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A STUDY ON MODE SHAPES OF BEAMS WITH INTERNAL HINGES AND INTERMEDIATE ELASTIC RESTRAINTS

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Abstract. Dynamic analysis of structural elements becomes an important design procedure. An adequate understanding of the free vibration is crucial to the design and performance evaluation of a mechanical system. This work deals with the problem of free vibrations of uniform beams with elastically restrained ends and with internal hinges and intermediate translational restraints.
The main objective of this work is to obtain the minimum stiffness of a elastic restraint that raises a natural frequency of a beam with an internal hinge, to its upper limit. The minimum stiffness is determined by using the derivative of the function which gives the natural frequencies, with respect to the support position. Additionally, the effects on natural frequencies of the presence of two internal hinges are analyzed.

1 INTRODUCTION

There has been extensive research into the vibration of Euler–Bernoulli beams with elastical restraints. It is not possible to give a detailed account by reason of the great amount of information; nevertheless, some relevant references will be cited. Particularly, several investigators have studied the influence of elastic restraints at the ends of vibrating beams (Mabie and Rogers, 1968; Mabie and Rogers, 1972; Lee, 1973; Mabie and Rogers, 1974; Grant, 1975; Hibbeler, 1975; Maurizi et al., 1976; Goel, 1976a, b; Grossi and Laura, 1982; Laura and Grossi, 1982; Cortinez and Laura, 1985; Laura and Gutierrez, 1986; Grossi and Bhat, 1991; Grossi et al., 1993; Nallim and Grossi, 1999). Exact frequency and normal mode shape expressions have been derived for uniform beams with ends elastically restrained against rotation and translation (Rao and Mirza, 1989). Excellent handbooks have appeared in the literature giving frequencies, tables and mode shape expressions (Blevins, 1979; Karnovsky and Lebed, 2004).

The problem of vibrations of beams elastically restrained at intermediate points has also been extensively treated. One of the earliest works has been performed by Lee and Saibel (1952), who analyzed the problem of free vibrations of a constrained beam with intermediate elastic supports. Rutemberg (1978) presented eigenfrequencies for a uniform cantilever beam with a rotational restraint at an intermediate position. Lau (1984) extended Rutemberg´s results including an additional spring to against translation. Maurizi and Bambill (1987) analyzed the transverse vibrations of clamped beams with an intermediate translational restraint. Rao (1989) analyzed the frequencies of a clamped-clamped uniform beam with intermediate elastic support. De Rosa et al. (1995) studied the free vibrations of stepped beams with intermediate elastic supports. Ewing and Mirsafian (1996) analyzed the forced vibrations of two beams joined with a non-linear rotational joint. Arenas and Grossi (1999) presented exact and approximate frequencies of a uniform beam, with one end spring-hinged and a rotational restraint in a variable position. Grossi and Albarracín (2003) determined the exact eigenfrequencies of a uniform beam with intermediate elastic constraints.

The minimum stiffness of a point support that raises a natural frequency of a beam to its upper limit has been investigated by several researchers. Courant and Hilbert (1953) has demonstrated that the optimum location of a rigid support should be at the nodal points of a higher vibration mode. Akesson and Olhoff (1988) showed that in the case of elastic supports the optimum locations are the same as that of rigid supports and that there exists a minimum stiffness of an additional elastic support whenever the fundamental frequency of a uniform cantilever beam is increased to its maximum. Wang (2003) determined the minimum stiffness of an internal elastic support to maximize the fundamental frequency of a vibrating beam. Wang et al. (2006) derived the closed-form solution for the minimum stiffness of a simple point support that raises a natural frequency of a beam to its upper limit. Albarracín et al. (2004) detected a rather curious situation of changes in frequencies values and mode shapes when an intermediate translational restraint is placed in a beam simply supported at both ends.

There is only a limited amount of information for the vibration of beams with internal hinges. Wang and Wang (2001) studied the fundamental frequency of a beam with an internal hinge and subjected to an axial force. Chang et al. (2006) investigated the dynamic response of a beam with an internal hinge, subjected to a random moving oscillator. Grossi and Quintana (2008) investigated the natural frequencies and mode shapes of a non homogeneous tapered beam subjected to general axial forces, with arbitrarily located internal hinge and elastics supports, and ends elastically restrained against rotation and translation. Finally, Raffo and Grossi (2011) studied the effects on natural frequencies and mode shapes of beams with an intermediate elastic support obtaining the exact value of its rigidity when a modal shift occurs.

The above review of the literature reveals that many efforts had been devoted to the analysis of the influence of elastic restraints parameters, located at the ends and at intermediate points, on the dynamics characteristics of beams. However, the influence on frequencies and mode shapes of varying intermediate supports located at nodal points of higher modes has been studied only for classical end conditions. There is no paper that presents a complete analysis of the mentioned effects of intermediate elastic supports in a beam generally restrained at both ends. Also, in this subject the presence of internal hinges has not been treated yet.

The aim of the present paper is to investigate the natural frequencies and mode shapes of a beam with two arbitrarily located internal hinges, four intermediate elastic restraints and ends elastically restrained against rotation and translation. Adopting the adequate values of the rotational and translational restraints parameters at the ends, all the possible combinations of classical end conditions, (i.e.: clamped, simply supported, sliding and free) can be generated. The presence of the two hinges and the intermediate elastic restraints in particular, allows to include a hinge located at an intermediate point and a translational restraint located at a different point. From this property the study of the influence of a translational restraint located at a node of a higher mode of vibration can be inferred to be valuable. The existence of a critical value of the dimensionless restraint parameter which determines the interchange of roles of the corresponding modal shapes of two consecutive non-dimensional frequency parameters is demonstrated.

The classical method of separation of variables has been used for the determination of the exact frequencies and mode shapes. The algorithm developed can be applied to a wide range of elastic restraint conditions. The effects of the variations of the elastic restraints on the switching of the mode shape order and the influence of the internal hinges are investigated.

Tables and figures are given for frequencies, and two-dimensional plots for mode shapes are included. A great number of problems were solved and, since the number of cases is prohibitively large, results are presented for only a few cases.

2 THE BOUNDARY VALUE PROBLEM

Let us consider a beam of length l, which has elastically restrained ends, it is constrained at two intermediate points and has two internal hinges, as shown in Figure 1. The beam system is made up of three different spans, which correspond to the intervals $\left[0,c_1\right]$, $\left[c_1,c_2\right]$ and $\left[c_2,l\right]$ respectively. It is assumed that the ends and the internal hinges are elastically restrained against translation and rotation. The rotational restraints are characterized by the parameters $r_L, r_R, r_{c_i}, i=1,2$ and the translational restraints by $t_L, t_R, t_{c_i}, i=1,2.$ Adopting the adequate values of the parameters r_L, r_R and t_L, t_R all the possible combinations of classical end conditions can be generated. By using $t_{c_i}, r_{c_i}, i=1,2,$ the effects of the internal hinges and intermediate restraints are taken into account.

