Asociación Argentina
de Mecánica Computacional

Mecánica Computacional Vol XXXII, págs. 3433-3448 (artículo completo)
Carlos G. García Garino, Aníbal E. Mirasso, Mario A. Storti, Miguel E. Tornello (Eds.)
Mendoza, Argentina, 19-22 Noviembre 2013

# NATURAL VIBRATIONS OF PLANE FRAMES UNDER COMPRESSION THROUGH A POWER SERIES SOLUTION

**Héctor E. Goicoechea**[a,b], **Carlos P. Filipich**[a,c] **and Marta B. Rosales**[a,d]

[a]*Departamento de Ingeniería, Universidad Nacional del Sur, Av. Alem 1253, (8000) Bahía Blanca, Argentina*

[b]*Becario CIN (Becas Estímulo a las Vocaciones Científicas), h.e.goicoechea@gmail.com*

[c]*CIMTA, FRBB, Universidad Tecnológica Nacional, 11 de Abril 461, 8000 Bahía Blanca, Argentina, cfilipich@gmail.com*

[d]*CONICET, Argentina , mrosales@criba.edu.ar*

**Keywords:** natural vibrations, critical load, plane frames

**Abstract.** The use of plane frames is very common within Structural Engineering. Standard loads derive in bending, shear and axial internal forces being the compression ones of special interest. Thus, a buckling study is mandatory in this structural type. A previous work of the research group addressed the frames buckling loads determination by means of a power series technique (in practice, polynomials are used in the applications). Separately, the natural vibration of frames was also solved. Now, an extension to the natural vibration problem of plane frames with members under compression is presented. The frames are open and no branches are considered. The governing linear differential equations are first stated together with the boundary and continuity conditions at each node linking consecutive members with arbitrary slope. The equations take into account the second order effect of the axial load. The power series algorithm is then introduced and a systematization is proposed. Various frame configurations are studied and the natural frequencies are found. The frequency decreasing effect due to the variation of the external load as it approaches to the buckling limit, is shown. Also, the convergence of the solution is assessed for increasing number of terms in the polynomials. This technique is useful as an alternative to other popular methods, such as finite elements. Here, each member of the frame is not divided into elements. Instead, the mode shape of each member is represented by the power series with arbitrary accuracy. Finally, it should be mentioned that the resulting matrix of the eigenvalue problem is always of dimension 6 by 6 disregarding the number of members of the open frame, contrary to other methods. Numerical examples and comparisons illustrate the proposed technique.

## 1 INTRODUCTION

A dynamic study of framed structures is almost always necessary in order to help to prevent damage or excessive deformations during dynamic actions such as wind and earthquakes. Usually, these structures lead to discrete models with a considerable number of degrees of freedom and their analysis involves high computational costs. Despite the evolution of the computer systems to larger capacities, the availability of reduced models is always desirable. In this sense, the vibration and the buckling studies are commonly carried out by means of the stiffness method in which approximated functions are employed. With the aid of the finite element method (FEM), a large number of subdivisions helps achieve acceptable results (e.g. Tsai, 2010; Ma, 2010). However, and depending on the case, such approach can be computational expensive. On the other hand, addressing the problem directly with transcendental solutions for each member, would result cumbersome. A previous work of the research group addresses the buckling loads determination by means of a power series technique (in practice, polynomials are used in the applications) (Filipich et al., 2003a). On the other hand, a natural vibration study on plane frames is also carried out (Filipich et al., 2003b). Now, an extension to the natural vibration problem of plane frames with members under compression is presented. A very simple methodology is herein presented to evaluate the natural frequencies of plane, single branched, open frames of $N$ members under external loads, taking into account the normal internal forces and their second effect. The idea is simple and it starts form the statement of the differential governing equations for each frame member, stating compatibility conditions at each node. After stating the differential equations that govern the bending-axial internal forces equilibrium and the respective exact solutions for each frame member, $6N$ unknown coefficients arise. The authors suggest a direct way of solving the problem that results in a 6 x 6 matrix for *all* plane frames and for an *arbitrary* number of members. Thus, the size of the matrix is constant disregarding the number of members. Actually, when dealing with open frames (with no branches), three conditions are stated at each end. Then, with exception to elastic boundaries, the boundary conditions (BC) at one end are stated and solved and then only three unknowns remain that are solved with a 3 x 3 system of equations. Some structural elements and loaded frames are numerically solved. Results are compared with those found with classical solutions when available and approximated results found with finite element method (FEM) models.

## 2 STATEMENT OF THE PROBLEM

In order to study the case of vibrations considering the second order effect due to the normal internal forces, the problem of a structure composed of beam-columns subjected only to internal normal force is first stated. Figure 1 shows the plane frame, in which the *i-th* bar properties are: length $L_i$, modulus of elasticity $E_i$, second area moment $J_i$, cross-sectional area $F_i$. Each bar is subjected to an axial load $P_i$. The elastica of each member will be denoted $\tilde{u}_i = \tilde{u}_i(\tilde{x})$ and $\tilde{v}_i = \tilde{v}_i(\tilde{x})$, with $0 \leq \tilde{x} \leq L_i$.

