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ON OBJECTIVITY AND THE PRINCIPLE OF MATERIAL FRAME-INDIFFERENCE

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Abstract. In the nineteen sixties and the seventies, rapid development of Modern Continuum Mechanics was based on the fundamental ideas set forth in the biblical treatise, The Non-Linear Field Theories of Mechanics, by Truesdell and Noll. Of them, one of the most important ideas is the principle of material frame-indifference (MFI). Unfortunately, due to the original somewhat loose statements, attempts for better interpretation of MFI appeared again and again throughout the following decades even until these days. Some involving serious misunderstandings and misinterpretations, hidden behind some seemingly plausible physical arguments or some impressive yet over-sophisticated mathematics.

As we understand, the essential meaning of MFI is the simple idea that material properties are independent of observers. In order to explain this, we shall describe what a frame of reference (regarded as an observer) is, and one shall never forget that any configuration/motion implies a previous choice of frame. Transformation properties for kinematic quantities can usually be derived from the deformation/motion under change of frame. For a non-kinematic quantity, such as force and stress, frame-indifference property (also known as objectivity) can not be derived and hence must be postulated. Frame-indifference postulate for the stress, sometimes unsuitably called the principle of frame-indifference, is a universal assumption which has nothing to do with material properties. This has caused some great confusions in the interpretation of “material” frame-indifference in the literature.

Mathematically, the principle of material frame-indifference can be stated as invariance of constitutive function under change of frame. However, great care must be observed of what such constitutive functions are. They should not simply be the constitutive functions relative to some reference configuration in two different frames, because a choice of reference configuration may change material properties. We shall carefully state the domain of constitutive functions and with clear and simple mathematical reasoning deduce the well-known condition of material objectivity as a consequence of objectivity postulate and the principle of material frame-indifference.

In this paper we show some misunderstandings and misstatements found in the very recent literature and we show how to correct them.

1 INTRODUCTION

Constitutive equations relates motion and forces of a material body. Hooke, in 1678 (Hooke (1931)), was the first to state a constitutive equation for a spring and to remark that the response of the spring is unaffected by a rigid motion. Later in 1829, Poisson and Cauchy (Poisson (1829); Cauchy (1829)), remarked also rotational-invariance of material response. Zaremba in 1905 (Zaremba (1903)) and Jaumann in 1906 (Jaumann (1906)), on the other hand, demanded invariance of response of a material for all observers. While all these ideas and some similar ones may be regarded as somewhat obvious, it might be more subtle to state them clearly. Various invariance ideas of material response were integrated for the first time in the fundamental treatise, The Non-Linear Field Theories of Mechanics, by Truesdell and Noll (Truesdell and Noll (2004)). In the nineteen sixties and the seventies, rapid development of Modern Continuum Mechanics was based on the fundamental ideas set forth in this treatise. Of them, one of the most important ideas is the principle of material frame-indifference (MFI). Unfortunately, due to the original somewhat loose statements in mathematical terms, attempts for better interpretation of MFI appeared again and again throughout the following decades even until these days (Frewer (2009); Gurtin et al. (2010); Murdoch (2003, 2005); Noll (2006); Noll and Seguin (2010); Ryskin (1985); Woods (1981)).

As we understand, the essential meaning of MFI is the simple idea that material properties are independent of observers. In order to explain this, we shall describe what a frame of reference (regarded as an observer) is, and one shall never forget that any configuration/motion implies a previous choice of frame (except some frame-free formulation (Noll (2006); Noll and Seguin (2010))). Transformation properties for kinematic quantities can usually be derived from the deformation/motion under change of frame. For a non-kinematic quantity, such as force and stress, frame-indifference property (also known as objectivity) cannot be derived and hence must be postulated. Frame-indifference postulate for the stress, sometimes unsuitably called the principle of frame-indifference, is a universal assumption which has nothing to do with material properties. This has caused some great confusions in the interpretation of "material" frame-indifference in the literature.

Mathematically, the principle of material frame-indifference can be stated as invariance of constitutive function under change of frame. However, great care must be observed of what such constitutive functions are. They should not simply be the constitutive functions relative to some reference configuration in two different frames, because a choice of reference configuration may change material properties. We shall carefully state the domain of constitutive functions and the notion of observer-independence to deduce the well-known condition of material objectivity as a consequence of objectivity postulate and the principle of material frame-indifference.

In this paper we shall remark on some misunderstandings and misstatements found in the very recent literature. Conventional notations now widely used in continuum mechanics textbooks will be followed. The main concepts, such as placement, configuration, deformation, that are essential are clearly described. For allowing different observers possibly belong to different Euclidean spaces as suggested by someone with more critical mind (Murdoch (2005)), our formulations take this into consideration, for which we introduce some notions of isometries in Euclidean spaces with changes in orientation as well as scaling.

Isometries in Euclidean spaces

For a Euclidean space $\mathbb{E}$, there is a vector space $\mathbb{V}$, called the translation space of $\mathbb{E}$, such that the difference $\boldsymbol{v} = \boldsymbol{x}_2 - \boldsymbol{x}_1$ of any two points $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{E}$ is a vector in $\mathbb{V}$. We also require that the vector space $\mathbb{V}$ be equipped with an inner product $(\cdot,\cdot)_{\mathbb{V}}$, so that length and angle can be defined.

