. The centroidal reference system and the other geometric properties are also depicted. To construct the mean or deterministic model the following hypotheses are invoked

- The model is developed in the context of small displacements of linear elasticity.
- The constitutive relations correspond to the Eringen's non-local formulation.
- The beam may be conceived for both isotropic or functional graded materials.
- The beam is subjected to bending motion in two orthogonal direction, extension and twisting motions as well.
- The annular cross section deforms according to classical beam theories.
- Initial normal stresses are taken into account through the .

Then, according to the previously formulated hypotheses, the following displacement field can be conceived:

$$\bar{U}_{\boldsymbol{P}} = \left\{ \begin{array}{c} u_x \\ u_y \\ u_z \end{array} \right\} = \left\{ \begin{array}{c} u_{xc} \\ u_{yc} \\ u_{zc} \end{array} \right\} + \left[\begin{array}{ccc} 0 & -\theta_z & \theta_y \\ \theta_z & 0 & -\theta_x \\ -\theta_y & \theta_x & 0 \end{array} \right] \left\{ \begin{array}{c} 0 \\ y \\ z \end{array} \right\} \tag{1}$$

where: u_x, u_y and u_z are the displacement coordinates of a generic point $\boldsymbol{P}$ in the nano-beam's domain, whereas u_{xc} is the axial displacement of the reference system u_{yc}, and u_{zc} are the lateral

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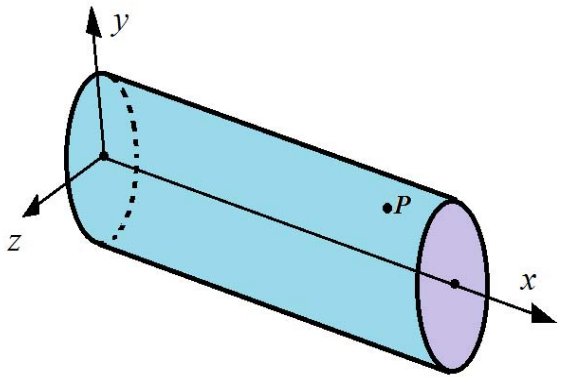

Figure 1: Sketch of the single walled nano-tube.

displacements of the reference system, θ_x is the twisting angle and θ_y and θ_z are the bending rotation parameters.

In the present article, the most representative linear (ε_{xx}, γ_{xy}, γ_{xz}) and second order (η_{xx}) strain components are:

$$\begin{aligned}
\{\varepsilon_{xx}\} &= \begin{bmatrix} 1 & z & -y \end{bmatrix} \begin{bmatrix} \varepsilon_{D1} & \varepsilon_{D5} & \varepsilon_{D3} \end{bmatrix}^T \\
\begin{Bmatrix} \gamma_{xy} \\ \gamma_{xz} \end{Bmatrix} &= \begin{bmatrix} 1 & 0 & -z \\ 0 & 1 & y \end{bmatrix} \begin{Bmatrix} \varepsilon_{D2} \\ \varepsilon_{D4} \\ \varepsilon_{D6} \end{Bmatrix} \\
\eta_{xx} &= \tfrac{1}{2}\left[(\varepsilon_{D1})^2 + \left(u'_{yc} - z\theta'_x\right)^2 + \left(u'_{zc} + y\theta'_x\right)^2 \right]
\end{aligned} \tag{2}$$

Where to contract notation the following definitions are made:

$$\{\varepsilon_{D1}, \varepsilon_{D2}, \varepsilon_{D3}, \varepsilon_{D4}, \varepsilon_{D5}, \varepsilon_{D6}\} = \left\{u'_{xc}, u'_{yc} - \theta_z, \theta'_z, u'_{zc} + \theta_y, \theta'_y, \theta'_x\right\} \tag{3}$$

2.2 Non Local constitutive relations

In the nonlocal elasticity theory, it is assumed that the stress at a given point depends on the strains of the whole continuum (Eringen, 1972) in relation that can be written as follows:

$$\bar{\sigma} - \mu\nabla^2\bar{\sigma} = \bar{\mathcal{C}} : \bar{\varepsilon} \tag{4}$$

where $\bar{\sigma}$ is the stress tensor, $\bar{\mathcal{C}}$ is the fourth order Hookean elasticity tensor, and $\bar{\varepsilon}$ is the strain tensor. The symbols ∇^2 and : stand for the Laplacian operator and double dot tensor product. The parameter $\mu = (e_o a)^2$ is a scale factor that depends on the material and geometric features. The coefficient e_o is estimated such that the non local elasticity matches the atomistic lattice models, and a is the so called internal characteristic lengths (Eringen, 1972; Simsek and Yurtcu, 2013).

The Eq. (4), in the case of slender beams or nano-tubes and according to the assumptions invoked in the previous paragraph, can be reduced to the following form (Peddison et al., 2003; Reddy, 2010; Simsek and Reddy, 2013):

$$\begin{aligned}
\sigma_{xx} - \mu\sigma''_{xx} &= E_{xx}\varepsilon_{xx} \\
\sigma_{xy} - \mu\sigma''_{xy} &= G_{xy}\gamma_{xy} \\
\sigma_{xz} - \mu\sigma''_{xz} &= G_{xz}\gamma_{xz}
\end{aligned} \tag{5}$$

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In Eq. (5), E_{xx} and $G_{xy} = G_{xz} = G$ are the longitudinal and transversal elasticity moduli, whereas ε_{xx}, γ_{xy} and γ_{xz} are the strain components given in Eq. (2).

2.3 Motion Equations

The motion equations of the beam can be deduced by means of the linearized principle of virtual works:

$$\int_V \bar{\sigma} : \delta\bar{\varepsilon} dV + \int_V \bar{\sigma}^0 : \delta\bar{\eta} dV + \int_V \rho\ddot{\bar{U}} \cdot \delta\bar{U} dV - \int_V \bar{X} \cdot \delta\bar{U} dV = 0 \tag{6}$$

where the first and second integral corresponds to the virtual works of the internal forces and initial stresses, respectively; the third integral corresponds to the virtual work of the inertial forces and the fourth integral corresponds to the virtual work of the applied forces volume $\bar{X}$. The over dots identify derivation with respect to the time.