Let us introduce the strain energy $U_i$ due to the bending, axial and second order effect. The kinetic energy $K_i$ is also written after assuming normal modes with natural circular frequency $\omega$ of the frame. I should be noted that the energy associated to the position of the load due to the bar shortening is neglected.

$$2U_i = \int_0^{L_i} E_i J_i \tilde{v}_i''^2 d\tilde{x}_i + \int_0^{L_i} E_i F_i \tilde{u}_i'^2 d\tilde{x}_i + P_i \int_0^{L_i} \tilde{v}_i'^2 d\tilde{x}_i \tag{1}$$

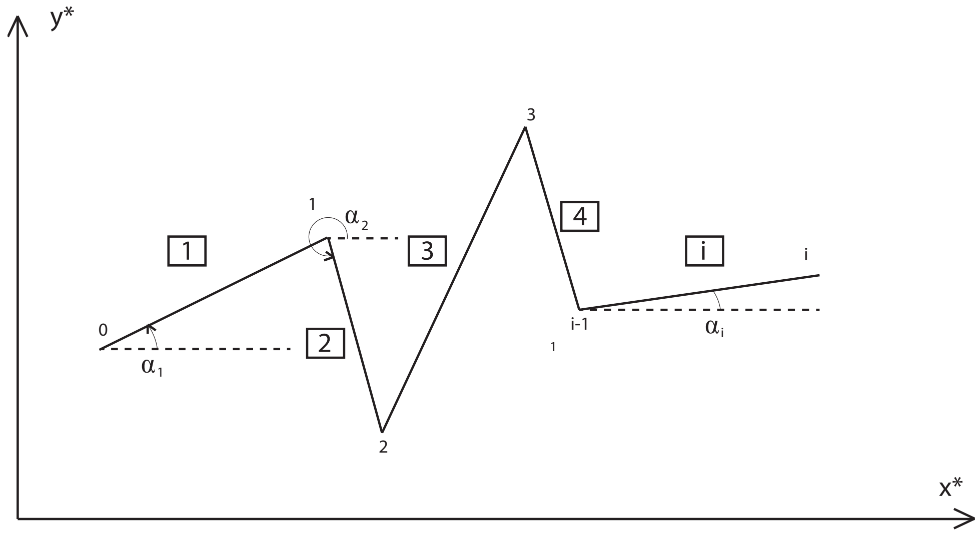



Figure 1: Scheme of the studied assembly.

$$2K_i = \rho_i F_i \omega^2 \int_0^{L_i} (\tilde{u}_i^2 + \tilde{v}_i^2) d\tilde{x}_i; \tag{2}$$

$\rho_i$ is the mass density of each bar and $L_i = \sqrt{[\tilde{x}_i - \tilde{x}_{i-1}]^2 + [\tilde{y}_i - \tilde{y}_{i-1}]^2}$. The principle of minimum total energy yields,

$$\delta \sum_{i=1}^{N} (U_i - K_i) = 0 \tag{3}$$

The following notation will be used in what follows.

$$\Delta\alpha = \alpha_i - \alpha_{i+1}$$

$$sin(\alpha_i) = \frac{\tilde{y}_i - \tilde{y}_{i-1}}{L_i}; \qquad cos(\alpha_i) = \frac{\tilde{x}_i - \tilde{x}_{i-1}}{L_i}$$

$$s = sin(\Delta\alpha) = sin(\alpha_i - \alpha_{i+1}); \qquad c = cos(\Delta\alpha) = cos(\alpha_i - \alpha_{i+1})$$

Let us introduce a nondimensionalization with $\tilde{x} = xL_i$ and $0 \le x \le 1$. From now on, all functions of the nondimensionalized variables will be denoted without the tilde. The prime indicates the derivative w.r.t. the $x$ variable.

The analysis of each rigid node connecting two contiguous bars $i$ and $i+1$ implies that the displacement vector of the node must be compatible for both. Then, the projection of the components of bar $i$ over the direction of bar $i+1$ yields the continuity conditions (see Figure 2).

$$v_{i+1}(0) = v_i(1)c - u_i(1)s \tag{4a}$$

$$u_{i+1}(0) = v_i(1)s + u_i(1)c \tag{4b}$$

$$v_i^{'}(1)L_{i+1} = v_{i+1}^{'}(0)L_i \tag{4c}$$

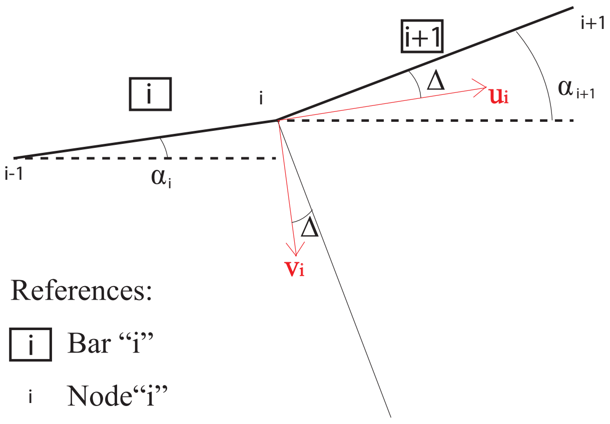



Figure 2: Continuity conditions at a node.