Let $\mathbb{V}^*$ be the translation space of another Euclidean space $\mathbb{E}^*$ and $\mathcal{L}(\mathbb{V},\mathbb{V}^*)$ be the space of linear transformations from $\mathbb{V}$ to $\mathbb{V}^*$. For $A \in \mathcal{L}(\mathbb{V},\mathbb{V}^*)$, the transpose (or more generally called adjoint) of A, denoted by $A^\top \in \mathcal{L}(\mathbb{V}^*,\mathbb{V})$ satisfies $(\boldsymbol{u}, A^\top \boldsymbol{v}^*)_{\mathbb{V}} = (A\boldsymbol{u}, \boldsymbol{v}^*)_{\mathbb{V}^*}$ for $\boldsymbol{u} \in \mathbb{V}$ and $\boldsymbol{v}^* \in \mathbb{V}^*$.

Remark. In an inner product space, since the norm is defined as $\|\boldsymbol{u}\|_{\mathbb{V}} = \sqrt{(\boldsymbol{u},\boldsymbol{u})_{\mathbb{V}}}$, from the identity, $\|\boldsymbol{u}+\boldsymbol{v}\|^2_{\mathbb{V}} = \|\boldsymbol{u}\|^2_{\mathbb{V}} + \|\boldsymbol{v}\|^2_{\mathbb{V}} + 2(\boldsymbol{u},\boldsymbol{v})_{\mathbb{V}}$, it follow that

$$\|\boldsymbol{u}\|^2_{\mathbb{V}} = \|\boldsymbol{u}^*\|^2_{\mathbb{V}^*} \iff (\boldsymbol{u},\boldsymbol{v})_{\mathbb{V}} = (\boldsymbol{u}^*,\boldsymbol{v}^*)_{\mathbb{V}^*},$$

for any corresponding $\boldsymbol{u},\boldsymbol{v} \in \mathbb{V}$ and $\boldsymbol{u}^*,\boldsymbol{v}^* \in \mathbb{V}^*$. $\Box$

Definition. *$\mathcal{I} \in \mathcal{L}(\mathbb{V},\mathbb{V}^*)$ is called an isometric transformation if $\|\mathcal{I}\boldsymbol{u}\|_{\mathbb{V}^*} = \|\boldsymbol{u}\|_{\mathbb{V}}$ for any $\boldsymbol{u} \in \mathbb{V}$. Let $\mathcal{O}(\mathbb{V},\mathbb{V}^*)$ denote the set of all isometric transformations in $\mathcal{L}(\mathbb{V},\mathbb{V}^*)$.*

From the above observation, an isometric transformation preserves the norm as well as the inner product, the length and the angle.

Definition. *(Isometry): A bijective map $i : \mathbb{E} \to \mathbb{E}^*$ is an isometry if for $\boldsymbol{x} \in \mathbb{E}$,*

$$\boldsymbol{x}^* = i(\boldsymbol{x}) = \mathcal{I}(\boldsymbol{x} - \boldsymbol{x}_0) + \boldsymbol{x}_0^*, \tag{1}$$

for some $\boldsymbol{x}_0 \in \mathbb{E}$, $\boldsymbol{x}_0^* \in \mathbb{E}^*$ and some $\mathcal{I} \in \mathcal{O}(\mathbb{V},\mathbb{V}^*)$.

Let $L(\mathbb{V}) = \mathcal{L}(\mathbb{V},\mathbb{V})$ be the space of linear transformations and $O(\mathbb{V}) = \mathcal{O}(\mathbb{V},\mathbb{V})$ be the group of orthogonal transformation on $\mathbb{V}$. Note that $\mathcal{O}(\mathbb{V},\mathbb{V}^*)$ does not have a group structure in general. The transformation (1) is often referred to as a *Euclidean* transformation when $\mathbb{E} = \mathbb{E}^*$ and $\mathbb{V} = \mathbb{V}^*$. In this case $\mathcal{I} \in O(\mathbb{V})$ is an orthogonal transformation.

For $\mathcal{I} \in \mathcal{O}(\mathbb{V},\mathbb{V}^*)$, it follows that $\mathcal{I}^\top\mathcal{I} = I_{\mathbb{V}}$ and $\mathcal{I}\mathcal{I}^\top = I_{\mathbb{V}^*}$ are identity transformations. Hence $\mathcal{I}^\top = \mathcal{I}^{-1}$ and $\mathcal{I}^\top \in \mathcal{O}(\mathbb{V}^*,\mathbb{V})$ is an isometric transformation from $\mathbb{V}^*$ to $\mathbb{V}$. Moreover, if $\mathcal{R} \in \mathcal{O}(\mathbb{V},\mathbb{V}^*)$, then $\mathcal{I}^\top\mathcal{R} \in O(\mathbb{V})$ and $\mathcal{I}\mathcal{R}^\top \in O(\mathbb{V}^*)$ are orthogonal transformations on $\mathbb{V}$ and $\mathbb{V}^*$ respectively. Indeed, isometric transformation is the counterpart of orthogonal transformation when two different vector spaces are involved.

2 FRAME OF REFERENCE

The event world $\mathcal{W}$ is a four-dimensional space-time in which physical events occur at some places and certain instants. Let $\mathcal{T}$ be the collection of instants and $\mathcal{W}_s$ be the placement space of simultaneous events at the instant s, then the neo-classical space-time (Noll (1973)) can be expressed as the disjoint union of placement spaces of simultaneous events at each instant,

$$\mathcal{W} = \bigcup_{s\in\mathcal{T}} \mathcal{W}_s\,.$$

A point $p_s \in \mathcal{W}$ is called an event, which occurs at the instant s and the place $p \in \mathcal{W}_s$. At different instants s and $\bar{s}$, the spaces $\mathcal{W}_s$ and $\mathcal{W}_{\bar{s}}$ are two disjoint spaces. Thus it is impossible to determine the distance between two non-simultaneous events at p_s and $p_{\bar{s}}$ if $s \neq \bar{s}$, and hence $\mathcal{W}$ is not a product space of space and time. However, it can be set into correspondence with a product space through a frame of reference on $\mathcal{W}$.

