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STRONG DATUM AND TERRESTRIAL REFERENCE FRAMES IN THE ADJUSTMENT OF A FREE GEODETIC NETWORK

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Abstract. The free geodetic network considered in this work is a free two-dimensional trilateration network which is a set of physical points accessible through occupation, direct or indirect observation that provides to the users of parameters that allows us to know the shape and size of the Earth. The goal of this work is to find a LEast Square Solution (LESS) to the inverse problem of estimate the network points coordinates using the observed distances in an adjustment model of type Gauss-Markov Model (GMM). The positions of the network points are defined in a *Local two-dimensional Geodetic Reference System (LGRS)* using only Cartesian or plane coordinates (x,y). The LGRS is *defined using attributes of a topocentric Terrestrial Reference System TRS* described by Boucher,C. (2001) as one of the major types of TRS in use, hence, the LGRS is *designated* here as *Terrestrial Reference Cartesian Coordinate System TRS(x,y)*.The origin "**o**" of the TRS(x,y) is on the Earth's surface and the orientation of the axis **ox** and **oy** uses the *local vertical direction* in **"o"**. Right-handed convention is adopted for the axis. The scale or length defined of the unit vectors along **ox** and **oy** of the TRS(x,y) is the meter (SI), and it is realized by the observed distances of the trilateration network. The TRS(x,y) has not defined its origin and orientation in a given epoch. Since the available observation (distances),do not carry the necessary information to realize completely these attributes of the coordinate system (origin and orientation) ,there will be a datum defect – datum problem- and therefore a rank-deficient Singular Gauss-Markov Model (SGMM) in the adjustment of the network.

Hence, to estimate the network points coordinates (unknown parameters), the datum problem must be solved. It can be done, by introducing in the SGMM the necessary information not contained in the observations, i.e. definition and realization of the origin and orientation in a given epoch of the TRS(x,y) in the form of independent linear equations on the unknown parameters which are known as "constraints".

In this work, the datum of the two-dimensional free trilateration network is defined by introducing in the adjustment model (SGMM) more constraints than the minimum required or necessaries – so called "Strong Datum" – namely, more linear equations than the network datum defect (which leads to an "over-constrained" adjustment problem) using a chosen Terrestrial Reference Frame TRFd(x,y) given by the coordinates (xd,yd) of all selected datum points according to Vacaflor, J.L (2012) and four parameters of a plane coordinate transformation : two translation , one differential rotation and one scale factor, with respect to a known "a priori" Terrestrial Reference Frame TRF(xo,yo).

The Weighted LEast Square Solution (W-LESS) of this "over-constrained" adjustment problem is developed following the general methodology given by Schaffrin, B (1985).

1 INTRODUCTION

The free geodetic network considered in this work is a free two-dimensional trilateration network which is a set of physical points accessible through occupation, direct or indirect observation that provides to the users of parameters that allows us to know the shape and size of the Earth. The goal of this work is to find a Weighted LEast Square Solution (W-LESS) to the inverse problem of estimate the network points coordinates using the observed distances in an adjustment model of type Gauss- Markov Model (GMM). The positions of the network points are defined in a *Local two-dimensional Geodetic Reference System (LGRS)* using only Cartesian or plane coordinates (x,y). The LGRS is *defined using attributes of a topocentric Terrestrial Reference System TRS* described by Boucher,C. (2001) as one of the major types of TRS in use, hence, the LGRS is *designated* here as *Terrestrial Reference Cartesian Coordinate System TRS(x,y)*.The origin "**o"** of the TRS(x,y) is on the Earth's surface and the orientation of the axis **ox** and **oy** uses the *local vertical direction* in **"o"**. Right-handed convention is adopted for the axis. The scale or length defined of the unit vectors along **ox** and **oy** of the TRS(x,y) and TRS(xo,yo) is the meter (SI), and it is realized by the observed distances of the trilateration network. The $TRS(x_0, y_0)$ is the terrestrial reference system where the coordinates of a known *"a priori"* $TRF(x_0, y_0)$ are expressed. The TRS(x,y) has not defined its origin and orientation in a given epoch. Since the available observation (distances),do not carry the necessary information to realize completely these attributes of the coordinate system (origin and orientation) ,there will be a datum defect – datum problem- and therefore a rank-deficient Singular Gauss-Markov Model (SGMM) in the adjustment of the network.