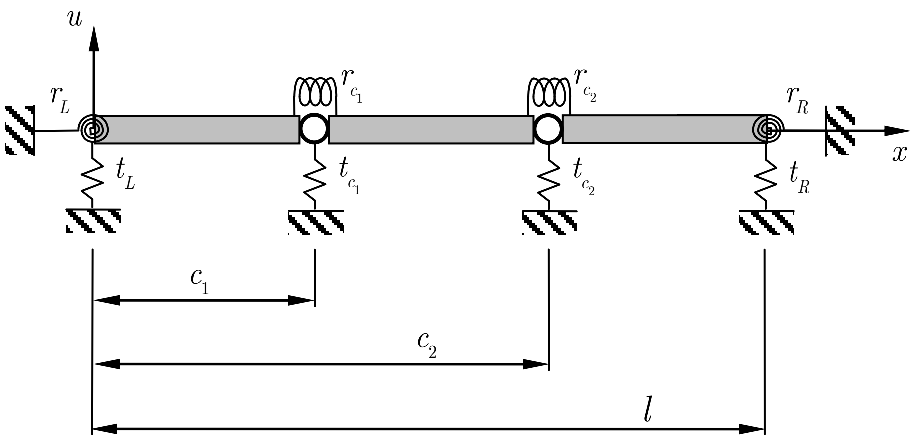


Figure 1: Mechanical system under study.

In order to analyze the transverse planar displacements of the system under study, we suppose that the vertical position of the beam at any time t is described by the function $u = u(x,t)$, $x \in [0,l]$. It is well known that at time t the kinetic energy of the beam can be expressed as

$$T_b = \frac{1}{2}\sum_{i=1}^{3}\int_{c_{i-1}}^{c_i}(\rho A)_i(x)\left(\frac{\partial u}{\partial t}(x,t)\right)^2 dx \tag{1}$$

where $(\rho A)_i = \rho_i A_i$ denotes the mass per unit length of the $i-$th span and $c_0 = 0, c_3 = l.$

The total potential energy due to the elastic deformation of the beam, the elastic restraints at the ends and the intermediate elastic restraints, is given by:

$$\begin{aligned} U = \frac{1}{2}\Bigg\{&\sum_{i=1}^{3}\int_{c_{i-1}}^{c_i}(EI)_i(x)\left(\frac{\partial^2 u}{\partial x^2}(x,t)\right)^2 dx \\ &+\sum_{i=0}^{3}\left(r_{c_i}\left(\frac{\partial u}{\partial x}(c_i^+,t)-\frac{\partial u}{\partial x}(c_i^-,t)\right)^2 + t_{c_i}u^2(c_i,t)\right)\Bigg\}, \end{aligned} \tag{2}$$

where $(EI)_i = E_i I_i$ denotes the flexural rigidity of the $i-$th span, $r_{c_0} = r_L, t_{c_0} = t_L, r_{c_3} = r_R,$ $t_{c_3} = t_R$ and the notations $0^+, c_i^+, c_i^-$ and l^- imply the use of lateral limits and lateral derivatives. Since $c_0^- = 0^- \notin [0,l]$ and $c_3^+ = l^+ \notin [0,l],$ in Eq. (2) it is assumed that $\frac{\partial u}{\partial x}(0^-,t) \equiv 0, \frac{\partial u}{\partial x}(l^+,t) \equiv 0.$

Hamilton´s principle requires that between times t_a and $t_b,$ at which the positions are known, the motion will make stationary the action integral $F(u) = \int_{t_a}^{t_b} L dt$ on the space of admissible functions, where the Lagrangian L is given by $L = T_b - U$. In consequence, the energy functional to be considered is given by

$$F(u) = \frac{1}{2}\int_{t_a}^{t_b}\left[\sum_{i=1}^{3}\int_{c_{i-1}}^{c_i}\left((\rho A)_i(x)\left(\frac{\partial u}{\partial t}(x,t)\right)^2 - (EI)_i(x)\left(\frac{\partial^2 u}{\partial x^2}(x,t)\right)^2\right)dx\right]dt - \\ -\frac{1}{2}\int_{t_a}^{t_b}\sum_{i=0}^{3}\left(r_{c_i}\left(\frac{\partial u}{\partial x}(c_i^+,t) - \frac{\partial u}{\partial x}(c_i^-,t)\right)^2 + t_{c_i}u^2(c_i,t)\right)dt. \tag{3}$$

The stationary condition for the functional given by Eq. (3) requires that

$$\delta F(u;v) = 0, \forall v \in D_a, \tag{4}$$

where $\delta F(u,v)$ is the first variation of F at u in the direction v and D_a is the space of admissible directions at u for the space D of admissible functions. In order to make the mathematical developments required by the application of the techniques of the calculus of variations, we assume that $(\rho A)_i \in C([c_{i-1},c_i])$, $(EI)_i \in C^2([c_{i-1},c_i])$, $i = 1,2,3.$

The space D is the set of functions $u(x,\bullet) \in C^2[t_a,t_b]$, $u(\bullet,t) \in C([0,l])$, $u(\bullet,t)\big|_{[c_{i-1},c_i]} \in C^4([c_{i-1},c_i])$, $i = 1,2,3.$

In view of all these observations and since Hamilton´s principle requires that at times t_a and t_b the positions are known, the space D is given by

$$D = \Big\{u; u(x,\bullet) \in C^2[t_a,t_b], u(\bullet,t) \in C([0,l]), u(\bullet,t)\big|_{[c_{i-1},c_i]} \in C^4([c_{i-1},c_i]), \\ i = 1,2,3, u(x,t_a), u(x,t_b) \text{ prescribed}, \forall x \in [0,l]\Big\}. \tag{5}$$

The only admissible directions v at $u \in D$ are those for which $u + \varepsilon v \in D$ for sufficiently small ε and $\delta F(u;v)$ exists. In consequence, and in view of Eq. (5), v is an admissible direction at u for D if, and only if, $v \in D_a$ where

$$D_a = \Big\{v; v(x,\bullet) \in C^2[t_a,t_b], v(\bullet,t) \in C([0,l]), v(\bullet,t)\big|_{[c_{i-1},c_i]} \in C^4([c_{i-1},c_i]), \\ i = 1,2,3, v(x,t_a) = v(x,t_b) = 0, \forall x \in [0,l]\Big\}. \tag{6}$$

The definition of the first variation of F at u in the direction v, is given by

$$\delta F(u;v) = \left.\frac{dF(u+\varepsilon v)}{d\varepsilon}\right|_{\varepsilon=0}. \tag{7}$$

The application of Eq. (7) with the expression of F given by Eq. (3), leads to

$$\delta F(u;v) = \\ = \int_{t_a}^{t_b}\left[\sum_{i=1}^{3}\int_{c_{i-1}}^{c_i}\left((\rho A)_i(x)\frac{\partial u}{\partial t}(x,t)\frac{\partial v}{\partial t}(x,t)-(EI)_i(x)\frac{\partial^2 u}{\partial x^2}(x,t)\frac{\partial^2 v}{\partial x^2}(x,t)\right)dx - \right. \\ \left. -\sum_{i=0}^{3}\left(r_{c_i}\left(\frac{\partial u}{\partial x}(c_i^+,t)-\frac{\partial u}{\partial x}(c_i^-,t)\right)\left(\frac{\partial v}{\partial x}(c_i^+,t)-\frac{\partial v}{\partial x}(c_i^-,t)\right)+t_{c_i}u(c_i,t)v(c_i,t)\right)\right]dt. \tag{8}$$