Definition. *(Frame of reference): A frame of reference is a one-to-one mapping*

$\phi : \mathcal{W} \to \mathbb{E} \times \mathbb{R},$

taking $p_s \mapsto (\boldsymbol{x}, t)$*, where* $\mathbb{R}$ *is the space of real numbers and* $\mathbb{E}$ *is a three-dimensional Euclidean space. We shall denote the map taking* $p \mapsto \boldsymbol{x}$ *as the map* $\phi_s : \mathcal{W}_s \to \mathbb{E}$*.*

In general, the Euclidean spaces of different frames of reference may not be the same. Therefore, for definiteness, we shall denote the Euclidean space of the frame ϕ by $\mathbb{E}_\phi$, and its translation space by $\mathbb{V}_\phi$. We assume that $\mathbb{V}_\phi$ is an inner product space.

Of course, there are infinite many frames of reference. Each one of them may be regarded as an *observer*, since it can be depicted as a person taking a snapshot so that the image of ϕ_s is a picture (three-dimensional at least conceptually) of the placements of the events at some instant s, from which the distance between two simultaneous events can be measured. A sequence of events can also be recorded as video clips depicting the change of events in time by an observer.

Now, suppose that two observers are recording the same events with video cameras. In order to compare their video clips regarding the locations and time, they must have a mutual agreement that the clock of their cameras must be synchronized so that simultaneous events can be recognized and since during the recording two observers may move independently while taking pictures with their cameras from different angles, there will be a relative motion, a scaling and a relative orientation between them. We shall make such a consensus among observers explicit mathematically.

$$\begin{tikzcd}
& p_s \in \mathcal{W} \arrow[dl, "\phi"'] \arrow[dr, "\phi^*"] & \\
(\boldsymbol{x}, t) \in \mathbb{E}_\phi \times \mathbb{R} \arrow[rr, "*"] & & (\boldsymbol{x}^*, t^*) \in \mathbb{E}_{\phi^*} \times \mathbb{R}
\end{tikzcd}$$

Figure 1: A change of frame

Let ϕ and ϕ^* be two frames of reference. They are related by the composite map $* := \phi^* \circ \phi^{-1}$,

$* : \mathbb{E}_\phi \times \mathbb{R} \to \mathbb{E}_{\phi^*} \times \mathbb{R}, \quad$ taking $\quad (\boldsymbol{x}, t) \mapsto (\boldsymbol{x}^*, t^*),$

where $(\boldsymbol{x}, t)$ and $(\boldsymbol{x}^*, t^*)$ are the position and time of the same event observed by ϕ and ϕ^* simultaneously. In general $\mathbb{E}_\phi$ and $\mathbb{E}_{\phi^*}$ are different Euclidean spaces. Physically, an arbitrary map would be irrelevant as long as we are interested in establishing a consensus among observers, which requires preservation of distance between simultaneous events and time interval as well as the sense of time.

Definition. *(Euclidean change of frame): A change of frame (observer) from* ϕ *to* ϕ^* *taking* $(\boldsymbol{x}, t) \mapsto (\boldsymbol{x}^*, t^*)$*, is an isometry of space and time given by*

$$\boldsymbol{x}^* = \mathcal{Q}(t)(\boldsymbol{x} - \boldsymbol{x}_0) + \boldsymbol{c}^*(t), \qquad t^* = t + a, \tag{2}$$

for some constant time difference $a \in \mathbb{R}$*, some relative translation* $\boldsymbol{c}^* : \mathbb{R} \to \mathbb{E}_{\phi^*}$ *with respect to the reference point* $\boldsymbol{x}_0 \in \mathbb{E}_\phi$ *and some* $\mathcal{Q} : \mathbb{R} \to \mathcal{O}(\mathbb{V}_\phi, \mathbb{V}_{\phi^*})$*.*

We have denoted $\mathcal{O}(\mathbb{V}_\phi, \mathbb{V}_{\phi^*}) = \{\mathcal{Q} \in \mathcal{L}(\mathbb{V}_\phi, \mathbb{V}_{\phi^*}) : \|\mathcal{Q}\boldsymbol{u}\|_{\mathbb{V}_{\phi^*}} = \|\boldsymbol{u}\|_{\mathbb{V}_\phi},\ \forall \boldsymbol{u} \in \mathbb{V}_\phi\}$. Euclidean changes of frame will often be called *changes of frame* for simplicity, since they are the only changes of frame among consenting observers of our concern for the purpose of discussing frame-indifference in continuum mechanics.

All consenting observers form an equivalent class, denoted by $\mathfrak{C}$, among the set of all observers, i.e., for any $\phi, \phi^* \in \mathfrak{C}$, there exists a Euclidean change of frame from $\phi \to \phi^*$. From now on, only classes of consenting observers will be considered. Therefore, any observer, would mean any observer in some $\mathfrak{C}$, and a change of frame, would mean a Euclidean change of frame.