Hence, to estimate the network points coordinates (unknown parameters), the datum problem must be solved. It can be done, by introducing in the SGMM the necessary information not contained in the observations, i.e. definition and realization of the origin and orientation in a given epoch of the TRS(x,y) in the form of independent linear equations on the unknown parameters which are known as "constraints".

2 STRONG DATUM AND TERRESTRIAL REFERENCE FRAMES IN THE ADJUSTMENT OF A FREE GEODETIC NETWORK

In this work, the datum of the two-dimensional free trilateration network is defined by introducing in the adjustment model (SGMM) more constraints than the minimum required or necessaries – so called "Strong Datum" – namely, more linear equations than the network datum defect (which leads to an "over-constrained" adjustment problem) using a chosen Terrestrial Reference Frame TRFd(x,y) given by the coordinates (xd,yd) of all selected datum points according to Vacaflor, J.L (2012) and four parameters of a plane coordinate transformation : two translation , one differential rotation and one scale factor, with respect to a known *"a priori"* Terrestrial Reference Frame TRF(xo,yo).

The Weighted LEast Square Solution (W-LESS) of this "over-constrained" adjustment problem is developed following the general methodology given by Schaffrin, B (1985).

Let us consider the TRS(x,y) where its origin is a point **P** not specified of the Earth's surface and it is the origin **o** of the Cartesian coordinate system (x,y), the first and second axis **ox** and **oy** respectively are mutually orthogonal with not specified orientations.

Hence, we are dealing with: a) a two-dimensional trilateration network constituted by "k" physical points P_i with coordinates (x_i, y_i), $i = 1...k$ in the $TRS(x, y)$, where "n" distances between these points have been observed, b) the position and orientation of the $TRS(x, y)$ are not *defined* for any epoch causing a datum defect and therefore, a *free* geodetic network, and c) the coordinates (x_i^0, y_i^0), $i = 1...k$ "a priori" or "approximated" are available from the known reference frame $TRF(x_0, y_0)$.

The lack of definition in the origin and orientation of the TRS(x,y) cause a datum defect and a rank-deficient Singular Gauss-Markov Model (SGMM) for the adjustment of the network (Schaffrin,1985) :

$$y - e = A\xi \quad , \; r(A) =: q < m < n \; , \; d =: m - q = 3 \, , \; e \sim (0, \sigma_0^2 P^{-1} =: D\{y\}) \tag{1}$$

with,

$n =$ Number of observations; $m =$ Number of unknown parameters; $r =$ Rank
$d =$ Number of datum defect = 3; $D =$ Dispersion; $P_{nxn} =$ Symmetric positive-definite weight matrix; $o =$ Order; $\sigma_0^2 =$ Unknown (observational) variance component; E=Expectation.

$y_{nx1} =$ Vector of observations (increments)

$$y_{nx1} = [y_{ij}] = [(s_{12}^{obs} - s_{12}^0), (s_{13}^{obs} - s_{13}^0), ..., (s_{ij}^{obs} - s_{ij}^0), ..., (s_{k-1,k}^{obs} - s_{k-1,k}^o)]^T$$

$e_{nx1} = [e_{ij}] =$ Vector of random errors (unknown)

$$E\{e_{nx1}\} = 0$$

$$s_{ij}^0 = \sqrt{(\Delta x_{ij}^0)^2 + (\Delta y_{ij}^0)^2} \; , \; i = 1..k \; , \; j = 1..k \; , \; i < j$$

$$\Delta x_{ij}^0 = x_j^0 - x_i^0 \; ; \; \Delta y_{ij}^0 = y_j^0 - y_i^0$$

$A_{nxm} =$ Design or coefficient matrix ("Jacobian")

$$A_{nxm} = \begin{bmatrix} \alpha_{12} \\ \cdots \\ \alpha_{ij} \\ \cdots \\ \alpha_{k-1,k} \end{bmatrix} ; \; \alpha_{ij_{1xm}} = [0, \ldots, -\Delta x_{ij}^0, -\Delta y_{ij}^0, \ldots, \Delta x_{ij}^0, \Delta y_{ij}^0, \ldots 0].(1/s_{ij}^0)$$