After the condition 3, the governing differential equations are derived as follows

$$\left(\frac{EJ}{L^3}\right)_i\left[v_i^{(IV)}-\left(\frac{PL^2}{EJ}\right)_i v_i^{''}\right]-(\rho FL)_i\,\omega^2 v_i=0 \tag{5a}$$

$$-\left(\frac{EF}{L^2}\right)_i u_i^{''}L_i-(\rho FL)_i\,\omega^2 u_i=0 \tag{5b}$$

They can be rewritten in a more compact version as

$$v_i^{IV}-k_i^2 v_i^{''}-\Omega_i^2 v_i=0 \tag{6a}$$

$$u_i^{''}+\Omega_i^2\left(\frac{J}{FL^2}\right)_i u_i=0 \tag{6b}$$

in which the nondimensionalized parameters are defined as

$$\Omega_i^2=\left(\frac{\rho FL^4}{EJ}\right)_i\omega^2; \qquad k_i^2=\left(\frac{PL^2}{EJ}\right)_i$$

The differential system also contains the boundary conditions at the extreme ends and they are found from the calculus of variation and the extreme condition on the energy functional, together with the differential equations. The boundary conditions are:

$$(EJ)_i\left[\frac{v^{''}}{L_i^2}\frac{\delta v^{'}}{L_i}\bigg|_0^1-\frac{v^{'''}}{L_i^3}\delta v\bigg|_0^1\right]=0 \tag{7a}$$

$$(EF)_i\left[\frac{u^{'}}{L_i}\delta u\bigg|_0^1\right]+\left[P_i\frac{v^{'}}{L_i}\delta v\bigg|_0^1\right]=0 \tag{7b}$$

Additionally, the continuity conditions at each node lead to the next relationships (Equations 8 to 10)

$$\left(\frac{EJ}{L^3}\right)_i v_{i1}^{'''}c-\left(\frac{EF}{L}\right)_i u_{i1}^{'}s+\frac{P_i}{L_i}v_{i1}^{'}c+\left(\frac{EJ}{L^3}\right)_{(i+1)} v_{(i+1)0}^{'''}-\frac{P_{(i+1)}}{L_{(i+1)}}v_{(i+1)0}^{'}=0 \tag{8}$$

$$\left(\frac{EJ}{L^3}\right)_i v_{i1}^{'''} s + \left(\frac{EF}{L}\right)_i u_{i1}^{'} c + \frac{P_i}{L_i} v_{i1}^{'} s - \left(\frac{EF}{L}\right)_{(i+1)} u_{(i+1)0}^{'} = 0 \tag{9}$$

$$\left(\frac{EJ}{L^2}\right)_{(i+1)} v''_{(i+1)0} = \left(\frac{EJ}{L^2}\right)_i v''_{i1} \tag{10}$$

where $(\cdot)_{i0} \equiv (\cdot)_i(0)$ and $(\cdot)_{i1} \equiv (\cdot)_i(1)$.

## 3 SOLUTION OF THE PROBLEM VIA POWER SERIES

In order to state an efficent numerical means to solve the differential system, a solution based on power series is proposed. The response of each *i-th* bar is stated as follows

$$u_i = \sum_{j=0}^{\infty} B_{ij} x^j \tag{11}$$

$$v_i = \sum_{j=0}^{\infty} A_{ij} x^j \tag{12}$$

with $i \in \mathbb{Z}$, $i$ denotes the bar under study and $j$ is the summation and power index. Since the variable has been nondimensionalized ($0 \leq x \leq 1$), the variable $x$ is employed for all bars. The exact solution would be given by the infinite series (11) and (12). However, in practice finite series will be used, giving an approximate solution. The convergence of the solution to attain a desirable number of exact digits will be done by increasing the number $N$ of terms in the series. In order to obtain the unknowns $A_{ij}$, $B_{ij}$, the proposed solution and its derivatives should be introduced in the differential system. Let us write generically, the series $\alpha = \sum_{j=0}^{N} C_j x^j$, and define a coefficient $\phi_{(m)j} = \frac{(j+m)!}{j!}$. The expression of the *m-th* derivative of a series "$\alpha$" can be written as

$$\alpha^{(m)} = \sum_{j=0}^{N-m} \phi_{(m)j} C_{(j+m)} x^j, \text{ con } \phi_{(m)j} = \frac{(j+m)!}{j!} \tag{13}$$

Explicitly, the functions and derivatives are detailed below,

$$\begin{aligned}
v_i &= \sum_{j=0}^{N} A_{ij} x^j & u_i &= \sum_{j=0}^{N} B_{ij} x^j \\
v_i^{'} &= \sum_{j=0}^{N-1} A_{i(j+1)} \phi_{1j} x^j & u_i^{'} &= \sum_{j=0}^{N-1} B_{i(j+1)} \phi_{1j} x^j \\
v_i^{''} &= \sum_{j=0}^{N-2} A_{i(j+2)} \phi_{2j} x^j & u_i^{''} &= \sum_{j=0}^{N-2} B_{i(j+2)} \phi_{2j} x^j \\
v_i^{'''} &= \sum_{j=0}^{N-3} A_{i(j+3)} \phi_{3j} x^j \\
v_i^{(IV)} &= \sum_{j=0}^{N-4} A_{i(j+4)} \phi_{4j} x^j
\end{aligned}$$

The differential equations that govern the problem (6a) and (6b) after substituting the power series, read

$$\left(\frac{EJ}{L^3}\right)_i v_i^{(IV)} - \frac{P_i}{L_i} v_i'' - (\rho F L)_i \, \omega^2 v_i = 0$$

$$\Rightarrow \sum_{j=0}^{N-4} A_{i(j+4)} \phi_{4j} x^j - k_i^2 \left[ \sum_{j=0}^{N-2} A_{i(j+2)} \phi_{2j} x^j \right] - \Omega_i^2 \sum_{j=0}^{N} A_{ij} x^j = 0 \tag{14}$$

$$\left(\frac{EF}{L}\right)_i u_i'' + (\rho F L)_i \, \omega^2 u_i = 0$$

$$\Rightarrow \sum_{j=0}^{N-1} B_{i(j+2)} \phi_{2j} x^j + \Omega_i^2 \left(\frac{J}{FL^2}\right)_i \sum_{j=0}^{N} B_{ij} x^j = 0 \tag{15}$$