Applying the conventional steps of variational calculus, the motion equations can be written as follows:

$$\begin{array}{l} -Q'_x + \mathcal{G}_1 + \mathcal{M}_1 - \mathcal{F}_1 = 0 \\ -Q'_y + \mathcal{G}_2 + \mathcal{M}_2 - \mathcal{F}_2 = 0 \\ -M'_z - Q_y + \mathcal{G}_3 + \mathcal{M}_3 - \mathcal{F}_3 = 0 \\ -Q'_z + \mathcal{G}_4 + \mathcal{M}_4 - \mathcal{F}_4 = 0 \\ -M'_y + Q_z + \mathcal{G}_5 + \mathcal{M}_5 - \mathcal{F}_5 = 0 \\ -M'_x + \mathcal{G}_6 + \mathcal{M}_6 - \mathcal{F}_6 = 0 \end{array} \tag{7}$$

These equations are subjected to conventional boundary conditions at $x = 0$ and $x = L$ by prescribing the appropriate kinematic variables or forces/moments depending on the case.

In Eq. (7), Q_x is the axial force, M_y and M_z are the bending moments, Q_y and Q_z are the shear forces and M_x is the twisting moment, whereas $\mathcal{G}_i$, $\mathcal{M}_i$ and $\mathcal{F}_i$ with $i = 1, ..., 6$ correspond to the terms of initial forces (employed for solving static instability or linearized buckling problems), inertial forces and applied forces respectively. For the sake of simplified legibility, the extended expressions of $\mathcal{G}_i$, $\mathcal{M}_i$ and $\mathcal{F}_i$ with $i = 1, ..., 6$ are given in the Appendix I. The internal forces are defined as stress resultants in the following form:

$$\begin{array}{l} \{Q_x, M_y, M_z\} = \displaystyle\int_A \sigma_{xx} \{1, z, -y\} \, dydz \\ \{Q_y, Q_z, M_x\} = \displaystyle\int_A \{\sigma_{xy}, \sigma_{xx}, -z\sigma_{xy} + y\sigma_{xz}\} \, dydz \end{array} \tag{8}$$

where: A is the domain of the cross-section of the beam.

Employing Eq. (8) together with Eq. (5), the non-local constitutive form in terms of beam forces/moments can be written as:

$$\begin{array}{lll} Q_x - \mu Q''_x = J_{11}\varepsilon_{D1}, & Q_y - \mu Q''_y = J_{22}\varepsilon_{D2}, & Q_z - \mu Q''_z = J_{44}\varepsilon_{D4}, \\ M_z - \mu M''_z = J_{33}\varepsilon_{D3}, & M_y - \mu M''_y = J_{55}\varepsilon_{D5}, & M_x - \mu M''_x = J_{66}\varepsilon_{D6}, \end{array} \tag{9}$$

The definition of the stiffness coefficients J_{ik}, $i, k = 1, ..., 6$ can be followed in the Appendix I. Now, with the aid of Eq. (9) and the differential equations of motion (7) it is possible (Reddy, 2007-2010; Simcek and Reddy, 2013) to derive the non-local form for the beam forces and

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moments as:

$$\begin{array}{l} Q_x = J_{11}\varepsilon_{D1} + \mu\mathcal{N}_1', \\ Q_y = J_{22}\varepsilon_{D2} + \mu\mathcal{N}_2', \\ M_z = J_{33}\varepsilon_{D3} + \mu\left(\mathcal{N}_3' - \mathcal{N}_2\right), \\ Q_z = J_{44}\varepsilon_{D4} + \mu\mathcal{N}_4', \\ M_y = J_{55}\varepsilon_{D5} + \mu\left(\mathcal{N}_5' - \mathcal{N}_4\right), \\ M_x = J_{66}\varepsilon_{D6} + \mu\mathcal{N}_6', \end{array} \tag{10}$$

where for simplification purposes $\mathcal{N}_i = \mathcal{G}_i + \mathcal{M}_i - \mathcal{F}_i$, $i = i, ..., 6$.

Finally the non-local governing equations can be written in the following form:

$$\begin{array}{l} -J_{11}\varepsilon_{D1}' - \mu\mathcal{N}_1'' + \mathcal{N}_1 = 0, \\ -J_{22}\varepsilon_{D2}' - \mu\mathcal{N}_2'' + \mathcal{N}_2 = 0, \\ -J_{33}\varepsilon_{D3}' - J_{22}\varepsilon_{D2} - \mu\mathcal{N}_3'' + \mathcal{N}_3 = 0, \\ -J_{44}\varepsilon_{D4}' - \mu\mathcal{N}_4'' + \mathcal{N}_4 = 0, \\ -J_{55}\varepsilon_{D5}' + J_{44}\varepsilon_{D4} - \mu\mathcal{N}_5'' + \mathcal{N}_5 = 0, \\ -J_{66}\varepsilon_{D6}' - \mu\mathcal{N}_6'' + \mathcal{N}_6 = 0, \end{array} \tag{11}$$

Note that when the coefficient $\mu = 0$ the previous non-local expressions of governing equations and the beam forces and moments, the corresponding local counterparts can be retrieved. The governing equations of the present model contains many of the recent approaches (Reddy, 2007, 2010; Di Paola et al., 2011; Thai, 2012; Shen et al., 2012) for buckling and vibration of nano-beams for Timoshenko theories in one plane.

2.4 Analytical Solution

An analytical solution of Eq. (11) can be derived for the eigenvalue problem in the case of a simply supported beam (with the boundary conditions: $u_{yc} = u_{zc} = \theta_x = 0$ and $Q_x = Q_x^0$, $M_z = M_z^0$, $M_y = M_y^0$ at $x = 0$ and $x = L$), that can be satisfied with the following displacements:

$$\begin{array}{l} u_{xc} = U_x \cos\left[k_n x\right] \sin\left[\omega t\right] \\ u_{yc} = U_y \sin\left[k_n x\right] \sin\left[\omega t\right] \\ \theta_z = \Theta_z \cos\left[k_n x\right] \sin\left[\omega t\right] \\ u_{zc} = U_z \sin\left[k_n x\right] \sin\left[\omega t\right] \\ \theta_y = \Theta_y \cos\left[k_n x\right] \sin\left[\omega t\right] \\ \theta_x = \Theta_x \sin\left[k_n x\right] \sin\left[\omega t\right] \end{array} \tag{12}$$

with $k_n = \pi n/L$ and ω is the circular frequency.