$\xi_{mx1} =$ Vector of unknown parameters (coordinate increments).

$$\xi_{mx1} = X_{mx1} - X_{mx1}^0$$

$X_{mx1} =$ Vector of unknown coordinates of the points P_i of the TRF(x,y) expressed in the $TRS(x, y)$.

$X_{mx1} = [x_1, y_1 ... x_k, y_k]^T$

$X^0_{mx1} =$ Vector of known coordinates of P_i of the "a priori" or "approximated" $TRF(x_0, y_0)$.

$X^0_{mx1} = [x^0_1, y^0_1 ... x^0_k, y^0_k]^T$

$\xi_{mx1} = [dx_1 \quad dy_1 \quad ... \quad dx_k \quad dy_k]^T$; $dx_i = x_i - x^0_i$; $dy_i = y_i - y^0_i$, $i = 1..k$, $m = 2k$

To complete the datum definition of the network in (1), it is necessary to introduce as *minimum three* independent condition equations to *define* and *realize* in a given epoch the origin and orientation of the $TRS(x, y)$.

In this work, the datum of the network is defined by introducing in (1) *more constraints* than the *three minimum* required or necessaries: so called *"Strong Datum"* leading to an *"over-constrained"* adjustment problem.

Following Vacaflor, J.L. (2008), it is assumed that the coordinate transformation model from the $TRS(x_0, y_0)$ to $TRS(x, y)$ is given by (a "right-handed" convention is adopted for the axis):

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \end{bmatrix} + (1+ds)\begin{bmatrix} 1 & -d\delta \\ d\delta & 1 \end{bmatrix} . \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \tag{2}$$

Where the four transformation parameters are: two translations t_x and t_y, one differential rotation $d\delta$ (of the $TRS(x, y)$) and one scale factor ds which are arranged into the vector PT_{3x1}:

$$PT = [t_x, t_y, d\delta, ds]^T \tag{3}$$

Since the scale definition of the TRS(x,y) and the TRS(xo,yo) is the meter (SI) – which is realized by the observed distances of the trilateration network - , then $ds = 0$ in (2) and (3).The definition of the origin and orientation of the $TRS(x, y)$ for a given epoch with respect to the $TRS(x_0, y_0)$ is done here by assigning to the transformation parameters t_x, t_y, and $d\delta$ of (2) or (3) the values: t^*_x, t^*_y and $d\delta^*$ respectively.

Hence, by definition it is established that:

$$PT = PT^* \tag{4}$$

$$PT^* = [t^*_x, t^*_y, d\delta^*, 0]^T$$

Then, for (x_i, y_i) of the $TRF(x, y)$ and (x^0_i, y^0_i) of the $TRF(x_0, y_0)$, $i = 1...k$ according to Eq. (2) and (4) leads to:

$$\begin{bmatrix} dx_i \\ dy_i \end{bmatrix} = \begin{bmatrix} 1 & 0 & -y^0_i & x^0_i \\ 0 & 1 & x^0_i & y^0_i \end{bmatrix} . \begin{bmatrix} t^*_x \\ t^*_y \\ d\delta^* \\ 0 \end{bmatrix} \tag{5}$$

$\therefore$ For the "k" points of the network:

$$\xi = E^T PT^* \tag{6}$$

With:

$$E_{4xm} = \begin{bmatrix} 1 & 0 & \ldots & 1 & 0 \\ 0 & 1 & \ldots & 0 & 1 \\ -y_1^0 & x_1^0 & \ldots & -y_k^0 & x_k^0 \\ x_1^0 & y_1^0 & \ldots & x_k^0 & y_k^0 \end{bmatrix} \quad (7)$$