3 MOTION AND DEFORMATION

In the space-time, a physical event is represented by its placement at a certain instant so that it can be observed in a frame of reference. Let a body $\mathcal{B}$ be a set of material points.

Definition. *(Configuration): Let $\xi : \mathcal{B} \to \mathcal{W}_t$ be a placement of the body $\mathcal{B}$ at the instant t, and let ϕ be a frame of reference, then the composite map $\xi_{\phi_t} := \phi_t \circ \xi$,*

$$\xi_{\phi_t} : \mathcal{B} \to \mathbb{E}_\phi$$

is called a configuration of the body $\mathcal{B}$ at the instant t in the frame ϕ.

A configuration thus identifies the body with a region in the Euclidean space of the observer. In this sense, the motion of a body can be viewed as a continuous sequence of events such that at any instant t, the placement of the body $\mathcal{B}$ in $\mathcal{W}_t$ is a one-to-one mapping

$$\chi_t : \mathcal{B} \to \mathcal{W}_t,$$

and the composite mapping $\chi_{\phi_t} := \phi_t \circ \chi_t$,

$$\chi_{\phi_t} : \mathcal{B} \to \mathbb{E}_\phi, \qquad \boldsymbol{x} = \chi_{\phi_t}(p) = \phi_t(\chi_t(p)), \quad p \in \mathcal{B},$$

is the configuration of the body at time t, with $\mathcal{B}_{\chi_t} := \chi_{\phi_t}(\mathcal{B}) \subset \mathbb{E}_\phi$ (see the right part of Figure 2). The motion can then be regarded as a sequence of configurations of $\mathcal{B}$ in time, $\chi_\phi = \{\chi_{\phi_t},\, t \in \mathbb{R} \mid \chi_{\phi_t} : \mathcal{B} \to \mathbb{E}_\phi\}$. We can also express a motion as

$$\chi_\phi : \mathcal{B} \times \mathbb{R} \to \mathbb{E}_\phi, \qquad \boldsymbol{x} = \chi_\phi(p, t) = \chi_{\phi_t}(p), \quad p \in \mathcal{B}.$$

Note that in our discussions, we have been using $t \in \mathbb{R}$ as the time in the frame ϕ corresponding to the instant $s \in \mathcal{T}$ with $s = t$ for simplicity without loss of generality.

$$\begin{tikzcd}
 & p \in \mathcal{B} \arrow[dl, "\kappa"'] \arrow[ddl, "\kappa_\phi"] \arrow[dr, "\chi_t"] \arrow[ddr, "\chi_{\phi_t}"'] & \\
\mathcal{W}_{t_0} \arrow[d, "\phi_{t_0}"'] & & \mathcal{W}_t \arrow[d, "\phi_t"] \\
\boldsymbol{X} \in \mathcal{B}_\kappa \subset \mathbb{E}_\phi \arrow[rr, "\chi_{\kappa_\phi}(\,\cdot\,, t)"] & & \boldsymbol{x} \in \mathcal{B}_{\chi_t} \subset \mathbb{E}_\phi
\end{tikzcd}$$

Figure 2: Motion χ_{ϕ_t}, reference configuration κ_ϕ and deformation $\chi_{\kappa_\phi}(\,\cdot\,, t)$

Reference configuration

We regard a body $\mathcal{B}$ as a set of material points. Although it is possible to endow the body as a manifold with a differentiable structure and topology for doing mathematics on the body, to avoid such mathematical subtleties, usually a particular configuration is chosen as reference (see the left part of Figure 2),

$$\kappa_\phi : \mathcal{B} \to \mathbb{E}_\phi, \qquad \boldsymbol{X} = \kappa_\phi(p), \qquad \mathcal{B}_\kappa := \kappa_\phi(\mathcal{B}) \subset \mathbb{E}_\phi,$$

so that the motion at an instant t is a one-to-one mapping

$$\chi_{\kappa_\phi}(\cdot, t) : \mathcal{B}_\kappa \to \mathcal{B}_{\chi_t}, \qquad \boldsymbol{x} = \chi_{\kappa_\phi}(\boldsymbol{X}, t) = \chi_{\phi_t}(\kappa_\phi^{-1}(\boldsymbol{X})), \quad \boldsymbol{X} \in \mathcal{B}_\kappa,$$

from a region into another region in the same Euclidean space $\mathbb{E}_\phi$ for which topology and differentiability are well defined. This mapping is called a *deformation* from κ to χ_t in the frame ϕ and a motion is then a sequence of deformations in time.

For the reference configuration κ_ϕ, there is some instant, say t_0, at which the reference placement of the body is chosen, $\kappa : \mathcal{B} \to \mathcal{W}_{t_0}$ (see Figure 2). On the other hand, the choice of a reference configuration is arbitrary, and it is not necessary that the body should actually occupy the reference place in its motion under consideration. Nevertheless, in most practical problems, t_0 is usually taken as the initial time of the motion.

4 FRAME-INDIFFERENCE

The change of frame (2) gives rise to a linear mapping on the translation space, in the following way: Let $\boldsymbol{u}(\phi) = \boldsymbol{x}_2 - \boldsymbol{x}_1 \in \mathbb{V}_\phi$ be the difference vector of $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{E}_\phi$ in the frame ϕ, and $\boldsymbol{u}(\phi^*) = \boldsymbol{x}_2^* - \boldsymbol{x}_1^* \in \mathbb{V}_{\phi^*}$ be the corresponding difference vector in the frame ϕ^*, then from (2), it follows immediately that

$$\boldsymbol{u}(\phi^*) = \mathcal{Q}(t)\boldsymbol{u}(\phi),$$

where $\mathcal{Q}(t) \in \mathcal{O}(\mathbb{V}_\phi, \mathbb{V}_{\phi^*})$ is the isometric transformation associated with the change of frame $\phi \to \phi^*$.