Substituting Eq. (12)in Eq. (11) it is possible to arrive to the following eigenvalue equation:

$$\left(\bar{\mathbf{K}}_{\mathbf{EA}} + \zeta\bar{\mathbf{K}}_{\mathbf{GA}} - \omega^2\bar{\mathbf{M}}_{\mathbf{M}}\right)\bar{\mathbf{V}} = \mathbf{0} \tag{13}$$

where $\bar{\mathbf{K}}_{\mathbf{EA}}$, $\mathbf{K}_{\mathbf{GA}}$ and $\bar{\mathbf{M}}_{\mathbf{A}}$ are matrices of elastic stiffness coefficients, geometric stiffness coefficients and mass coefficients, respectively, whereas the vector $\bar{\mathbf{V}} = \{U_x, U_y, \Theta_z, U_z, \Theta_y, \Theta_x\}$ contains the displacement amplitudes of Eq. (12) and $\mathbf{0}$ is the null vector. ζ parameterizes the initial stresses/forces, which is useful for buckling problems and ω is the circular frequency. The extended form of matrices $\bar{\mathbf{K}}_{\mathbf{EA}}$, $\bar{\mathbf{K}}_{\mathbf{GA}}$ and $\bar{\mathbf{M}}_{\mathbf{A}}$ are described in the Appendix I.

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The previous equation can be employed to calculate the buckling eigenvalues (by setting $\omega = 0$) as well as to calculate natural free vibration frequencies (by setting $\zeta = 0$) or vibration frequencies under normal initial stresses (by setting $\zeta = 1$).

3 VARIATIONAL FORMULATION

3.1 Functional of the non-local beam

In order to develop a finite element for the non-local governing equations (10), it is necessary to construct the quadratic functional form or the weak formulation of the problems by integrating the non-local motion equations, as it is mentioned in the works of Phadikar and Pradhan (2010) and Hemmatnezhad and Ansari (2013), since the principle of virtual works and/or principle of minimum total potential/complementary energy can not be used directly to construct the weak formulation nor the quadratic forms in the case of non-local elasticity (Phadikar and Pradhan, 2010). Then employing a weighted-integral statement in Eq. (11) and after performing integration by parts, the following functional can be attained:

$$
\begin{aligned}
&\frac{1}{2}\int_L \left[\sum_{i=1}^{6}\left(J_{ii}\varepsilon_{Di}^2 - \rho_{ii}\dot{\mathbf{W}}_i^{\,2} - \rho_{ii}\mu\dot{\mathbf{W}}_i'^{\,2}\right)\right] dx + \\
&+\frac{1}{2}\int_L Q_x^0\left(\mathbf{W}_1'^{\,2} + \mathbf{W}_2'^{\,2} + r_y^2\mathbf{W}_3'^{\,2} + \mathbf{W}_4'^{\,2} + r_z^2\mathbf{W}_5'^{\,2} + r_p^2\mathbf{W}_6'^{\,2}\right) dx + \\
&+\frac{1}{2}\int_L M_z^0\left(\mathbf{W}_1'\mathbf{W}_3' - \mathbf{W}_4'\mathbf{W}_6'\right) dx + \frac{1}{2}\int_L M_y^0\left(\mathbf{W}_1'\mathbf{W}_5' - \mathbf{W}_2'\mathbf{W}_6'\right) dx + \\
&+\frac{\mu}{2}\int_L Q_x^0\left(\mathbf{W}_1''^2 + \mathbf{W}_2''^2 + r_y^2\mathbf{W}_3''^2 + \mathbf{W}_4''^2 + r_z^2\mathbf{W}_5''^2 + r_p^2\mathbf{W}_6''^2\right) dx + \\
&+\frac{\mu}{2}\int_L M_z^0\left(\mathbf{W}''_1\mathbf{W}''_3 - \mathbf{W}''_4\mathbf{W}''_6\right) dx + \frac{\mu}{2}\int_L M_y^0\left(\mathbf{W}''_1\mathbf{W}''_5 - \mathbf{W}''_2\mathbf{W}''_6\right) dx + \\
&-\int_L \left[\sum_{i=1}^{6}\mathcal{F}_i\left(\mathbf{W}_i - \mu\mathbf{W}''_i\right)\right] dx + \left[\sum_{i=1}^{6}\mathcal{B}_i\mathbf{W}_i\right]_{x=0}^{x=L} = 0
\end{aligned}
\tag{14}
$$

where $\mathcal{B}_i$, $i = 1, ..., 6$ are the natural boundary conditions (not shown in the article due to their extension and size limitations), $\mathbf{W} = \{u_{xc}, u_{yc}, \theta_z, u_{zc}, \theta_y, \theta_x\}$ is the vector of kinematic variables. Q_x^0, M_y^0 and M_z^0 are the initial forces and moments due to the initial normal stresses. r_y^2, and r_z^2 are the axial radii of gyration of the cross-section r_p^2 is the polar radius of gyration of the cross-section.

Notice that Eq. (14) contains spatial derivatives of second order, which means that the shape functions have to be at least of Class $C(2)$ in order to avoid miss-representation of the functional. Another aspect to be accounted for in the selection of the shape functions, is that the shear deformability is considered, so a special cares has to be taken to avoid shear-locking phenomenon.

3.2 Finite element equations

The kinematic variables of the problem can be discretized within the domain of a finite element by means of the following general form:

$$\mathbf{W} = \mathbf{N}\mathbf{q}_e \tag{15}$$

In which, $\mathbf{N}$ is matrix whose rows ($\mathbf{N}_i$, $i = 1, ..., 6$) contains the shape functions for each kinematic variable. $\mathbf{q}_e$ is the vector of nodal variables. Then substituting Eq. (15) into Eq. (14),