From these equations, it is possible to establish relationships among the coefficients for each $j$. Thus the following recurrences are obtained

$$A_{i(j+4)} = k_i^2 A_{i(j+2)} \frac{\phi_{2j}}{\phi_{4j}} + \frac{\Omega_i^2 A_{ij}}{\phi_{4j}} \tag{16}$$

$$B_{i(j+2)} = -\Omega_i^2 \left(\frac{J_i}{F_i L_i^2}\right) \frac{B_{ij}}{\phi_{2j}} \tag{17}$$

In turn, the geometrical conditions derive in the next expressions

$$A_{(i+1)0} = c \left[ \sum_{j=0}^{N} A_{ij} \right] - s \left[ \sum_{j=0}^{N} B_{ij} \right] \tag{18}$$

$$B_{(i+1)0} = s \left[ \sum_{j=0}^{N} A_{ij} \right] + c \left[ \sum_{j=0}^{N} B_{ij} \right] \tag{19}$$

$$A_{(i+1)1} = \left[ \sum_{j=0}^{N-1} A_{i(j+1)} \phi_{1j} \right] \frac{L_{(i+1)}}{L_i} \tag{20}$$

Finally, the continuity conditions (8) to (10) at the intermediate nodes yield

$$\left(\frac{EJ}{L^2}\right)_{(i+1)} v_{(i+1)0}'' = \left(\frac{EJ}{L^2}\right)_i v_{i1}'' \tag{21}$$

$$A_{(i+1)2} = \frac{\left(\frac{EJ}{L^2}\right)_i}{\phi_{20} \left(\frac{EJ}{L^2}\right)_{(i+1)}} \sum_{j=0}^{N-2} A_{i(j+2)} \phi_{2j} \tag{22}$$

$$\left(\frac{EF}{L}\right)_{(i+1)} u_{(i+1)0}' = -\left(\frac{EJ}{L^3}\right)_i s \left[ v_{i1}''' - k_i^2 v_{i1}' \right] + c \left(\frac{EF}{L}\right)_i u_{i1}' \tag{23}$$

$$B_{(i+1)1} = - \frac{\left(\frac{EJ}{L^3}\right)_i s \left[ \sum_{j=0}^{N-3} A_{i(j+3)} \phi_{3j} - k_i^2 \sum_{j=0}^{N-1} A_{i(j+1)} \phi_{1j} \right] + c \left(\frac{EF}{L}\right)_i \left[ \sum_{j=0}^{N-1} B_{i(j+1)\phi_{1j}} \right]}{\left(\frac{EF}{L}\right)_{(i+1)}} \tag{24}$$

$$\left(\frac{EJ}{L^3}\right)_{(i+1)}\left\{v'''_{(i+1)0}-k^2_{(i+1)}v'_{(i+1)0}\right\}=c\left(\frac{EJ}{L^3}\right)_i\left[v'''_{i1}-k_i^2v'_{i1}\right]+\left(\frac{EF}{L}\right)_i u'_{i1}s \tag{25}$$

$$\begin{aligned} A_{(i+1)3} \quad = \quad & \frac{1}{\phi_{30}\left(\frac{EJ}{L^3}\right)_{(i+1)}}\left\{c\left(\frac{EJ}{L^3}\right)_i\left[\sum_{j=0}^{N-3}A_{i(j+3)}\phi_{3j}-k_i^2\sum_{j=0}^{N-1}A_{i(j+1)}\phi_{1j}\right]+\right. \\ & \left. s\left(\frac{EF}{L}\right)_i\left[\sum_{j=0}^{N-1}B_{i(j+1)}\phi_{1j}\right]+k^2_{(i+1)}A_{(i+1)1}\phi_{1j}\left(\frac{EJ}{L^3}\right)_{(i+1)}\right\} \end{aligned} \tag{26}$$

From the above equations, it is observed that the recurrences start from $A_{i4}$ and $B_{i2}$. Then, and in order to solve the problem, the coefficients $A_{i0}$, $A_{i1}$, $A_{i2}$, $A_{i3}$, $B_{i0}$ and $B_{i1}$ should be known. Three of them are determined by stating the boudary conditions at one of the end nodes (eg. left, corresponding to the first bar of the frame at $x = 0$. The other three will be found from the boundary conditions at the other end node, corresponding to the last bar, at $x = 1$. Thus, instead of solving a 6x6 system of equations, one 3x3 system is solved in terms of three remaining unknowns and finally a 3x3 homogeneous system is stated for the eigenproblem. Its solution yield the sought eigenvalue (*i.e.* proportional to the critical load or the natural frequency). Another shortcut to solve this problem is mentioned in what follows. Since the problem is linear, the superposition principle is applicable. Thus, if, for instance, the frame is hinged at its left end, the procedure is as follows. The boundary conditions are $v = 0$, $v'' = 0$ and $u = 0$ which yield to $A_{10} = 0$, $A_{12} = 0$ and $B_{10} = 0$. Then, for these BC, the recurrence equations 16 and 17 are in terms of $A_{11}$, $A_{13}$ and $B_{11}$ that are chosen as unknowns. The 3x3 system can be written in the following way. Always, for the given BC at $x = 0$ of the bar 1 (i=1), the next procedure is followed. First, the unknowns are set as

$$\begin{aligned} A_{11}=1,\;\; A_{13}=0,\;\; B_{11}=0 \\ A_{11}=0,\;\; A_{13}=1,\;\; B_{11}=0 \\ A_{11}=0,\;\; A_{13}=0,\;\; B_{11}=1 \end{aligned} \tag{27}$$

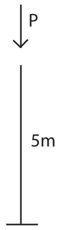
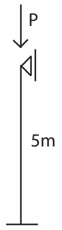
Each column of the matrix is found correspondingly after stating, for each alternative, the three BC of the last bar (i=N) at $x = 1$. Then, the null condition set on its determinant yields the sought eigenvalues.