The well known procedure of integration by parts, gives:

$$\int_{c_{i-1}}^{c_i}\left((EI)_i(x)\frac{\partial^2 u}{\partial x^2}(x,t)\frac{\partial^2 v}{\partial x^2}(x,t)\right)dx = (EI)_i(x)\frac{\partial^2 u}{\partial x^2}(x,t)\frac{\partial v}{\partial x}(x,t)\Big|_{c_{i-1}^+}^{c_i^-} - \\ -\frac{\partial}{\partial x}\left((EI)_i(x)\frac{\partial^2 u}{\partial x^2}(x,t)\right)v(x,t)\Big|_{c_{i-1}^+}^{c_i^-} + \int_{c_{i-1}}^{c_i}\frac{\partial^2}{\partial x^2}\left((EI)_i(x)\frac{\partial^2 u}{\partial x^2}(x,t)\right)v(x,t)dx, \\ i=1,2,3. \tag{9}$$

$$\int_{t_a}^{t_b}\left(\int_{c_{i-1}}^{c_i}\left((\rho A)_i(x)\frac{\partial u}{\partial t}(x,t)\frac{\partial v}{\partial t}(x,t)\right)dx\right)dt = \\ = -\int_{t_a}^{t_b}\left(\int_{c_{i-1}}^{c_i}(\rho A)_i(x)\frac{\partial^2 u}{\partial t^2}(x,t)v(x,t)dx\right)dt, i=1,2,3. \tag{10}$$

By replacing Eqs. (9) and (10) into Eq. (8) we get

$$\delta F(u;v) \\ = -\int_{t_a}^{t_b}\left[\sum_{i=1}^{3}\int_{c_{i-1}}^{c_i}\left((\rho A)_i(x)\frac{\partial^2 u}{\partial t^2}(x,t)+\frac{\partial^2}{\partial x^2}\left((EI)_i(x)\frac{\partial^2 u}{\partial x^2}(x,t)\right)\right)v(x,t)dx + \right. \\ +\sum_{i=1}^{3}\left((EI)_i(x)\frac{\partial^2 u}{\partial x^2}(x,t)\frac{\partial v}{\partial x}(x,t)\Big|_{c_{i-1}^+}^{c_i^-} - \frac{\partial}{\partial x}\left((EI)_i(x)\frac{\partial^2 u}{\partial x^2}(x,t)\right)v(x,t)\Big|_{c_{i-1}^+}^{c_i^-}\right) + \\ \left. +\sum_{i=0}^{3}\left(r_{c_i}\left(\frac{\partial u}{\partial x}(c_i^+,t)-\frac{\partial u}{\partial x}(c_i^-,t)\right)\left(\frac{\partial v}{\partial x}(c_i^+,t)-\frac{\partial v}{\partial x}(c_i^-,t)\right)+t_{c_i}u(c_i,t)v(c_i,t)\right)\right]dt. \tag{11}$$

Now it is convenient to consider the directions v which satisfy

$$v(c_i,t) = \frac{\partial v}{\partial x}(c_i^-,t) = \frac{\partial v}{\partial x}(c_i^+,t) = 0, i=0,1,2,3, \forall t \in (t_a,t_b). \tag{12}$$

Substituting Eqs. (12) into Eq. (11) and applying the stationary condition required by Hamilton´s principle given by Eq. (4), leads to

$$\delta F(u;v) = \int_{t_a}^{t_b}\left[\sum_{i=1}^{3}\int_{c_{i-1}}^{c_i}\left((\rho A)_i(x)\frac{\partial^2 u}{\partial t^2}(x,t) \right.\right. \\ \left.\left. +\frac{\partial^2}{\partial x^2}\left((EI)_i(x)\frac{\partial^2 u}{\partial x^2}(x,t)\right)\right)v(x,t)dx\right]dt = 0, \forall v \in D_a. \tag{13}$$

Let us assume $t_a = 0$, then as v is an arbitrary smooth function, the fundamental lemma of the calculus of variations can be applied to Eq. (13) to conclude that the function u must satisfy the following differential equations:

$$\begin{gathered}\frac{\partial^2}{\partial x^2}\left(\left(EI\right)_i\left(x\right)\frac{\partial^2 u}{\partial x^2}\left(x,t\right)\right)+\left(\rho A\right)_i\left(x\right)\frac{\partial^2 u}{\partial t^2}\left(x,t\right)=0,\\ \forall x\in\left(c_{i-1},c_i\right), i=1,2,3, t\geq 0.\end{gathered} \tag{14}$$

Now it is possible to remove the restrictions given by Eqs. (12) and since the function u must satisfy the differential equations stated above, Eq. (11) is reduced to

$$\begin{aligned}&\delta F\left(u;v\right)\\ &=-\int_{t_a}^{t_b}\left[\sum_{i=1}^{3}\left(\left(EI\right)_i\left(x\right)\frac{\partial^2 u}{\partial x^2}\left(x,t\right)\frac{\partial v}{\partial x}\left(x,t\right)\Bigg|_{c_{i-1}^+}^{c_i^-}-\frac{\partial}{\partial x}\left(\left(EI\right)_i\left(x\right)\frac{\partial^2 u}{\partial x^2}\left(x,t\right)\right)v\left(x,t\right)\Bigg|_{c_{i-1}^+}^{c_i^-}\right)\right.\\ &\left.+\sum_{i=0}^{3}\left(r_{c_i}\left(\frac{\partial u}{\partial x}\left(c_i^+,t\right)-\frac{\partial u}{\partial x}\left(c_i^-,t\right)\right)\left(\frac{\partial v}{\partial x}\left(c_i^+,t\right)-\frac{\partial v}{\partial x}\left(c_i^-,t\right)\right)+t_{c_i}u\left(c_i,t\right)v\left(c_i,t\right)\right)\right]dt.\end{aligned} \tag{15}$$