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the following matrices of a finite element of the present nano-tube model with generic shape functions are deduced:

$$\mathbf{K}_e = \int_{l_e} \left[\frac{J_{11}}{l_e} \mathbf{N'}_1^T \mathbf{N'}_1 + \frac{J_{22}}{l_e} \left(\mathbf{N'}_2^T - l_e \mathbf{N}_3^T \right) \left(\mathbf{N'}_2 - l_e \mathbf{N}_3 \right) + \frac{J_{33}}{l_e} \mathbf{N'}_3^T \mathbf{N'}_3 \right] d\bar{x} + \\ + \int_{l_e} \left[\frac{J_{66}}{l_e} \mathbf{N'}_6^T \mathbf{N'}_6 + \frac{J_{44}}{l_e} \left(\mathbf{N'}_4^T - l_e \mathbf{N}_5^T \right) \left(\mathbf{N'}_4 - l_e \mathbf{N}_5 \right) + \frac{J_{55}}{l_e} \mathbf{N'}_5^T \mathbf{N'}_5 \right] d\bar{x} \tag{16}$$

$$\mathbf{M}_e = \int_{l_e} \sum\nolimits_{i=1}^{6} \rho_{ii} \left[l_e \mathbf{N}_i^T \mathbf{N}_i + \frac{\mu}{l_e} \mathbf{N'}_i^T \mathbf{N'}_i \right] d\bar{x} \tag{17}$$

$$\mathbf{K}_{ge} = \int_{l_e} \frac{Q_x^0}{l_e} \left[\mathbf{N'}_1^T \mathbf{N'}_1 + \mathbf{N'}_2^T \mathbf{N'}_2 + r_y^2 \mathbf{N'}_3^T \mathbf{N'}_3 + \mathbf{N'}_4^T \mathbf{N'}_4 + r_z^2 \mathbf{N'}_5^T \mathbf{N'}_5 + r_p^2 \mathbf{N'}_6^T \mathbf{N'}_6 \right] d\bar{x} + \\ + \int_{l_e} \frac{M_z^0}{l_e} \left[\mathbf{N'}_3^T \mathbf{N'}_1 + \mathbf{N'}_1^T \mathbf{N'}_3 - \mathbf{N'}_6^T \mathbf{N'}_4 + \mathbf{N'}_4^T \mathbf{N'}_6 \right] d\bar{x} + \\ + \int_{l_e} \frac{M_y^0}{l_e} \left[\mathbf{N'}_5^T \mathbf{N'}_1 + \mathbf{N'}_1^T \mathbf{N'}_5 - \mathbf{N'}_6^T \mathbf{N'}_2 + \mathbf{N'}_2^T \mathbf{N'}_6 \right] d\bar{x} + \\ + \int_{l_e} \frac{\mu Q_x^0}{l_e^3} \left[\mathbf{N''}_1^T \mathbf{N''}_1 + \mathbf{N''}_2^T \mathbf{N''}_2 + r_y^2 \mathbf{N''}_3^T \mathbf{N''}_3 + \mathbf{N''}_4^T \mathbf{N''}_4 + r_z^2 \mathbf{N''}_5^T \mathbf{N''}_5 + r_p^2 \mathbf{N''}_6^T \mathbf{N''}_6 \right] d\bar{x} + \\ + \int_{l_e} \frac{\mu M_z^0}{l_e^3} \left[\mathbf{N''}_3^T \mathbf{N''}_1 + \mathbf{N''}_1^T \mathbf{N''}_3 - \mathbf{N''}_6^T \mathbf{N''}_4 + \mathbf{N''}_4^T \mathbf{N''}_6 \right] d\bar{x} + \\ + \int_{l_e} \frac{\mu M_y^0}{l_e^3} \left[\mathbf{N''}_5^T \mathbf{N''}_1 + \mathbf{N''}_1^T \mathbf{N''}_5 - \mathbf{N''}_6^T \mathbf{N''}_2 + \mathbf{N''}_2^T \mathbf{N''}_6 \right] d\bar{x} \tag{18}$$

$$\mathbf{F}_e = \int_{l_e} \left[\sum\nolimits_{i=1}^{6} \mathcal{F}_i \left(\mathbf{N}_i^T - \mu \mathbf{N''}_i^T \right) \right] d\bar{x} \tag{19}$$

where $\mathbf{K}_{ge}$, $\mathbf{K}_{ge}$ and $\mathbf{M}_e$ are respectively the elastic stiffness matrix, geometric stiffness matrix and mass matrix of the element, whereas $\mathbf{F}_e$ is the volume forces vector of the element.

Assembling these matrices in the usual way leads to the following finite element equation:

$$\left(\mathbf{K} + \mathbf{K_G} \right) \bar{\mathbf{Q}} + \mathbf{M} \ddot{\bar{\mathbf{Q}}} = \bar{\mathbf{F}}, \tag{20}$$

where $\mathbf{K}$, $\mathbf{K_G}$ and $\mathbf{M}$ are the global matrices of elastic stiffness, geometric stiffness and mass, respectively; whereas $\bar{\mathbf{Q}}$, $\ddot{\bar{\mathbf{Q}}}$ and $\bar{\mathbf{F}}$ are the global vectors of nodal displacements, nodal accelerations and nodal forces, respectively.

Eq. (20) can be modified in order to account for "a posteriori" structural proportional Rayleigh damping given by:

$$\mathbf{C}_{RD} = \eta_1 \mathbf{M} + \eta_2 \left(\mathbf{K} + \mathbf{K_G} \right). \tag{21}$$

The coefficients η_1 and η_2 in Eq. (21) can be computed employing two given damping coefficients (namely, ξ_1 and ξ_2) for the first and second modes, according to the common methodology presented in the bibliography related to finite element procedures (Meirovitch, 1997). This leads to:

$$\left(\mathbf{K} + \mathbf{K_G} \right) \bar{\mathbf{Q}} + \mathbf{C}_{RD} \dot{\bar{\mathbf{Q}}} + \mathbf{M} \ddot{\bar{\mathbf{Q}}} = \bar{\mathbf{F}}. \tag{22}$$

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The response in the frequency domain of the linear dynamic system given by Eq. (24) can be written:

$$\widehat{\mathbf{Q}}(\omega) = \left[-\omega^2\mathbf{M} + i\omega\mathbf{C}_{RD} + (\mathbf{K} + \mathbf{K_G})\right]^{-1} \widehat{\mathbf{F}}(\omega), \tag{23}$$

where $\widehat{\mathbf{Q}}$ and $\widehat{\mathbf{F}}$ are the Fourier transform of the displacement vector and force vector, respectively; whereas ω is the circular frequency measured in [rad/sec].

From Eq. (24) it is also possible to calculate the eigenvalues for the buckling and free vibration problems by neglecting damping and forces and assuming harmonic motion:

$$(\mathbf{K} + \zeta\mathbf{K_G})\,\bar{\mathbf{Q}} - \omega^2\mathbf{M}\bar{\mathbf{Q}} = \bar{\mathbf{0}}. \tag{24}$$

in which ζ identifies the eigenvalue for buckling problems and ω identifies the eigenvalue for free vibrations. In the calculation of buckling eigenvalues the parameter ω is set to zero, on the other hand for the dynamic eigenvalue, the parameter ζ is set to one or zero in order to include or not the initial stresses and forces.