Any vector quantity in $\mathbb{V}_\phi$, which has this transformation property, is said to be objective with respect to Euclidean transformations, *objective* in the sense that it pertains to a quantity of its real nature rather than its values as affected by different observers. This concept of objectivity can be generalized to any tensor spaces of $\mathbb{V}_\phi$.

Let

$$s : \mathfrak{E} \to \mathbb{R}, \qquad \boldsymbol{u} : \mathfrak{E} \to \mathbb{V}_\mathfrak{E}, \qquad T : \mathfrak{E} \to \mathbb{V}_\mathfrak{E} \otimes \mathbb{V}_\mathfrak{E},$$

where $\mathfrak{E}$ is the Euclidean class of frames of reference and $\mathbb{V}_\mathfrak{E} = \{\mathbb{V}_\phi : \phi \in \mathfrak{E}\}$. They are scalar, vector and (second order) tensor *observable* quantities respectively. We call $f(\phi)$ the value of the quantity f observed in the frame ϕ.

Definition. *(Frame-indifference): Relative to a change of frame from ϕ to ϕ^*, the observables s, $\boldsymbol{u}$ and T are called frame-indifferent (or objective) scalar, vector and tensor quantities respectively, if they satisfy the following transformation properties:*

$$\begin{aligned} &s(\phi^*) = s(\phi),\\ &\boldsymbol{u}(\phi^*) = \mathcal{Q}(t)\,\boldsymbol{u}(\phi),\\ &T(\phi^*) = \mathcal{Q}(t)\,T(\phi)\,\mathcal{Q}(t)^\top, \end{aligned}$$

where $\mathcal{Q}(t) \in \mathcal{O}(\mathbb{V}_\phi, \mathbb{V}_{\phi^*})$ is the isometric transformation of the change of frame from ϕ to ϕ^*.

More precisely, they are also said to be frame-indifferent with respect to Euclidean transformations or simply Euclidean objective. For simplicity, we often write $f = f(\phi)$ and $f^* = f(\phi^*)$.

The definition of objective scalar is self-evident. For the definition of objective tensors, we consider an inner product $(\boldsymbol{u}, T\boldsymbol{v})_{\mathbb{V}}$. For any objective vectors $\boldsymbol{u}^* = \mathcal{Q}\boldsymbol{u}$, $\boldsymbol{v}^* = \mathcal{Q}\boldsymbol{v}$, it follows that

$$(\boldsymbol{u}^*, T^*\boldsymbol{v}^*)_{\mathbb{V}^*} = (\mathcal{Q}\boldsymbol{u}, T^*\mathcal{Q}\,\boldsymbol{v})_{\mathbb{V}^*} = (\boldsymbol{u}, \mathcal{Q}^{\top}T^*\mathcal{Q}\,\boldsymbol{v})_{\mathbb{V}}.$$

Therefore, if $s = (\boldsymbol{u}, T\boldsymbol{v})_{\mathbb{V}}$ is an objective scalar, that is, $(\boldsymbol{u}^*, T^*\boldsymbol{v}^*)_{\mathbb{V}^*} = (\boldsymbol{u}, T\boldsymbol{v})_{\mathbb{V}}$, then it implies that T is an objective tensor satisfying $T^* = \mathcal{Q}\,T\,\mathcal{Q}^{\top}$.

Transformation properties of motion

Let χ_ϕ be a motion of the body in the frame ϕ, and χ_{ϕ^*} be the corresponding motion in ϕ^*,

$$\boldsymbol{x} = \chi_\phi(p,t), \quad \boldsymbol{x}^* = \chi_{\phi^*}(p,t^*), \quad p \in \mathcal{B}.$$

Then from (2), we have

$$\chi_{\phi^*}(p,t^*) = \mathcal{Q}(t)(\chi_\phi(p,t) - \boldsymbol{x}_o) + \boldsymbol{c}^*(t), \quad p \in \mathcal{B},$$

from which, one can easily show that the velocity and the acceleration are not objective quantities,

$$\begin{aligned} \dot{\boldsymbol{x}}^* &= \mathcal{Q}\dot{\boldsymbol{x}} + \dot{\mathcal{Q}}(\boldsymbol{x} - \boldsymbol{x}_o) + \dot{\boldsymbol{c}}^*, \\ \ddot{\boldsymbol{x}}^* &= \mathcal{Q}\ddot{\boldsymbol{x}} + 2\dot{\mathcal{Q}}\dot{\boldsymbol{x}} + \ddot{\mathcal{Q}}(\boldsymbol{x} - \boldsymbol{x}_0) + \ddot{\boldsymbol{c}}^*. \end{aligned} \tag{3}$$

A change of frame (2) with constant $\mathcal{Q}$ and $\boldsymbol{c}^*(t) = \boldsymbol{c}_0 + \boldsymbol{c}_1 t$, for constant $\boldsymbol{c}_0$ and $\boldsymbol{c}_1$ (so that $\dot{\mathcal{Q}} = 0$ and $\ddot{\boldsymbol{c}}^* = 0$), is called a *Galilean transformation*. Therefore, from (3) we conclude that the acceleration is not Euclidean objective but it is objective with respect to Galilean transformation. Moreover, it also shows that the velocity is neither a Euclidean nor a Galilean objective vector quantity.