## 4 EXAMPLES

In this section, various examples are presented. First, some simple structural elements and frames are studied to find their critical loads. Also various frames are analyzed with loads under the critical one to find the natural frequency. Comparisons with other known results, when available, are also presented. Alternatively, the comparison is made with a finite element solution.

To apply the proposed method, it is required a previous analysis with any structural technique (usually a structural analysis software) to find the internal forces for a reference load that will be affected by a $\beta$ coefficient. With the knowledge of the axial internal force for each member, the algorithm is stated to find the value of $\beta$ as the eigenvalue of the problem. Afterwards, the value of the critical load is found. The desired accuracy is attained by fixing a number of digits and performing a convergence study. Thus, the number of terms $N$ is increased until these digits are repeated.

Table 1: Critical loads of three Euler columns (Clamped: C, Hinged: H, Free: F) found with the power series solution using $N = 40$.

| Case | Scheme | $\beta$ | Euler formula kg | Power series solution kg |
|---|---|---|---|---|
| C-F |   | 2.00 | 4145,23 | 4145,23 |
| C-H |   | 0.70 | 33838,6 | 33920.4 |
| H-H |   | 1.00 | 16580,9 | 16580.9 |

## 4.1 Example 1. Columns

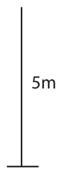
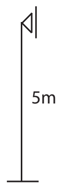
The well-known Euler expression for the critical load is $\pi^2 EI_{min}/(\beta L)^2$, where $\beta$ depends of the boundary conditions of the column (see, for instance, Timoshenko and Gere, 1961). Three cases were analyzed with the present algorithm to evaluate the accuracy. The modulus of elasticity is $E = 2.1\text{x}10^6 kg/cm^2$ and area second moment $I_{min} = 200cm^4$. The values of the critical loads were found with $N = 40$ (Table 1). On the other hand, the natural frequencies for the same structural elements are depicted in Table 2. Results found with the power series algorithm using $N = 15$ are compared with exact values reported in the open literature (e.g. Paz, 1992).

## 4.2 Example 2: Frame I

A frame composed of two members (Figure 3) is considered. Both ends are supported by hinges. The two member have the same geometric and material characteristics: modulus of elasticity $E = 2.1$ x $10^6 kg/cm^2$, cross-section area $F = 18.2$ cm$^2$, length of each member 4 m and area second moment $I = 573cm^4$, corresponding to a normal steel section (European classification) I14.

Using analytical procedures, Filipich (1981) found the exact value of the critical load as $Pcr = 12.25EJ/h^2$, with $h = 4$ m. Table 3 depicts the value of this critical load and the one

Table 2: Natural frequencies of three columns (Clamped: C, Hinged: H, Free: F) found with the power series solution using $N = 15$.

| Case | Scheme | Exact solution Hz | Power series solution Hz |
|---|---|---|---|
| C-F |   | 3.837806834 | 3.837823552 |
| C-H |   | 16.82238663 | 16.83007591 |
| H-H |   | 10.77293380 | 10.77300365 |

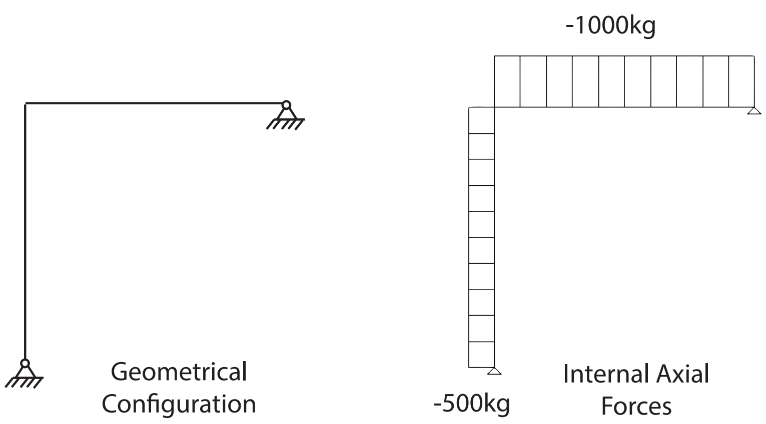



Figure 3: Example 2. Frame I geometrical configuration and internal axial forces

Table 3: Critical load (kg) of Frame I (Fig. 3) found with the power series method using $N = 40$.

| Analytical | Power series solution |
|---|---|
| 92127.66 | 92284.26 |

Table 4: First natural frequencies of Frame I (Fig. 2) under loads below the critical value, found with the power series method using $N = 15$.

| $P$ kg | Power series solution Hz |
|---|---|
| 0 | 28.43608492 |
| 10000 | 26.95676493 |
| 20000 | 25.36536934 |
| 30000 | 23.63855007 |
| 40000 | 21.74341298 |
| 50000 | 19.63070833 |
| 60000 | 17.2200968 |
| 70000 | 14.36204609 |
| 80000 | 10.70384325 |
| 90000 | 4.631056092 |
| 92000 | 1.628081434 |

found with the power series algorithm with $N = 40$. Afterwards, after knowing the critical load, the frame was subjected to loads under the critical value and the natural frequencies were found. The Results are reported in Table 4.