4 PROBABILISTIC APPROACH

The Maximum Entropy Principle (MEP) is used to construct the probabilistic models taking into account the uncertain parameters (Jaynes, 2003). The construction of the probability density functions of the random variables is quite sensitive in stochastic analysis and they should be deduced according to the given information (sensitively scarce) about the uncertain parameters. The MEP offers a consistent strategy, in the authors' opinion, to construct the probabilistic model despite the lack of experimental data. Thus, the MEP allows to derive the probability density functions of the random variables guaranteeing consistence with the available information and the physics of the problem. The deterministic model developed in the previous sections has many parameters that can be uncertain, however the most relevant for a non local beam theory are the non-local scale parameter $\mu = (e_0 a)^2$, the elasticity moduli and the mass density, and a parameter for positioning the buckling load $e_b \in [-R_c, R_c]$, with R_c the radius of the circular cross-section. Let introduce $\{V_1, V_2, V_3, V_4\}$ as the random variables associated to the parameters $\{E, \rho, e_0 a, e_b\}$. Thus, according to the MEP the probability functions of the random variables are obtained solving the following optimization problem:

$$p_V^* = \arg \max_{p_V \in C^p} S(p_V) \tag{25}$$

where p_V^* is the optimal probability density function such that $\forall\ p_V \in C^p$, $S(p_V^*) \geq S(p_V)$, which is the measure of entropy and C^p is the set of the admissible probability density functions that satisfy the information available. The entropy is defined as (Shannon,1948):

$$S(p_V) = \int_{\Xi} p_V \ln(p_V)\, dv \tag{26}$$

where Ξ is the support of the probability distributions. In the case in which scarce information is available about the dependency among random variables, the Maximum Entropy Principle states that the random variables involved have to be independent.

The available information to obtain the probability density functions of both sets of random variables is related to some information extracted from the technical literature, the expected mistakes in the construction process and some assumptions. Thus, the nominal values of the parameters in the deterministic model are assumed to be the mean of the random variables,

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consequently the expected values are $\mathcal{E}\{V_i\} = \underline{V}_i$, $i = 1, ..., 4$. The random variables V_i, $i = 1, ..., 4$ have bounded supports.

The expected value and the coefficient of variation are necessary to calculate the variance of the random variable, $\text{var}(V_i) = (\underline{V}_i \delta_{V_i})^2$, $i = 1, ..., 4$, that has to be kept finite in order to guarantee the physical consistency of the problem. If there is no information about the relation or dependency among random variables, the MEP states that the random variables must be independent. Moreover if the random variable is bounded, the MEP states that the probability distribution is uniform. Consequently, according to the aforementioned background, the probability density functions of the random variables V_i can be written as:

$$p_{V_i}(v_i) = \mathfrak{S}_{[\mathcal{L}_{V_i}, \mathcal{U}_{V_i}]}(v_i) \frac{1}{2\sqrt{3}\underline{V}_i \delta_{V_i}}, i = 1, ..., 3 \tag{27}$$

$$p_{V_4}(v_4) = \mathfrak{S}_{[\mathcal{L}_{V_i}, \mathcal{U}_{V_i}]}(v_4) \frac{1}{2R_c} \tag{28}$$

The Matlab function unifrnd$\left(\underline{V}_i \left(1 - \delta_{V_i}\sqrt{3}\right), \underline{V}_i \left(1 + \delta_{V_i}\sqrt{3}\right)\right)$ can be used to generate realizations of the random variables V_i, $i = 1, ..., 4$, where δ_{V_i} is the coefficient of variation of the random variable V_i; whereas unifrnd$(-R_c, R_c)$ is used to generate realizations for the random variable V_4 (R_c is the radius of the cross-section).

Then, using Eq. (27) and Eq. (28) in the assembly of the matrices of finite element model given in Eq. (23) the stochastic finite element model can be written as:

$$\widehat{\mathbb{W}}(\omega) = \left[-\omega^2 \mathbb{M} + i\omega \mathbb{C}_{RD} + (\mathbb{K} + \mathbb{K}_G)\right]^{-1} \widehat{\mathbb{F}}(\omega). \tag{29}$$

In Eq. (29) the math-blackboard typeface is employed to indicate stochastic entities, thus $\mathbb{K}$, $\mathbb{K}_G$, $\mathbb{M}$ and $\mathbb{F}$ are stochastic because Eq. (27) is employed in its derivation, and $\mathbb{C}_{RD}$ is stochastic through the stochastic nature of matrices employed in Eq. (21), hence $\widehat{\mathbb{W}}$ is stochastic.

The Monte Carlo method is used the simulate the stochastic dynamics, which implies the integration of a deterministic system for each independent realization of random variables V_i, $i = 1, ..., 4$. The convergence of the stochastic response $\widehat{\mathbb{W}}$ is calculated appealing to the following function:

$$conv(N_{MS}) = \sqrt{\frac{1}{N_{MS}} \sum_{j=1}^{N_{MS}} \int_\Omega \left\| \widehat{\mathbb{W}}_j(\omega) - \widehat{\mathbf{W}}(\omega) \right\|^2 d\omega}, \tag{30}$$

where N_{MS} is the number of Monte Carlo samplings.

5 NUMERICAL STUDIES

In Tab. 1 it is possible to see the material and geometric properties of a single walled carbon nano beam. The values of the material properties can be considered a mean among the ones available in the international literature, as for example: Lu et al. (2007), Hemmatnezhad and Ansari (2013), Shen et al. (2012). The bounds of the random variables associated to material properties can fall into the range defined by a maximum coefficient of variation $\delta_{V_i} = 0.1$ according to the data collected in the available papers.

In the subsequent paragraphs a couple of studies are carried out in order to evaluate different aspects of the uncertainty propagation in the dynamics/buckling of nano-beams. These studies are performed employing two types of iso-parametric finite elements approaches: one

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Table 1: Material and geometric properties of the carbon nano-beam.

Property Name	Value
Elasticity Modulus [GPa]	1006
Poisson Coefficient	0.25
Material Density $[kg/m^3]$	2100
Beam Length [nm]	10.00
Radius of the wall [nm]	1.00
Thickness of the wall [nm]	0.20

approach using elements with 3 nodes and quadratic shape functions and the other approach using elements with 5 nodes and quartic shape functions.