Since v and $\dfrac{\partial v}{\partial x}$ in Eq. (15) are smooth functions of t and are arbitrary, the stationary condition given by Eq. (4) applied to Eq. (15) leads to the boundary and transitions conditions:

$$r_L\frac{\partial u}{\partial x}\left(0^+,t\right)=\left(EI\right)_1\frac{\partial^2 u}{\partial x^2}\left(0^+,t\right), \tag{16}$$

$$t_L u\left(0^+,t\right)=-\frac{\partial}{\partial x}\left(\left(EI\right)_1\frac{\partial^2 u}{\partial x^2}\left(0^+,t\right)\right), \tag{17}$$

$$w\left(c_i^-,t\right)=w\left(c_i^+,t\right), i=1,2, \tag{18}$$

$$r_{c_i}\left(\frac{\partial u}{\partial x}\left(c_i^+,t\right)-\frac{\partial u}{\partial x}\left(c_i^-,t\right)\right)=\left(EI\right)_i\frac{\partial^2 u}{\partial x^2}\left(c_i^-,t\right), i=1,2, \tag{19}$$

$$r_{c_i}\left(\frac{\partial u}{\partial x}\left(c_i^+,t\right)-\frac{\partial u}{\partial x}\left(c_i^-,t\right)\right)=\left(EI\right)_{i+1}\frac{\partial^2 u}{\partial x^2}\left(c_i^+,t\right), i=1,2, \tag{20}$$

$$t_{c_i}u\left(c_i,t\right)=\frac{\partial}{\partial x}\left(\left(EI\right)_i\frac{\partial^2 u}{\partial x^2}\left(c_i^-,t\right)\right)-\frac{\partial}{\partial x}\left(\left(EI\right)_{i+1}\frac{\partial^2 u}{\partial x^2}\left(c_i^+,t\right)\right), i=1,2, \tag{21}$$

$$r_R\frac{\partial u}{\partial x}\left(l^-,t\right)=-\left(EI\right)_3\frac{\partial^2 u}{\partial x^2}\left(l^-,t\right), \tag{22}$$

$$t_R u\left(l^-,t\right)=\frac{\partial}{\partial x}\left(\left(EI\right)_3\frac{\partial^3 u}{\partial x^3}\left(l^-,t\right)\right),\tag{23}$$

where $t\geq 0$.

Different situations can be generated by substituting values and/or limiting values of the restraint parameters r_{c_i} and t_{c_i}. When we consider $r_{c_i}=\infty, t_{c_i}=0, i=1,2,$ there are no internal hinges and the articulations are perfectly rigid. Now if we consider $r_{c_i}=0, t_{c_i}=0, i=1,2,$ there are internal hinges located at c_1 and c_2 and the articulations are perfect. Finally if we have $0<r_{c_i}<\infty, 0<t_{c_i}<\infty, i=1,2,$ there are internal hinges elastically restrained against rotation and supported by the respective translational restraints.

It is worth noting that this mathematical model allows the inclusion of a hinge located at a point c_i and a translational restraint located at a different point c_j. As stated above this property will probe it is valuable to study the influence of a translational restraint located at a node of a higher vibration mode.

3 NATURAL FREQUENCIES AND MODE SHAPES

Using the well-known method of separation of variables, when the mass per unit length and the flexural rigidity at the spans are the same, we assume as solutions of Eqs. (14) the functions given by the series

$$u_i\left(x,t\right)=\sum_{n=1}^{\infty}u_{i,n}\left(x\right)\cos\omega t,\ i=1,2,3,\tag{24}$$

where $u_{i,n}$ are the corresponding nth modes of natural vibration. Introducing the change of variable $\overline{x}=x/l$ into Eqs. (14) and (16)-(23), the functions $u_{i,n}$ are given by

$$u_{1,n}\left(\overline{x}\right)=A_1\cosh\lambda\overline{x}+A_2\sinh\lambda\overline{x}+A_3\cos\lambda\overline{x}+A_4\sin\lambda\overline{x},\quad\forall\overline{x}\in\left[o,\overline{c}_1\right],\tag{25}$$

$$u_{2,n}\left(\overline{x}\right)=A_5\cosh\lambda\overline{x}+A_6\sinh\lambda\overline{x}+A_7\cos\lambda\overline{x}+A_8\sin\lambda\overline{x},\quad\forall\overline{x}\in\left[\overline{c}_1,\overline{c}_2\right],\tag{26}$$

$$u_{3,n}\left(\overline{x}\right)=A_9\cosh\lambda\overline{x}+A_{10}\sinh\lambda\overline{x}+A_{11}\cos\lambda\overline{x}+A_{12}\sin\lambda\overline{x},\quad\forall\overline{x}\in\left[\overline{c}_2,1\right],\tag{27}$$

where $\overline{c}_i=c_i/l$ and

$$\lambda^4=\frac{\rho A}{EI}\omega^2 l^4.\tag{28}$$

Substituting Eqs. (25)-(27) into Eq. (24) and then in the boundary conditions given by Eqs. (16), (17), (22), (23) and transition conditions defined by Eqs. (18)-(21), expressed in the new variable $\overline{x}$, we obtain a set of twelve homogeneous equations in the constants A_i. Since the system is homogeneous in order to obtain a non-trivial solution the determinant of coefficients must be equal to zero. This procedure yields the frequency equation:

$$G\left(T_L,R_L,T_R,R_R,T_{\overline{c}_i},R_{\overline{c}_i},\lambda,\overline{c}_i\right)=0,\ i=1,2,\tag{29}$$

where

$$T_L = \frac{t_L l^3}{EI},\ R_L = \frac{r_L l}{EI},\ T_R = \frac{t_R l^3}{EI},\ R_R = \frac{r_R l}{EI},\ T_{\bar{c}_i} = \frac{t_{c_i} l^3}{EI},\quad R_{\bar{c}_i} = \frac{r_{c_i} l}{EI},\quad i = 1,2. \tag{30}$$

The values of the frequency parameter $\lambda = \left(\left(\rho A / EI\right)\omega^2\right)^{1/4} l$, were obtained with the classical bisection method and rounded to eight decimal digits.

In order to describe the corresponding boundary conditions the symbolism SS identifies a simply supported end, C a clamped end, F a free end and ER identifies an elastically restrained end. Since the number of cases which can be analyzed by the developed algorithm is prohibitively large, results are presented only for a few cases.

4 THE INFLUENCE OF INTERMEDIATE TRANSLATIONAL RESTRAINTS WITH AN INTERNAL HINGE

As stated in Section 1, in the determination of an additional translational restraint required to maximize a natural frequency, Courant and Hilbert (1953) for rigid supports and Akesson and Olhoff (1988) for elastic supports, have demonstrated that the optimum location of a support should be at the nodal points of a higher vibration mode. Raffo and Grossi (2011) studied the influence of internal traslational restraints that shifts modal shapes. For this reason in all the described cases in this study the restraints locations coincide with nodal points of some higher modes.

In this work, the minimum stiffness of an elastic restraint that raises a natural frequency of a beam with an internal hinge, to its upper limit is obtained. The minimum stiffness is determined by using the derivative of the function which gives the natural frequencies, with respect to the support position. Additionally, sensitivity analyses of the effects on the two lower natural frequencies of the optimally supported beam to variations in position are considered. The analyzed cases are a SS-SS beam, a C-SS beam and a ER-ER beam with the adopted values $T_L = R_L = 1000,\ T_R = R_R = 100$.

4.1 Imperfection sensitivity of a simply supported beam (SS-SS).

In order to obtain the minimum value of the elastic restriction and its location, a sensitivity analysis is performed for the first two values of the frequency parameter.