Transformation properties of deformation gradient

Let $\kappa : \mathcal{B} \to \mathcal{W}_{t_0}$ be a reference placement of the body at some instant t_0 (see Figure 3), then

$$\kappa_\phi = \phi_{t_0} \circ \kappa \quad \text{and} \quad \kappa_{\phi^*} = \phi^*_{t_0} \circ \kappa \tag{4}$$

are the corresponding reference configurations of $\mathcal{B}$ in the frames ϕ and ϕ^* at the same instant, and

$$\boldsymbol{X} = \kappa_\phi(p), \quad \boldsymbol{X}^* = \kappa_{\phi^*}(p), \quad p \in \mathcal{B}.$$

Let us denote by $\gamma = \kappa_{\phi^*} \circ \kappa_\phi^{-1}$ the change of reference configuration from κ_ϕ to κ_{ϕ^*} in the change of frame, then it follows from (4) that $\gamma = \phi^*_{t_0} \circ \phi^{-1}_{t_0}$ and by (2), we have

$$\boldsymbol{X}^* = \gamma(\boldsymbol{X}) = \mathcal{Q}(t_0)(\boldsymbol{X} - \boldsymbol{x}_o) + \boldsymbol{c}^*(t_0). \tag{5}$$

$$
\begin{tikzcd}
& p \in \mathcal{B} \arrow[ddl, "\kappa_\phi"'] \arrow[d, "\kappa"] \arrow[ddr, "\kappa_{\phi^*}"] & \\
& \mathcal{W}_{t_0} \arrow[dl, "\phi_{t_0}"'] \arrow[dr, "\phi^*_{t_0}"] & \\
\boldsymbol{X} \in \mathcal{B}_\kappa \subset \mathbb{E}_\phi \arrow[rr, "\gamma"] & & \boldsymbol{X}^* \in \mathcal{B}_{\kappa^*} \subset \mathbb{E}_{\phi^*}
\end{tikzcd}
$$

Figure 3: Reference configurations κ_ϕ and κ_{ϕ^*} in the change of frame from ϕ to ϕ^*

On the other hand, the motion in referential description relative to the change of frame is given by $\boldsymbol{x} = \chi_\kappa(\boldsymbol{X}, t)$ and $\boldsymbol{x}^* = \chi_{\kappa^*}(\boldsymbol{X}^*, t^*)$. Hence from (2), we have

$$\chi_{\kappa^*}(\boldsymbol{X}^*, t^*) = \mathcal{Q}(t)(\chi_\kappa(\boldsymbol{X}, t) - \boldsymbol{x}_o) + \boldsymbol{c}^*(t).$$

Therefore we obtain for the deformation gradients, $F = \nabla_{\boldsymbol{X}}\chi_\kappa$ in the frame ϕ and $F^* = \nabla_{\boldsymbol{X}^*}\chi_{\kappa^*}$ in the frame ϕ^*, by the chain rule and the use of (5),

$$F^*(\boldsymbol{X}^*, t^*) = \mathcal{Q}(t)F(\boldsymbol{X}, t)\,\mathcal{Q}(t_0)^\top, \quad \text{or simply} \quad F^* = \mathcal{Q}F\mathcal{Q}_0^\top, \tag{6}$$

where $\mathcal{Q}_0 = \mathcal{Q}(t_0)$ is the isometric transformation due to the change of frame at the instant t_0 when the reference configuration is chosen.

The deformation gradient F is not a Euclidean objective tensor. However, the property (6) also shows that it is objective with respect to Galilean transformations, since in this case, $\mathcal{Q}(t)$ is a constant isometric transformation.

Remark 1. The transformation property (6) stands in contrast to $F^* = \mathcal{Q}F$, the widely used formula which is obtained "provided that the reference configuration is unaffected by the change of frame" as usually implicitly assumed, so that $\mathcal{Q}_0$ reduces to the identity transformation. On the other hand, the transformation property (6) has also been derived elsewhere in the literature (Murdoch (2000) and Ogden (1997) Sec. 2.2.8). □

Remark 2. Remember that a configuration is a placement of a body *relative* to a frame of reference and any *reference* configuration is not an exception. Therefore, to say that the "reference configuration is unaffected by the change of frame" is at best an assumption (Truesdell and Noll (2004) p. 308), or simply a misunderstanding (Gurtin et al. (2010) Sec. 20.1). For the same reason, two consenting observers cannot independently choose their reference configurations, because they must choose the configuration of the body at the *same* instant of their respective frames of reference (see discussions in (Liu (2005); Murdoch (2005))). □

5 GALILEAN INVARIANCE OF BALANCE LAWS

In classical mechanics, Newton's first law, often known as the law of inertia, is essentially a definition of inertial frame.

Definition. *(Inertial frame): A frame of reference is called an inertial frame if, relative to it, the velocity of a body remains constant unless the body is acted upon by an external force.*

We present the first law in this manner in order to emphasize that the existence of inertial frames is essential for the formulation of Newton's second law, which asserts that relative to an inertial frame, the equation of motion takes the simple form:

$$\mathfrak{m}\,\ddot{\boldsymbol{x}} = \mathfrak{f}. \tag{7}$$

Now, we shall assume that there is an inertial frame $\phi_0 \in \mathfrak{E}$, for which the equation of motion of a particle is given by (7), and we are interested in how the equation is transformed under a change of frame.