5.1 Quality of the FEM approaches

In Tab. 2 a comparison of the finite element approaches with respect to the analytical solution in the case of a simply supported nano-beam is shown. The first five natural frequencies in absence of initial stresses are calculated with the analytical solution given in Eq. (13) and the two FEM approaches. In this example $\mu = 4\ nm^2$.

Table 2: Comparison of FEM approaches with the analytical solution. Frequencies in [GHz].

Method	Nr.Elem	f_1	f_2	f_3	f_4	f_5
Analytical Solution	$---$	198	198	485	485	613
FEM, 5N-element	2	198	198	485	485	613
	6	198	198	485	485	613
FEM, 3N-element	2	203	203	512	512	615
	6	199	199	487	487	614
	12	198	198	485	485	613

As one can see, both FEM approaches behave quite well even with coarser meshes. In the following examples the 5-node element is employed with meshes of at least 10 elements, that guarantee approximation errors less than 1% as it was shown in the work of Piovan and Cortinez (2007) for the iso-parametric class of elements in the context of classical models of local elasticity.

5.2 Relation between buckling loads and vibratory patterns

In this example the uncertainty in the buckling parameter e_b is evaluated. A simply supported nano-beam with the properties shown in Tab. 1 is selected. The beam is subjected to an axial initial load whose buckling parameter is known and calculated considering the material properties and the non-local parameter known and deterministic. The non-local parameter for this example was assumed as $\mu = (e_o a)^2 = 4\ nm^2$. The stochastic analysis is performed over a set of 200 simulations.

In Fig. 2 one can see the variation of the first mode with respect to the normalized initial axial force. The parameter e_b involves an initial bending moment such that $M_y^0 = e_b Q_x^0$ modifying in vibration behavior specially nearly the buckling point. Note that the Q_{xref}^0 corresponds to the buckling load.

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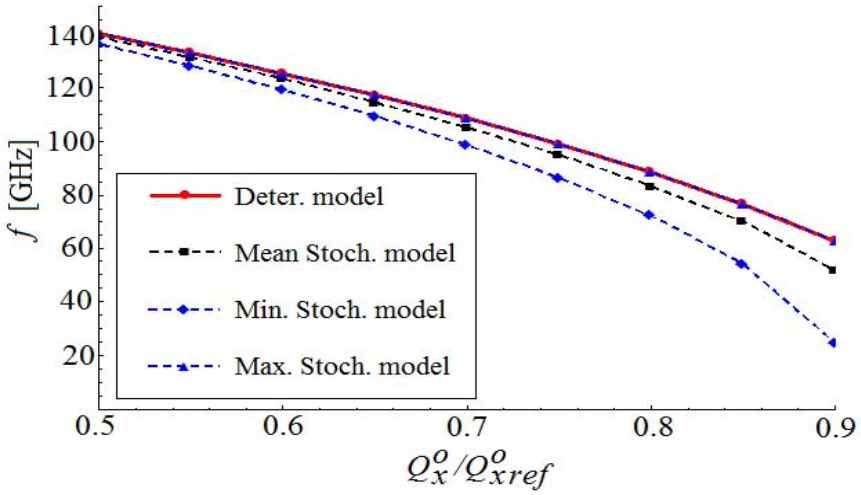

Figure 2: Variation of the first mode with respect to the normalized initial axial force.

5.3 Uncertainty propagation in the frequency response

The stochastic study in this example is concerned with the analysis of the uncertainty propagation in the frequency response function of the nano-beam subjected to a sudden arbitrary small force F applied in a given point $\boldsymbol{P}$ and used to perturb the structure. The nano-beam is subjected to clamped-free boundary conditions. The response is measured at the same point of perturbation, and evaluated by means of the following frequency response function:

$$H_F(\omega) = \frac{\left\| \widehat{\mathbf{Q}}_{\boldsymbol{P}}(\omega) \right\|}{\widehat{F}(\omega)}. \tag{31}$$

In Eq. (31), $\left\| \widehat{\mathbf{Q}}_{\boldsymbol{P}} \right\|$ is the norm of the Fourier transform of the displacement vector of the point $\boldsymbol{P}$, that is the one where the force is applied, and $\widehat{F}$ is the Fourier transform of the applied force F. The perturbation force F is located at the free end of the beam ($x = L$), oriented an angle $\varphi = 30^o$ with respect to y-axis to excite as many modes as possible. The effect of initial stresses/forces was not evaluated in this set of examples, and the band of frequencies was normalized with respect to the first natural frequency, i.e. $\lambda = f/f_1$. The calculations were performed employing 100 simulations in the Monte Carlo Method. This number is enough to attain the desired convergence as it can be seen in Fig. 3.

In Fig. 4 it is possible to see the frequency response function of the nano-beam, considering a low dispersion in material properties and non-local parameter. This meas that the parameter of variation has been taken with the value $\delta_{V_i} = 0.05$ for the mentioned random variables. Fig. 5 reflects the frequency response function of the nano-beam considering a low dispersion parameter $\delta_{V_i} = 0.05$ in the material properties and a higher dispersion parameter for the random variable associated to the non-local elasticity: $\delta_{V_i} = 0.30$. As it can be seen the propagation of uncertainty has increased notably with respect to the previous case.

Finally, in Fig. 5 one can see the frequency response function for the nano-beam in which a higher variation coefficient is employed ($\delta_{V_i} = 0.10$) for the material random variables whereas for the random variable associated to non-locality, the variation coefficient was retained in the same value of the previous case (i.e $\delta_{V_i} = 0.30$).