First the case of a SS-SS beam with one flexible support located at the first node of the second modal shape, $\bar{c}_1^{(1,2)} = 0.598855$, and with a hinge located at $\bar{c}_2 = 0.75$, is considered. In Figure 2, the first two exact values of the frequency parameter λ are plotted against the restraint parameter $T_{\bar{c}} \big/ T^{(1,2)}$. It is observed that the curves have a contact point denoted by P and that to this point it corresponds a value, namely $T^{(1,2)}$ of $T_{\bar{c}}$, such that over it the values of λ_1 cannot be raised further whereas the values of the coefficient λ_2 increases. More specifically, in the interval $\left[0,1\right]$ the parameter λ_1 increases from the value which corresponds to $T_{\bar{c}} \big/ T^{(1,2)} = 0$ to its maximum $\lambda_1 = \lambda_2$ for $T_{\bar{c}} \big/ T^{(1,2)} = 1$ and λ_2 remains constant as $T_{\bar{c}} \big/ T^{(1,2)}$ increase. In the interval $\left(1, 1.10^4\right]$ the parameter λ_2 increases and λ_1 remains constant as $T_{\bar{c}} \big/ T^{(1,2)}$ increases. This phenomenon suggests the possibility of a change in the corresponding mode shapes.

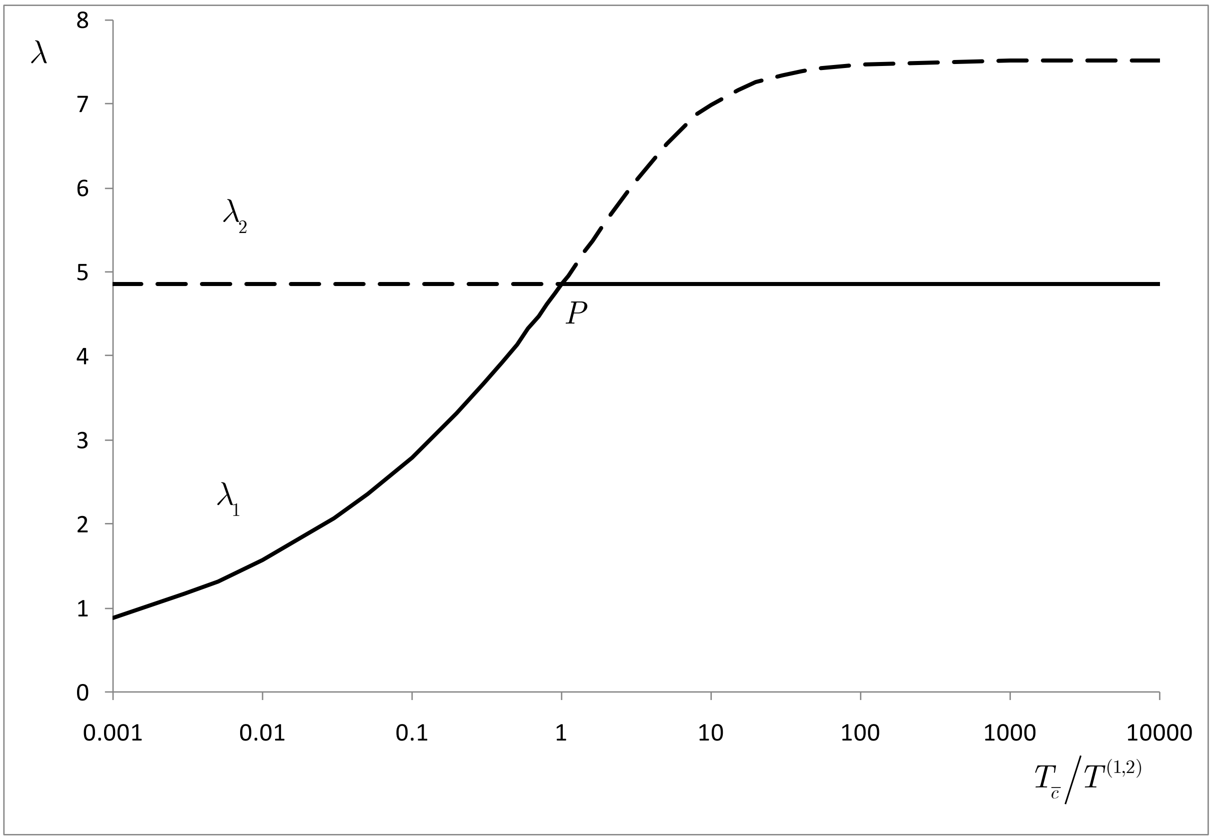


Figure 2: Variation of the first two exact values of the frequency parameter λ as a function of the translational restraint parameter $T_{\bar{c}} / T^{(1,2)}$ located at $\bar{c}_1^{(1,2)} = 0.598855$ which corresponds to a SS-SS beam with a hinge located at $\bar{c}_2 = 0.75$.

Based on the concepts presented, a numerical procedure has been developed with the purpose of determining the critical value $T^{(1,2)}$ of $T_{\bar{c}}$, which consists in the replacement of the values of λ_2 and $\bar{c}$ into Eq. (29). In this case the value $T^{(1,2)} = 325.337632$ has been obtained. Table 1 depicts the first two exact values of the frequency coefficient λ and the corresponding mode shapes of a SS-SS beam with a traslational restriction at $\bar{c}_1^{(1,2)} = 0.598855$ and a hinge located at $\bar{c}_2 = 0.75$, for different values of the restraint stiffness $T_{\bar{c}}$. It is observed that the modes which correspond to λ_1 for $T_{\bar{c}} = 0$ and $T_{\bar{c}} = T^{(1,2)} - \varepsilon$ do not have a nodal point, while the second mode has it. Nevertheless, when $T_{\bar{c}} = T^{(1,2)} + \varepsilon$ the first mode has a nodal point and the second mode do not has it. Obviously ε assumes a small value. It can also be observed that as $T_{\bar{c}}$ increases, the first modal shape presents inflection points as it is illustrated by the figure which corresponds to the case $T^{(1,2)} - \varepsilon$. The corresponding mode shapes are analog until $T_{\bar{c}} = T^{(1,2)}$. In this process we have that $\lambda_1 \to \lambda_2$ from the left as $T_{\bar{c}}$ increases in the interval $\left[0, T^{(1,2)}\right]$. When $T_{\bar{c}} > T^{(1,2)}$ there is a change: the values of λ_1 remain constant meanwhile the values of λ_2 increase as $T_{\bar{c}}$

increases and the original second mode ($T_{\bar{c}} = 0$) becomes the new first mode, i.e.: the mode shape which corresponds to λ_1 when $T_{\bar{c}} > T^{(1,2)}$, is identical to the mode shape which corresponds to λ_2 when $T_{\bar{c}} = 0$.