### 4.3 Ejemplo 3: Frame II

Now, a three member frame (Figure 4) with columns of height $h$ and girder of length $L$ and $L = h$, having all the members the same cross-section and material, with the same values of Frame I, is analyzed. As reported by Filipich (1981), the critical load of this frame found with analytical procedures is $P_{cr} = 1,823EJ/h^2$. Table 5 shows the resulting critical load and Table 6, the variation of the first natural frequencies with the increase of the load.

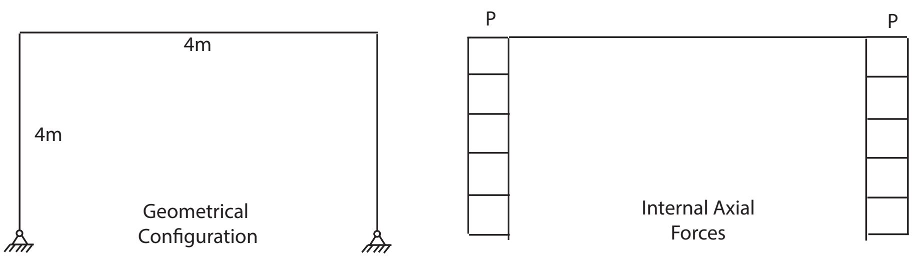



Figure 4: Example 3. Frame II geometrical configuration and internal axial forces

Table 5: Critical load (kg) of Frame II (Fig. 4) found with the power series method using $N = 40$.

| Analytical | Power series solution |
|---|---|
| 13710.10 | 13679.56 |

Table 6: First natural frequencies of Frame II (Fig. 4) under loads below the critical value, found with the power series method using $N = 15$.

| $P$ kg | Power series solution Hz |
|---|---|
| 0 | 4.220246881 |
| 15000 | 3.982394329 |
| 30000 | 3.729343256 |
| 45000 | 3.457756411 |
| 60000 | 3.1628603 |
| 75000 | 2.837393786 |
| 90000 | 2.46929511 |
| 105000 | 2.035562135 |
| 120000 | 2.035562135 |
| 135000 | 0.4838114928 |

### 4.4 Example 4: Frame III

In this example, the frame is similar to Example 3 but the girder has twice the length. The modulus of elasticity is $E = 2.1$ x $10^6 kg/cm^2$, cross-section area is $F = 18.2$ cm$^2$, $h = 4$ m, $L = 8$ m, the area second moments of the columns and the girder are $I = 573$ cm $^4$ and $I = 1146$ cm$^4$, respectively.

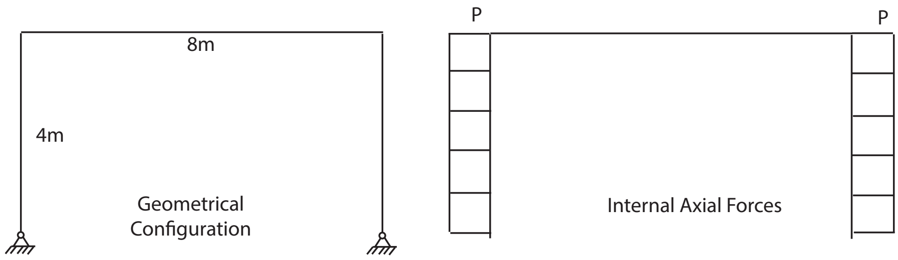



Figure 5: Example 4. Frame III geometrical configuration and internal axial forces

Filipich (1981) reports the analytical solution for this case $P_{cr} = 1,823EJ/h^2$. Tables 7 and 8 depict the critical load and the natural frequencies under compresive load, respectively.

### 4.5 Example 5: Frame IV

The frame in this example (Figure 6) is similar to Frame II, with a different internal axial force pattern. For this case, Filipich (1981) found that the value of the critical load is $P_{cr} = 3,591EJ/h^2$

Table 7: Critical load (kg) of Frame III (Fig. 5) found with the power series method using $N = 40$.

| Analytical | Power series solution |
|---|---|
| 13710.10 | 13692.83 |

Table 8: First natural frequencies of Frame III (Fig. 5) under loads below the critical value, found with the power series method using $N = 15$.

| $P$ kg | Power series solution Hz |
|---|---|
| 0 | 2.615132716 |
| 15000 | 2.468431862 |
| 30000 | 2.312275616 |
| 45000 | 2.144593539 |
| 60000 | 1.962426382 |
| 75000 | 1.761277293 |
| 90000 | 1.533686001 |
| 105000 | 1.265452961 |
| 120000 | 0.9217373965 |
| 135000 | 0.311198037 |

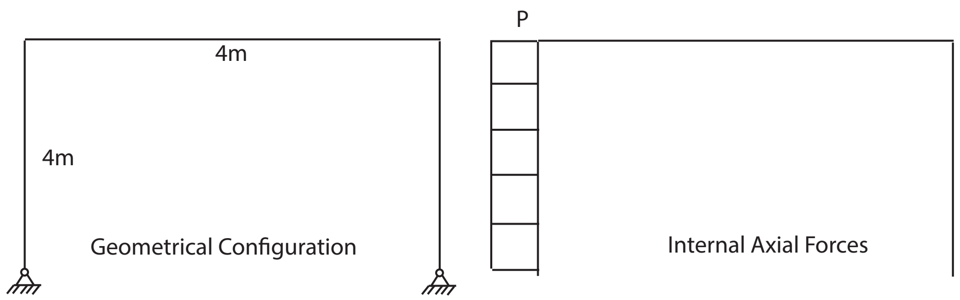



Figure 6: Example 5. Frame IV geometrical configuration and internal axial forces

Table 9: Critical load (kg) of Frame IV (Fig. 6) found with the power series method using $N = 40$.