Unlike the acceleration, transformation properties of non-kinematic quantities cannot be deduced theoretically. Instead, for the mass and the force, it is conventionally *postulated* that they are Euclidean objective scalar and vector quantities respectively, so that for any change from ϕ_0 to $\phi^* \in \mathfrak{E}$ given by (2), we have

$$\mathfrak{m}^* = \mathfrak{m}, \qquad \mathfrak{f}^* = \mathcal{Q}\,\mathfrak{f},$$

which together with (3), by multiplying (7) with $\mathcal{Q}$, we obtain the equation of motion in the (non-inertial) frame ϕ^*,

$$\mathfrak{m}^* \ddot{\boldsymbol{x}}^* = \mathfrak{f}^* + \mathfrak{m}^*\,\mathfrak{i}^*, \tag{8}$$

where $\mathfrak{i}^*$ is called the inertial force given by

$$\mathfrak{i}^* = \ddot{\boldsymbol{c}}^* + 2\Omega(\dot{\boldsymbol{x}}^* - \dot{\boldsymbol{c}}^*) + (\dot{\Omega} - \Omega^2)(\boldsymbol{x}^* - \boldsymbol{c}^*),$$

where $\Omega = \dot{\mathcal{Q}}\mathcal{Q}^\top : \mathbb{R} \to L(\mathbb{V}_{\phi^*})$ is called the spin tensor of the frame ϕ^* relative to the inertial frame ϕ_0.

Note that the inertial force vanishes if the change of frame $\phi_0 \to \phi^*$ is a Galilean transformation, i.e., $\dot{\mathcal{Q}} = 0$ and $\ddot{\mathbf{c}}^* = 0$, and hence the equation of motion in the frame ϕ^* also takes the simple form,

$$m^* \ddot{\boldsymbol{x}}^* = \mathfrak{f}^*,$$

which implies that the frame ϕ^* is also an inertial frame.

Therefore, any frame of reference obtained from a Galilean change of frame from an inertial frame is also an inertial frame and thus, all inertial frames form an equivalent class $\mathfrak{G}$, such that for any $\phi, \phi^* \in \mathfrak{G}$, the change of frame $\phi \to \phi^*$ is a Galilean transformation. The Galilean class $\mathfrak{G}$ is a subclass of the Euclidean class $\mathfrak{E}$.

In short, we can assert that physical laws, like the equation of motion, are in general not (Euclidean) frame-indifferent. Nevertheless, the equation of motion is Galilean frame-indifferent, under the assumption that mass and force are frame-indifferent quantities. This is usually referred to as *Galilean invariance* of the equation of motion.

Motivated by classical mechanics, the balance laws of mass, linear momentum, and energy for deformable bodies,

$$\begin{aligned} &\dot{\rho} + \rho \operatorname{div} \dot{\boldsymbol{x}} = 0, \\ &\rho\, \ddot{\boldsymbol{x}} - \operatorname{div} T = \rho\, \boldsymbol{b}, \\ &\rho\, \dot{\varepsilon} + \operatorname{div} \boldsymbol{q} - T \cdot \operatorname{grad} \dot{\boldsymbol{x}} = \rho\, r, \end{aligned} \tag{9}$$

in an inertial frame are required to be invariant under Galilean transformation. Since two inertial frames are related by a Galilean transformation, it means that the equations (9) should hold in the same form in any inertial frame. In particular, the balance of linear momentum takes the forms in the inertial frames $\phi, \phi^* \in \mathfrak{G}$,

$$\rho\, \ddot{\boldsymbol{x}} - \operatorname{div} T = \rho\, \boldsymbol{b}, \qquad \rho^* \ddot{\boldsymbol{x}}^* - (\operatorname{div} T)^* = \rho^* \boldsymbol{b}^*.$$

Since the acceleration $\ddot{\boldsymbol{x}}$ is Galilean objective, in order this to hold, it is usually assumed that the mass density ρ, the Cauchy stress tensor T and the body force $\boldsymbol{b}$ are objective scalar, tensor, and vector quantities respectively. Similarly, for the energy equation, it is also assumed that the internal energy ε and the energy supply r are objective scalars, and the heat flux $\boldsymbol{q}$ is an objective vector. These assumptions concern the non-kinematic quantities, including external supplies $(\boldsymbol{b}, r)$, and the constitutive quantities $(T, \boldsymbol{q}, \varepsilon)$.

In fact, for Galilean invariance of the balance laws, only frame-indifference with respect to Galilean transformation for all those non-kinematic quantities would be sufficient. However, similar to classical mechanics, it is *postulated* that they are not only Galilean objective but also Euclidean objective. Therefore, with the known transformation properties of the kinematic variables, the balance laws in any arbitrary frame can be deduced.

To emphasize the importance of the objectivity postulate for constitutive theories, it will be referred to as Euclidean objectivity for constitutive quantities:

Euclidean objectivity. *The constitutive quantities: the Cauchy stress T, the heat flux $\boldsymbol{q}$ and the internal energy density ε, are Euclidean objective (Euclidean frame-indifferent),*

$$T(\phi^*) = \mathcal{Q}(t)\, T(\phi)\, \mathcal{Q}(t)^\top, \qquad \boldsymbol{q}(\phi^*) = \mathcal{Q}(t)\, \boldsymbol{q}(\phi), \qquad \varepsilon(\phi^*) = \varepsilon(\phi), \tag{10}$$

where $\mathcal{Q}(t) \in \mathcal{O}(\mathbb{V}_\phi, \mathbb{V}_{\phi^*})$ is the isometric transformation of the change of frame from ϕ to ϕ^*.