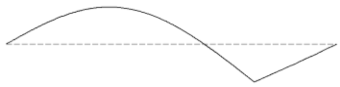
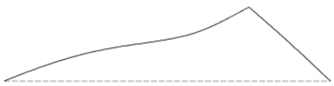
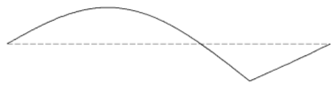
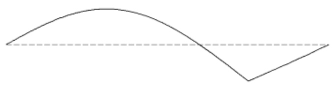
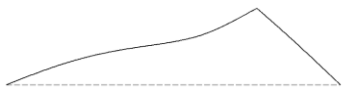
$T_{\bar{c}}$	λ_1	λ_2
0	0 	4.85792569 
$T^{(1,2)} - \varepsilon$	4.85781784 	4.85792569 
$T^{(1,2)} = 325.337632$	4.85792569	4.85792569
$T^{(1,2)} + \varepsilon$	4.85792569 	4.85803352 

Table 1: Values λ_1 and λ_2 of the frequency coefficient λ and mode shapes of a SS-SS beam with $T^{(1,2)} = 325.337632$ located at $\bar{c}_1 = 0.598855$, and with a hinge located at $\bar{c}_2 = 0.75$.

Considering previous works and present work, the described phenomenon can be generalized by arguing that there exists a critical value $T^{(i,i+1)}$ of $T_{\bar{c}}$ where $\lambda_i = \lambda_{i+1}, \forall i$.

The equality of eigenvalues can be explained through the existence of roots of multiplicity of the frequency Eq. (29). The procedure to obtain the values $T^{(i,i+1)}$, is analogous to that used for $T^{(1,2)}$.

In Figure 3 is analyzed the sensitivity of the first two values of the frequency parameter of the optimally supported beam to variations in position and stiffness around optimum point ($T_{\bar{c}} = T^{(1,2)}$, $\bar{c}_1 = \bar{c}_1^{(1,2)}$), the solid lines correspond to λ_1, the dashed lines correspond to λ_2, and it is considered $T_{\bar{c}} / T^{(1,2)} = 0.5, 0.75, 1.0, 1.25, 1.5, 5$ that represent six curves for λ_1 and six curves for λ_2 where the arrows in Figure 3 indicates the sense that $T_{\bar{c}} / T^{(1,2)}$ increases. From Figure 3, it is observed that the value of λ_1 increases when $T_{\bar{c}} / T^{(1,2)}$ increases. Also, when $T_{\bar{c}} / T^{(1,2)}$ increases, λ_1 is less sensible to the value of $\bar{c}_1 / \bar{c}_1^{(1,2)}$.

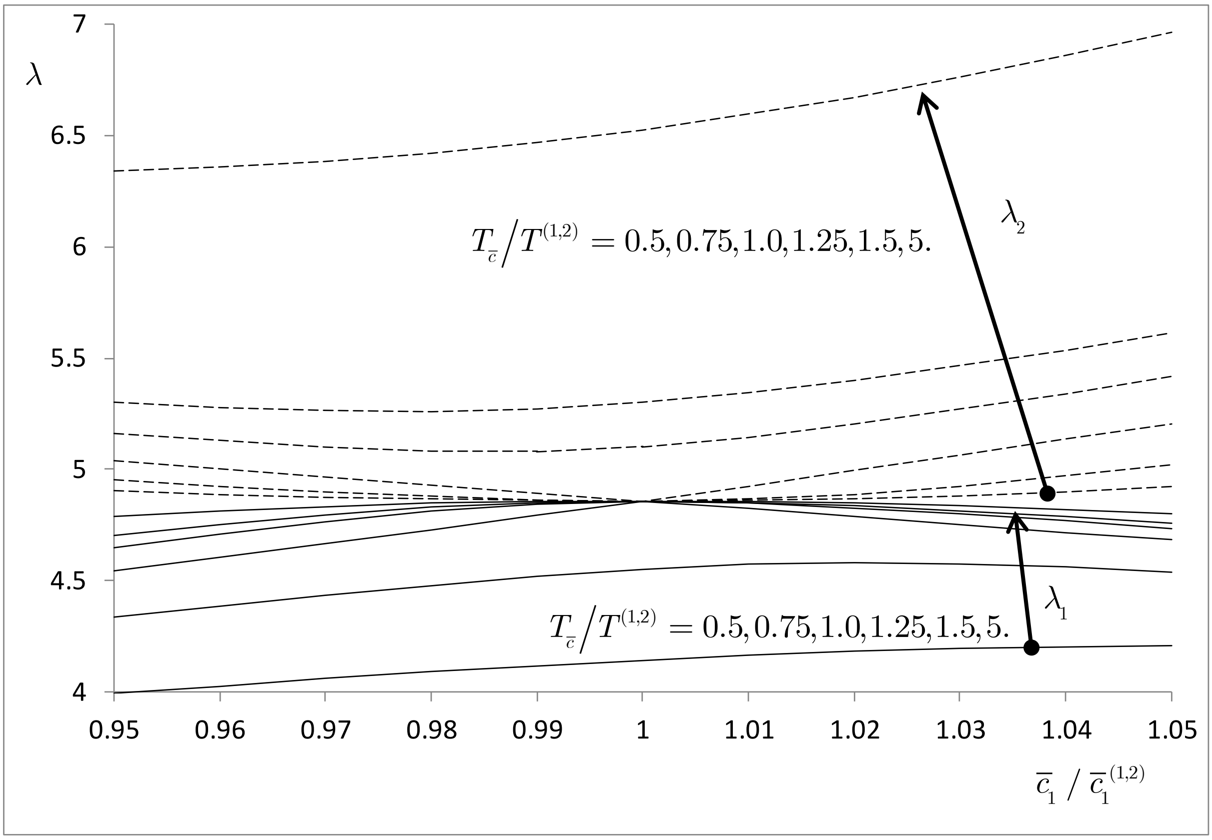


Figure 3: Sensitivity analysis of the first two exact values of the frequency parameter λ as a function of translational restraint parameter $T_{\bar{c}}/T^{(1,2)}$ when varying its location around optimum point of a SS-SS beam with a hinge located at $\bar{c}_2 = 0.75$.

4.2 Imperfection sensitivity of a clamped and simply supported beam (C-SS).

In this case, the analyzed beam consists in a C-SS beam with a hinge located at $\bar{c}_1 = 0.5$, and the traslational restraint is located at $\bar{c}_2 = 0.565059$. The optimal point that maximize the first value of the frequency parameter and minimize the value of the elastic traslational restriction was obtained with the same procedure as explained in the previous section, its value is $T^{(1,2)} = 1118.038045$. Figure 4 shows the sensitivity of the first two values of the frequency parameter of the optimally supported beam to variations in position and stiffness around optimum point ($T_{\bar{c}} = T^{(1,2)}$, $\bar{c}_2 = \bar{c}_2^{(1,2)}$). From Figure 4, it is observed that the value of λ_1 increases when $T_{\bar{c}}/T^{(1,2)}$ increases. Also, when $T_{\bar{c}}/T^{(1,2)}$ increases, λ_1 is less sensible to the value of $\bar{c}_2 / \bar{c}_2^{(1,2)}$.