To realize the position and orientation of the $TRS(x,y)$ it is provide the numerical values of a selected set of " l " datum coordinate differences, $dx_{id} = x_{id} - x_{id}^0$, $dy_{id} = y_{id} - y_{id}^0$, $i = 1 \ldots k'$, *or equivalently*, by providing the numerical values of a *selected set of "l" datum coordinates* $\{\ldots x_{id}, y_{id}, \ldots\}$ $i = 1 \ldots k'$ (since x_{id}^0, y_{id}^0 are known) of a Terrestrial Reference Frame constituted by a set of *k'* selected datum points P_{id}, $i = 1 \ldots k'$, designated as : *The Terrestrial Reference Frame chosen in the datum definition TRFd(x,y)* of the network.

The definition of the origin and orientation of the TRS(x,y) through t_x^*, t_y^*, $d\delta^*$ *and* its realization through the *TRFd(x,y)* , are introduced in (1) (the datum defect is eliminated*)* through l >3 linear condition equations or constraints (8) – *strong datum* - on a *set of "l" datum coordinates* $\{\ldots x_{id}, y_{id}, \ldots\}$ $i = 1 \ldots k'$ *selected of* ξ_{mx1} *through* $K_{lxm}\xi_{mx1}$ with the numerical values - taking into account (6) - as follows:

$$K\xi = KE^T PT^* \ , \ o(K) = lxm \, , \ r(K) = l > 3 \ , \ R(K^T) \cup R(A^T) = \Re^m \quad (8)$$

Where K_{lxm} is a matrix of zeros, *and ones* for those elements which multiply according to (8) to the *selected datum* datum coordinate differences, $dx_{id} = x_{id} - x_{id}^0$, $dy_{id} = y_{id} - y_{id}^0$, $i = 1 \ldots k'$, e.g., for $k_{11} := 1 \Rightarrow dx_1 := dx_{1d}$, $\Rightarrow x_1 := x_{1d}$, for $k_{22} := 1 \Rightarrow dy_1 := dy_{1d}$ $\Rightarrow y_1 := y_{1d} \Rightarrow P_1 := P_{1d}$ $\therefore P_{1d} \in TRFd(x,y)$.

with,

$$R(K^T) = \{K^T \alpha / \alpha \in \Re^l\} \ ; \ R(K^T) \subset \Re^m ; \ \dim R(K^T) = r(K) = l \quad (9)$$

$$R(A^T) = \{A^T \alpha / \alpha \in \Re^l\} \ ; \ R(A^T) \subset \Re^m ; \ \dim R(A^T) = r(A) = q \quad (10)$$

Or one of the equivalent conditions is fulfilled:

$$r[A^T, K^T] = m \, ; \, r\begin{bmatrix} A \\ K \end{bmatrix} = m \, ; \ R\left[A^T \quad , \quad K^T\right] = \Re^m \quad (11)$$

Hence, the model (1) becomes in:

$$\begin{aligned} & y - e = A\xi \quad , \ r(A) =: q < m < n \ , \ d =: m - q = 3 \, , \ e \sim (0, \sigma_0^2 P^{-1} =: D\{y\}) \\ & K\xi = KE^T PT^* \ , \ o(K) = lxm \, , \ r(K) = l > 3 \ , \ R(K^T) \cup R(A^T) = \Re^m \end{aligned} \quad (12)$$

If b:= $KE^T PT^*$

Then, following Schaffrin, B. (1985), the Weighted LEast Squares Solution $\hat{\xi}$ (W-LESS) is based on the target function:

$$\phi(\xi, \lambda) := (y - A\xi)^T P(y - A\xi) + 2\lambda^T (K\xi - b) = \min_{\xi, \lambda} \quad (13)$$

With the $(lx1)$ vector λ of "Lagrange Multipliers".

$$\frac{1}{2}\frac{\partial\phi}{\partial\xi} = N\hat{\xi} + K^T\hat{\lambda} - c = 0 \tag{14}$$

$$\frac{1}{2}\frac{\partial\phi}{\partial\lambda} = K\hat{\xi} - b = 0$$

with

$N := A^T PA$ = Normal matrix

$c := A^T Py$

$\Rightarrow$ The "extended normal equations":

$$\begin{bmatrix} N & K^T \\ K & 0 \end{bmatrix}\begin{bmatrix} \hat{\xi} \\ \hat{\lambda} \end{bmatrix} = \begin{bmatrix} c \\ b \end{bmatrix} \tag{15}$$