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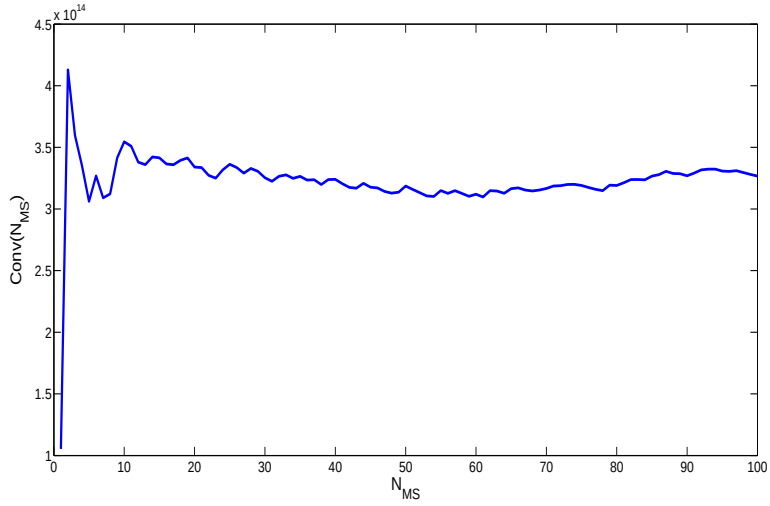

Figure 3: Example of Convergence of a Monte Carlo simulation for frequency response analysis.

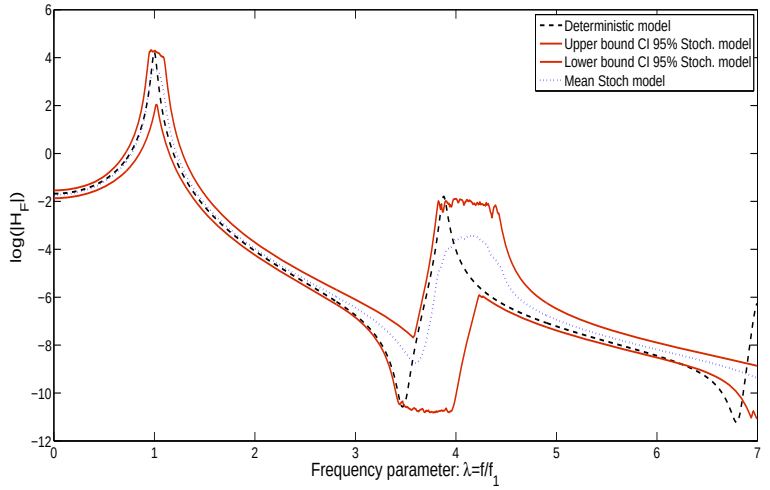

Figure 4: FRF for a nano-beam with low dispersion in non-local parameter and material properties.

As seen in the precedent figures the influence of the uncertainty in the dynamic response of nano-beams is quite sensitive. The propagation of uncertainty is higher in the higher modes since the material properties have more influence than other parameters. Fig. 7 depicts a comparison of the three cases evaluated in the present paragraph. It is seen that despite a huge increase in the variation of the non-local measure, its effective influence is not as higher as the one related to the material properties.

6 CONCLUSIONS

In this article a study about the propagation of uncertainty in the dynamics and buckling of single walled nano-beams has been performed. In this study a thin-walled nano-beam model for isotropic materials derived according to the non-local elasticity theory has been employed as the deterministic model, whose numerical outcome represents the expected response. Extensional, bending and twisting motions as well as initial states of normal stresses have been taken into account in a shear deformable beam conception. Thereafter a stochastic model has been

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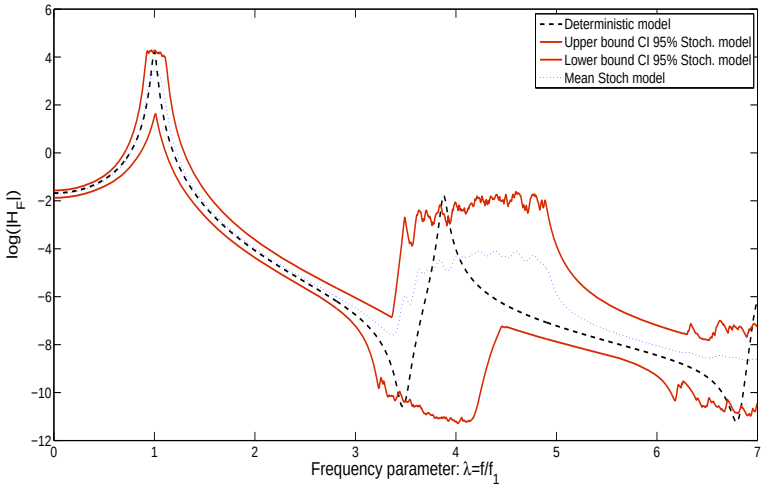

Figure 5: FRF for a nano-beam with high dispersion in non-local parameter and low dispersion in material properties

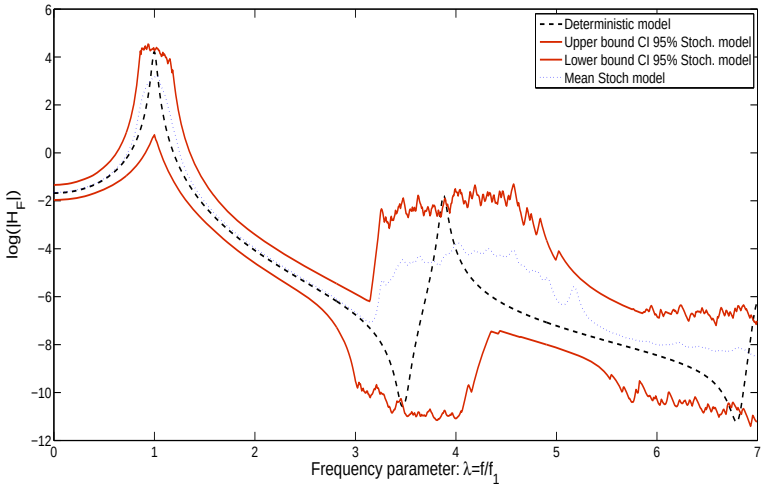

Figure 6: FRF for a nano-beam with high dispersion in non-local parameter and material properties.

constructed by introducing random variables connected with the uncertain parameters of the problem. The parameters selected for the studies of uncertainty propagation were the material properties and the non-local scale parameter. The probability density functions of each random variable have been derived according to the Maximum Entropy Principle accounting for given statistical expected information. A typical nano-tube with several types of loadings and boundary conditions has been evaluated. From the calculation carried out in the present work, the following topics can be concluded:

- The variation of the material properties propagates the uncertainty as usual, i.e. more sensitive in higher frequencies, however it is quite important.
- The variation of the non-local parameter is important but it not propagates the uncertainty as much as the variability of the material properties.
- The variation of the buckling parameter nearly the buckling point is quite important and