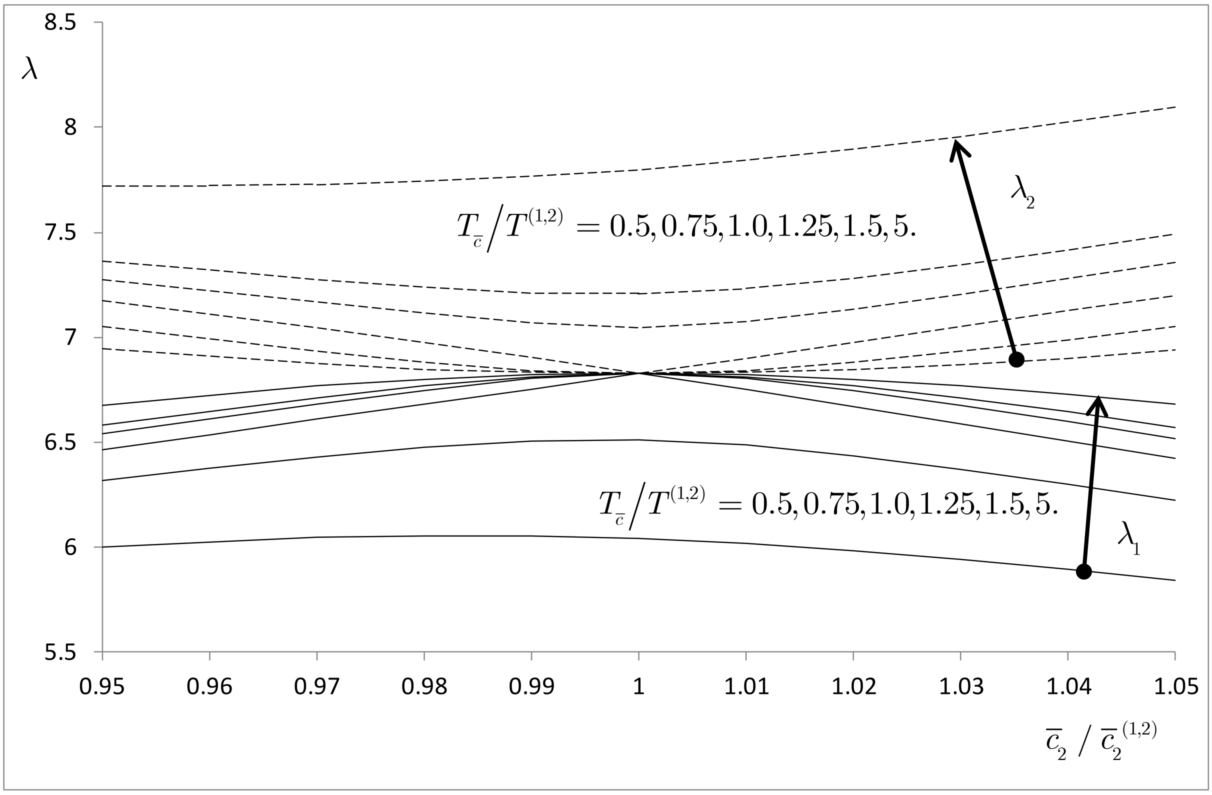


Figure 4: Sensitivity analysis of the first two exact values of the frequency parameter λ as a function of translational restraint parameter $T_{\bar{c}}/T^{(1,2)}$ when varying its location around optimum point of a C-SS beam with a hinge located at $\bar{c}_1 = 0.5$.

4.3 Imperfection sensitivity of an elastically restrained beam (ER-ER).

In this case, the analyzed beam consists in an ER-ER beam with the adopted values $T_L = R_L = 1000,\ T_R = R_R = 100$, with a hinge located at $\bar{c}_1 = 0.5$, and the traslational restraint located at $\bar{c}_2 = 0.617515$. The optimal point that maximize the first value of the frequency parameter and minimize the value of the elastic traslational restriction was obtained with the same procedure explained previously in this work. The value of the obtained traslational restriction is $T^{(1,2)} = 624.474974$. Figure 5 shows the sensitivity of the first two values of the frequency parameter around the optimal point ($T_{\bar{c}} = T^{(1,2)}$, $\bar{c}_2 = \bar{c}_2^{(1,2)}$), the solid lines correspond to λ_1, the dashed lines correspond to λ_2, and $T_{\bar{c}}/T^{(1,2)} = 0.5, 0.75, 1.0, 1.25, 1.5, 5$ representation increases in the sense of the arrows. From Figure 5, it is observed that when $T_{\bar{c}}/T^{(1,2)}$ increases, λ_1 increases and λ_1 is less sensible to $\bar{c}_2 \,/\, \bar{c}_2^{(1,2)}$.