| Analytical | Power series solution |
|---|---|
| 27006.56 | 26984.58 |

Table 10: First natural frequencies of Frame IV (Fig. 6) under loads below the critical value, found with the power series method using $N = 15$.

| $P$<br>kg | Power series solution<br>Hz |
|---|---|
| Pcr [N] | Series de Potencia [Hz] |
| 0 | 3.407365193 |
| 30000 | 3.215605727 |
| 60000 | 3.010940553 |
| 90000 | 2.790477237 |
| 120000 | 2.55005558 |
| 150000 | 2.283297249 |
| 180000 | 1.979477005 |
| 210000 | 1.61770091 |
| 240000 | 1.144023686 |
| 260000 | 0.6577296669 |

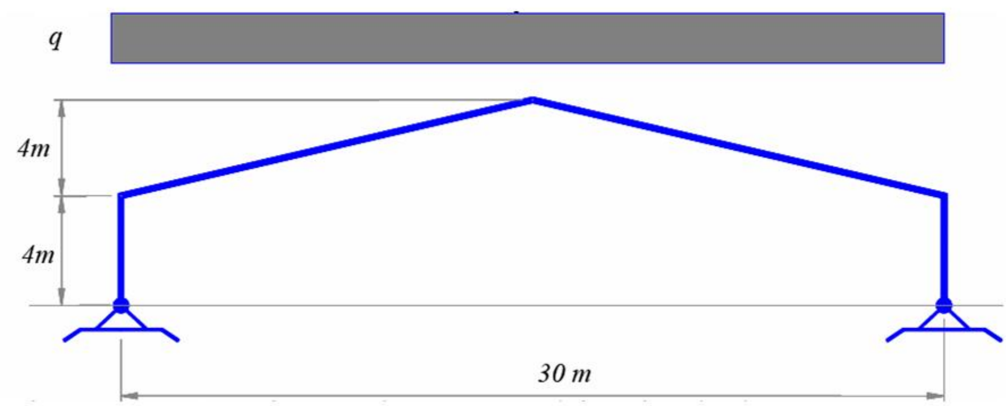



Figure 7: Example 6. Gable roof building.