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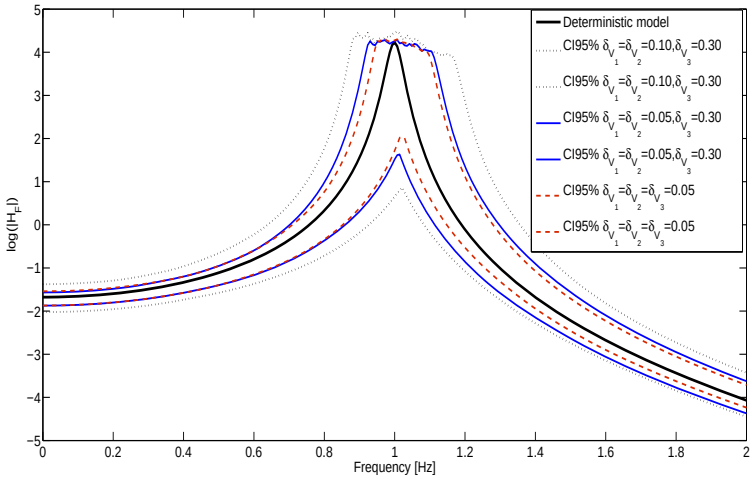

Figure 7: Comparison of the uncertainty propagation near the first mode.

has sensitive influence in the vibration patters of nano-beams subjected to initial normal stresses.

This study has shown the importance of the uncertainty quantification in the dynamics/buckling of nano-beams since this type of structures can be quite sensitive to variations in the loading conditions and the material parameters and even the non-local parameters. This type of studies is of strong importance in the case of bio-sensors modeled as nano-beam structures.

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APPENDIX I

The extended expressions and definitions of $\mathcal{G}_i$, $\mathcal{M}_i$ and $\mathcal{F}_i$ with $i = 1, ..., 6$ are following:

$$\begin{aligned}
\mathcal{G}_1 &= -Q_x^0 u''_{xc} - M_z^0 \theta''_z - M_y^0 \theta''_y, \\
\mathcal{G}_2 &= -Q_x^0 u''_{yc} + M_y^0 \theta''_x, \\
\mathcal{G}_3 &= -M_z^0 u''_{xc} - Q_x^0 r_z^2 \theta''_z, \\
\mathcal{G}_4 &= -Q_x^0 u''_{zc} + M_z^0 \theta''_x, \\
\mathcal{G}_5 &= -M_y^0 u''_{xc} - Q_x^0 r_y^2 \theta''_y, \\
\mathcal{G}_6 &= -M_z^0 u''_{zc} - M_y^0 u''_{yz} - Q_x^0 r_p^2 \theta''_x,
\end{aligned}$$

$$\begin{aligned}
&\mathcal{M}_1 = \rho_{11} \ddot{u}_{xc}, \quad \mathcal{M}_2 = \rho_{22} \ddot{u}_{yc}, \quad \mathcal{M}_4 = \rho_{44} \ddot{u}_{zc}, \\
&\mathcal{M}_3 = \rho_{33} \ddot{\theta}_z, \quad \mathcal{M}_5 = \rho_{55} \ddot{\theta}_y, \quad \mathcal{M}_6 = \rho_{66} \ddot{\theta}_x,
\end{aligned}$$

$$\begin{aligned}
\{\mathcal{F}_1, \mathcal{F}_3, \mathcal{F}_5\} &= \int_A \bar{X}_x \{1, -y, z\} \, dydz \\
\{\mathcal{F}_2, \mathcal{F}_4, \mathcal{F}_6\} &= \int_A \left\{\bar{X}_y, \bar{X}_z, -z\bar{X}_y + y\bar{X}_z\right\} dydz
\end{aligned}$$

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In the previous expression, Q_x^0, M_y^0 and M_z^0 identify the initial forces/moments under a uniform state of normal initial stresses. These forces/moments can be parameterized in order to calculate buckling loads, that is for example $Q_x^0 = \zeta F^0$ with $||F^0|| = 1$, and ζ as the parameter for the eigenvalue problem of static instability or buckling.

The stiffness coefficients J_{ih} and inertia coefficients ρ_{ih} are defined as:

$$\{J_{11}, J_{33}, J_{55}\} = \int_A E_{xx} \left\{1, y^2, z^2\right\} dydz$$
$$\{J_{22}, J_{44}, J_{66}\} = \int_A G \left\{\kappa, \kappa, y^2 + z^2\right\} dydz$$
$$\rho_{11} = \rho_{22} = \rho_{44} = \int_A \rho dydz$$
$$\{\rho_{33}, \rho_{55}, \rho_{66}\} = \int_A \rho \left\{y^2, z^2, y^2 + z^2\right\} dydz$$

where κ is the shear factor for a single walled nano tube.

The matrices of the analytical solution in the eigenvalue problem are given as follows:

$$\bar{\mathbf{K}}_{\mathbf{EA}} = \begin{bmatrix} J_{11}k_n^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & J_{22}k_n^2 & -J_{22}k_n & 0 & 0 & 0 \\ 0 & -J_{22}k_n & J_{22} + J_{33}k_n^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & J_{44}k_n^2 & J_{44}k_n & 0 \\ 0 & 0 & 0 & J_{44}k_n & J_{44} + J_{55}k_n^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & J_{66}k_n^2 \end{bmatrix}$$

$$\bar{\mathbf{K}}_{\mathbf{GA}} = \left(k_n^2 + \mu k_n^4\right) \begin{bmatrix} Q_x^0 & 0 & M_z^0 & 0 & M_y^0 & 0 \\ 0 & Q_x^0 & 0 & 0 & 0 & -M_y^0 \\ M_z^0 & 0 & r_z^2 Q_x^0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_x^0 & 0 & -M_z^0 \\ M_y^0 & 0 & 0 & 0 & r_y^2 Q_x^0 & 0 \\ 0 & -M_y^0 & 0 & -M_z^0 & 0 & r_p^2 Q_x^0 \end{bmatrix}$$

$$\bar{\mathbf{M}}_{\mathbf{A}} = \left(1 + \mu k_n^2\right) Diag\left(\rho_{11}, \rho_{22}, \rho_{33}, \rho_{44}, \rho_{55}, \rho_{66}\right)$$

Q_x^0, M_y^0 and M_z^0 are the initial forces and moments due to the initial normal stresses, r_y^2, and r_z^2 are the axial radius of gyration of the cross-section r_p^2 is the polar radius of gyration of the cross-section.

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