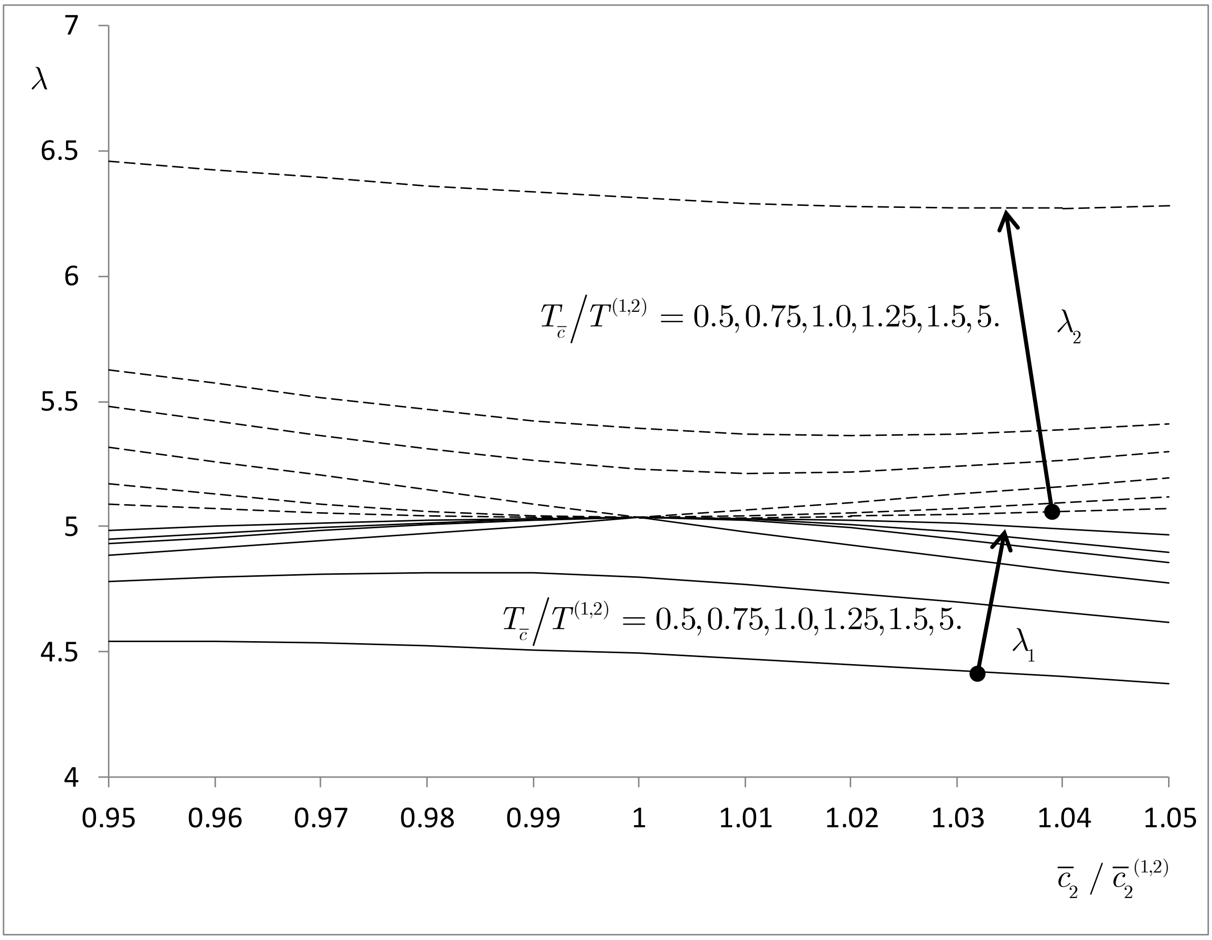


Figure 5: Sensitivity analysis of the first two exact values of the frequency parameter λ as a function of translational restraint parameter $T_{\bar{c}}/T^{(1,2)}$ when varying its location around optimum point of an ER-ER ($T_L = R_L = 1000$, $T_R = R_R = 100$) beam with a hinge located at $\bar{c}_1 = 0.5$.

5 THE INFLUENCE OF TWO INTERNAL HINGES

In this section, numerical results of the exact values of the frequency parameter of a beam with two internal line hinges are obtained. Tables 2 and Table 3 depict the first three exact values of the frequency parameter λ of a beam with two internal hinges. Different boundary conditions and values of the parameters $\bar{c}_i, i = 1,2$ are considered. The corresponding mode shapes are also included. It is worth noting that in order to avoid the zero frequency value which corresponds to a rigid body motion a small value of the restraint parameters $T_{\bar{c}_1}$ and $T_{\bar{c}_2}$ has been adopted. Tables 2 contains symmetrical boundary conditions and Table 3 includes non-symmetrical boundary conditions and in the case ER-ER the values $T_L = R_L = 1000$, $T_R = R_R = 100$ have been adopted. It is worth pointing that $u(\bullet, t) \in C[0,1]$, i.e. the deflection function is only continuous, but it has corner points only at the hinges locations. This property can be observed in the mode shapes included in Tables 2 and Table 3.

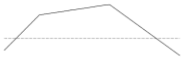
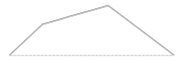
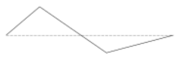
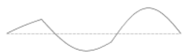
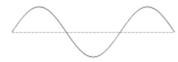
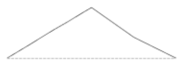
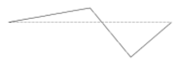
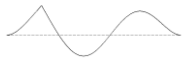
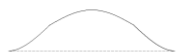
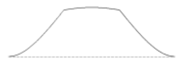
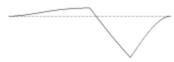
BC	$\bar{c}_1$	$\bar{c}_2$	λ_1		λ_2		λ_3	
F-F	0.2	0.6	1.375081		1.643200		9.362790	
	0.25	0.75	1.337032		1.618163		8.120665	
	1/3	2/3	1.414102		1.638014		10.669474	
	0.5	0.75	1.445185		1.681667		8.680980	
SS-SS	0.2	0.6	1.342282		1.583975		8.273242	
	0.25	0.75	1.315682		1.565026		7.855269	
	1/3	2/3	1.377333		1.564958		9.424777	
	0.5	0.75	1.390513		1.608228		7.173072	
C-C	0.2	0.6	3.772331		6.656591		9.738941	
	0.25	0.75	4.686472		6.031293		8.680536	
	1/3	2/3	4.248089		4.958357		10.740080	
	0.5	0.75	3.324057		6.124616		8.655705	

Table 2: Values λ_1, λ_2 and λ_3 of the frequency coefficient λ and mode shapes of a beam with two internal hinges with different symmetrical boundary conditions and values of the parameters $\bar{c}_i, i = 1, 2.$ $\left(T_{\bar{c}_1} = T_{\bar{c}_2} = 1\right)$.

6 CONCLUSIONS

The minimum stiffness of an elastic restraint that raises the first natural frequency of a beam with an internal hinge, to its upper limit was obtained. Additionally, a sensitivity analysis around the optimally supported beam to variations in position and stiffness of the intermediate traslational restraint was done. Also, the effects on natural frequencies of the presence of two internal hinges were analyzed.

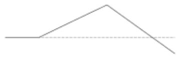
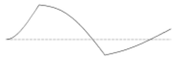
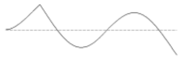
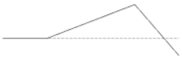
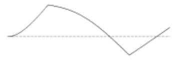
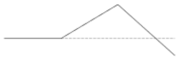
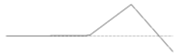
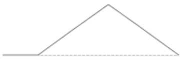
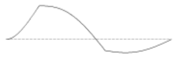
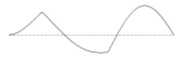
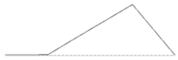
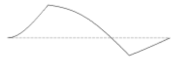
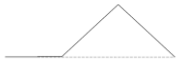
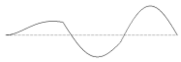
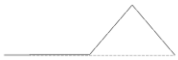
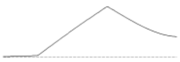
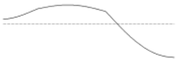
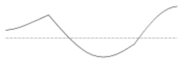
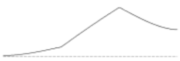
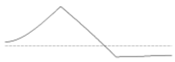
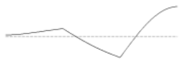
BC	$\overline{c}_1$	$\overline{c}_2$	λ_1		λ_2		λ_3	
C-F	0.2	0.6	1.438556		6.609714		9.735284	
	0.25	0.75	1.444605		5.385415		8.439022	
	1/3	2/3	1.505427		4.639534		10.705621	
	0.5	0.75	1.616772		3.382267		8.626570	
C-SS	0.2	0.6	1.391308		6.521151		8.583106	
	0.25	0.75	1.413622		5.353031		8.319202	
	1/3	2/3	1.456066		4.624047		9.810502	
	0.5	0.75	1.564002		3.375018		8.611874	
ER-ER	0.2	0.6	3.200393		5.459548		6.224496	
	0.25	0.75	3.661568		5.060301		7.209206	
	1/3	2/3	3.508990		4.478442		6.445352	
	0.5	0.75	3.266975		4.056564		7.364875	

Table 3: Values λ_1, λ_2 and λ_3 of the frequency coefficient λ and mode shapes of a beam with two internal hinges with different non-symmetrical boundary conditions and values of the parameters $\overline{c}_i, i = 1,2$. $\left(T_{\overline{c}_1} = T_{\overline{c}_2} = 1\right)$. The case $ER - ER$ is defined by $T_L = R_L = 1000, T_R = R_R = 100$.

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