Hence, the estimated coordinates can be obtained from the "extended normal equations" as follows:

From (15) $\Rightarrow$ $$K\hat{\xi} = b \therefore K^T QK\hat{\xi} = K^T Qb \tag{16}$$

For any positive-definite matrix Q_{lxl} .Adding (16) to $N\hat{\xi} + K^T\hat{\lambda} = c$ from (15), it is obtained

$$(N + K^T QK)\hat{\xi} + K^T\hat{\lambda} = c + K^T Qb \tag{17}$$

$$\Rightarrow \begin{bmatrix} N_K & K^T \\ K & 0 \end{bmatrix}\begin{bmatrix} \hat{\xi} \\ \hat{\lambda} \end{bmatrix} = \begin{bmatrix} c + K^T Qb \\ b \end{bmatrix} \tag{18}$$

With: $N_K := (N + K^T QK)$, regular (19)

$$\Rightarrow \begin{bmatrix} \hat{\xi} \\ \hat{\lambda} \end{bmatrix} = \begin{bmatrix} N_K & K^T \\ K & 0 \end{bmatrix}^{-1}\begin{bmatrix} c + K^T Qb \\ b \end{bmatrix} \tag{20}$$

By means of the inverse partitioned matrix in (20), the adjusted coordinates are obtained:

$$\hat{\xi} = N_K^{-1}c + N_K^{-1}K^T[KN_K^{-1}K^T]^{-1}[E^T PT^* - KN_K^{-1}c] \tag{21}$$

And the estimated vector $\hat{\lambda}$ of "Lagrange Multipliers":

$$\hat{\lambda} = Qb - \left[KN_K^{-1}K^T\right]^{-1}\left[b - KN_K^{-1}c\right] \tag{22}$$

The dispersion of the adjusted coordinates are given by:

$$D\left\{\hat{\xi}\right\} = \sigma_0^2 N_K^{-1} - \sigma_0^2 N_K^{-1}K^T\left[KN_k^{-1}K^T\right]^{-1}KN_K^{-1} \tag{23}$$

3 CONCLUSIONS

In this work , the datum of the two-dimensional free trilateration network is defined by introducing in a rank deficient Singular Gauss-Markov Model (SGMM) for the adjustment of the network, *more constraints than the minimum required* or necessaries – so called *"Strong Datum"* – namely, more linear equations than the network datum defect (l >3) which leads to an "over-constrained" adjustment problem.

The *definition* of the origin and orientation of the TRS(x,y) through the three parameters of a plane coordinate transformation model: two translations t_x^*, t_y^* and one differential rotation $d\delta^*$ with respect to the terrestrial reference system $TRS(x_0, y_0)$ where the coordinates of the "a priori" or "approximated" available $TRF(x_0, y_0)$ are expressed *and* its *realization* through the Terrestrial Reference Frame chosen in the datum definition *TRFd(x,y)*, are introduced in the SGMM (the datum defect is eliminated*)* through l >3 linear condition equations or constraints – *strong datum* - on a *set of "l" datum coordinates* $\{\ldots x_{id}, y_{id}, \ldots\}$ $i = 1 \ldots k'$ *selected of the* vector of unknown parameters (coordinate increments) ξ_{mx1} *through* $K_{lxm}\xi_{mx1}$ with the numerical values given by $K\xi = KE^T PT^*$, $o(K) = lxm$, $r(K) = l > 3$, $R(K^T) \cup R(A^T) = \Re^m$, with $PT^* = [t_x^*, t_y^*, d\delta^*, 0]^T$, $ds = 0$, the scale definition of the TRS(x,y) and the TRS(xo,yo) is the meter (SI).

The Weighted LEast Squares Solution $\hat{\xi}$ (W-LESS) of this "over-constrained" adjustment problem was developed following the general methodology given by Schaffrin, B (1985).

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