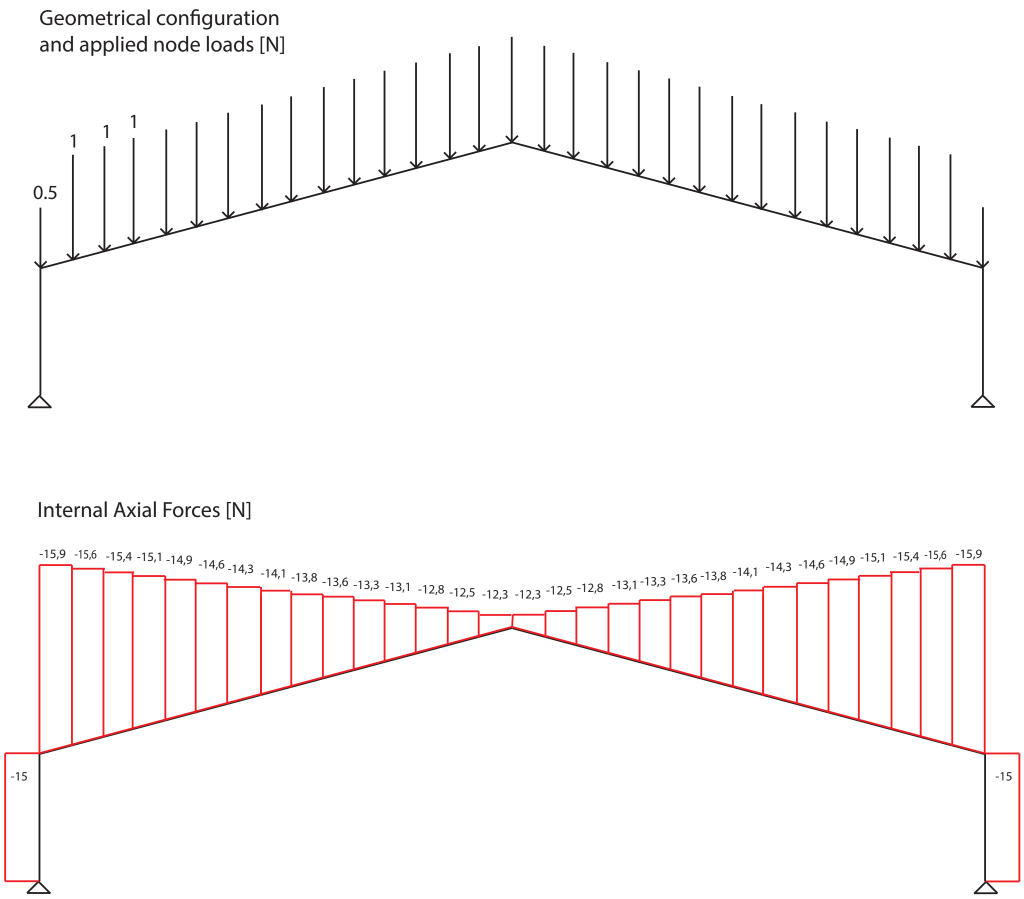



Figure 8: Example 6. Frame V geometrical configuration and internal axial forces

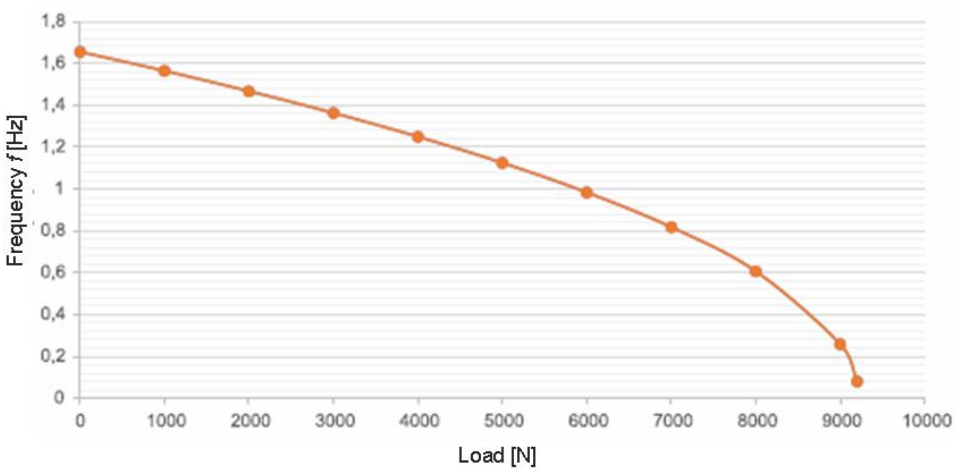



Figure 9: Example 6. Frecuency vs. Axial load curve.

### 4.6 Example 6: Frame V

Finally, a frame corresponding to a building with a gable roof is analyzed (Figures 7 and 8). The girder is discretized in elements such that the horizontal proyection is 1 m (i.e. 32 elements and 33 nodes). A comparison load of 1 N is applied at each node in order to attain a constant axial internal force. First, the internal forces are calculated as usual. Then, the power series algorithm is applied and the critical load is found. The data of the frame for both the columns and the girder are: modulus of elasticity $E = 2.1 \,\mathrm{x}10^{11}$ N/m$^2$, cross-sectional area $F = 0.0027868$ m$^2$ and area second moment $I = 27,868 \,\mathrm{x}10^{-6}$ m$^4$. The critical load found with the power series solution with $N = 40$ is $P_{cr} = 9221.22$. The study of the natural frequency problem was performed using the metholodology of the present work and, for the sake of comparison, with the finite elements software (ALGOR, 2009). This software contains a modulus named "Natural Frequencies with Load Stiffening" that permits to handle a situation when a part is under compression or tension at the same time that vibration is induced. In this case, it is a softening effect due to the compression load rather than a stiffening one. In both methods, the first natural frequency of the frame was found under loads smaller that the critical value. Table 11 contains the values of the frequencies obtained with Algor and with the power series solution. Figure 9 shows the curve that represents the decrease of the value of the first natural frequency as the load approaches its critical value.

## 5 FINAL COMMENTS

A simple analytical methodology was herein proposed to solve the natural frequency problem of open frames under compressive internal forces. The methodology is based on a power series

Table 11: First natural frequencies of Frame V (Fig. 8) under loads below the critical value, found with the power series method using $N = 15$ and compared with results from ALGOR (2009).

| $P$ kg | Finite Element solution ALGOR (2009) Hz | Power series solution Hz |
|---|---|---|
| 0 | 1.6548 | 1.6544 |
| 1000 | 1.5638 | 1.5634 |
| 2000 | 1.4668 | 1.4665 |
| 3000 | 1.3626 | 1.3623 |
| 4000 | 1.2494 | 1.2491 |
| 5000 | 1.1245 | 1.1242 |
| 6000 | 0.9832 | 0.9830 |
| 7000 | 0.8172 | 0.8170 |
| 8000 | 0.6066 | 0.6064 |
| 9000 | 0.2586 | 0.2584 |
| 9200 | 0.0808 | 0.0800 |

solution of the differencial system that governs the problem. Despite the availability of well-known techniques to solve linear frames, this technique is attractive due to its simplicity and very low computational cost. Disregarding the number of members of the frame, the solution is always found from the eigenproblem of a $6$ x$6$ system. Furthermore, if elastic supports are excluded, the solution may be found solving a $3$ x$3$ system. The results were compared with previous exact solutions for the critical loads and with a finite element software using a load stiffening modulus. It is shown that, as expected, the frequency values decrease with increased axial internal loads and the curve has an infinite asymtote as the load approaches to its critical value.

## REFERENCES


ALGOR. *ALGOR Software V21*, 2009.

Filipich C. Inestabilidad del equilibrio. In *Curso 813*. Secretaría Argentina para el Centro Internacional de Ciencias Mecánicas, 1981.

Filipich C., Bambill E., and Rosales M. Pandeo de pórticos planos. uso sistemático de series de potencias. In M.B. Rosales, V.H. Cortínez, and D.V. Bambill, editors, *Mecánica Computacional*, volume XXII, pages 454–468. 2003a.

Filipich C.P., Escalante M.R., and Rosales M.B. Series de potencias: una ventajosa alternativa aplicada a vibraciones libres de pórticos. In M.B. Rosales, V.H. Cortínez, and D.V. Bambill, editors, *Mecánica Computacional*, volume XXII, pages 908–920. 2003b.

Ma H. Exact solution of vibration problems of frame structures. *International journal for numerical methods in biomedical engineering*, 26(5):587–596, 2010.

Paz M. *Structural Dynamics, theory and computation*. Van Nostrand Reinhold Co., New York, 3rd edition, 1992.

Timoshenko S.P. and Gere J.M. *Theory of Elastic Stability*. McGraw-Hill, New York, 1961.

Tsai H.C. A distributed-mass approach for dynamic analysis of bernoulli–euler plane frames. *Journal of Sound and Vibration*, 329(18):3744–3758, 2010.