Note that this postulate concerns only frame-indifference properties of balance laws, so that it is a universal property for any deformable bodies, and therefore, do not concern any aspects of material properties of the body.

6 CONSTITUTIVE EQUATIONS IN MATERIAL DESCRIPTION

Physically a state of the thermomechanical behavior of a body is characterized by a description of the fields of density $\rho(p,t)$, motion $\chi(p,t)$ and temperature $\theta(p,t)$. The material properties of a body generally depend on the past history of its thermomechanical behavior.

Let us introduce the notion of the past history of a function. Let $h(\cdot)$ be a function of time. The history of h up to time t is defined by

$$h^t(s) = h(t-s),$$

where $s \in [0,\infty)$ denotes the time-coordinate pointed into the past from the present time t. Clearly $s = 0$ corresponds to the present time, therefore $h^t(0) = h(t)$.

Mathematical descriptions of material properties are called constitutive equations. We postulate that the history of thermomechanical behavior up to the present time determines the properties of the material body.

Principle of determinism. *Let ϕ be a frame of reference, and $\mathcal{C}$ be a constitutive quantity, then the constitutive equation for $\mathcal{C}$ is given by a functional of the form,*

$$\mathcal{C}(\phi, p, t) = \mathcal{F}_\phi(\rho^t, \chi^t, \theta^t; p), \qquad p \in \mathcal{B},\ t \in \mathbb{R}, \tag{11}$$

where the first three arguments are history functions:

$$\rho^t : \mathcal{B} \times [0,\infty) \to \mathbb{R}, \qquad \chi^t : \mathcal{B} \times [0,\infty) \to \mathbb{E}_\phi, \qquad \theta^t : \mathcal{B} \times [0,\infty) \to \mathbb{R}.$$

We call $\mathcal{F}_\phi$ the constitutive function of $\mathcal{C}$ in the frame ϕ. Such a functional allows the description of arbitrary non-local effect of an inhomogeneous body with a perfect memory of the past thermomechanical history. With the notation $\mathcal{F}_\phi$, we emphasize that the value of a constitutive function may depend on the frame of reference ϕ.

For simplicity, for further discussions on constitutive equations, we shall restrict our attention to material models for mechanical theory only, and only constitutive equations for the stress tensor will be considered. General results can be found elsewhere (Liu (2002, 2009)).

In order to avoid possible confusions arisen from the viewpoint of employing different Euclidean spaces, we shall be more careful about expressing relevant physical quantities in the proper space.

Let the set of history functions on a set $\mathcal{X}$ in some space $\mathbb{W}$ be denoted by

$$\mathfrak{H}(\mathcal{X}, \mathbb{W}) = \{h^t : \mathcal{X} \times [0, \infty) \to \mathbb{W}\}.$$

Then the constitutive equation for the stress tensor, $T(\phi, p, t) \in \mathbb{V}_\phi \otimes \mathbb{V}_\phi$, can be written as

$$T(\phi, p, t) = \mathcal{F}_\phi(\chi^t; p), \qquad \phi \in \mathfrak{E}, \quad p \in \mathcal{B}, \quad \chi^t \in \mathfrak{H}(\mathcal{B}, \mathbb{E}_\phi). \tag{12}$$

Condition of Euclidean objectivity

Let $\phi^* \in \mathfrak{E}$ be another observer, then the respective constitutive equation for the stress, $T(\phi^*, p, t^*) \in \mathbb{V}_{\phi^*} \otimes \mathbb{V}_{\phi^*}$, can be written as

$$T(\phi^*, p, t^*) = \mathcal{F}_{\phi^*}((\chi^t)^*; p), \qquad p \in \mathcal{B}, \quad (\chi^t)^* \in \mathfrak{H}(\mathcal{B}, \mathbb{E}_{\phi^*}),$$

where the corresponding histories of motion are related by

$$(\chi^t)^*(\bar{p}, s) = \mathcal{Q}^t(s)(\chi^t(\bar{p}, s) - \boldsymbol{x}_o) + \boldsymbol{c}^{*t}(s),$$

for any $s \in [0, \infty)$ and any $\bar{p} \in \mathcal{B}$ in the change of frame $\phi \to \phi^*$ given by (2).

We need to bear in mind that according to the assumption referred to as the Euclidean objectivity (10), the stress is a frame-indifferent quantity under a change of observer,

$$T(\phi^*, p, t^*) = \mathcal{Q}(t) T(\phi, p, t)) \mathcal{Q}(t)^\top.$$

Therefore, it follows immediately that

$$\mathcal{F}_{\phi^*}((\chi^t)^*; p) = \mathcal{Q}(t) \mathcal{F}_\phi(\chi^t; p) \mathcal{Q}(t)^\top, \tag{13}$$

where $\mathcal{Q}(t) \in \mathcal{O}(\mathbb{V}_\phi, \mathbb{V}_{\phi^*})$ is the isometric transformation of the change of frame $\phi \to \phi^*$.

The relation (13) will be referred to as the *condition of Euclidean objectivity*. It is a relation between the constitutive functions relative to two different observers. In other words, different observers cannot independently propose their own constitutive equations. Instead, the condition of Euclidean objectivity (13) determines the constitutive function $\mathcal{F}_{\phi^*}$ once the constitutive function $\mathcal{F}_\phi$ is given or vice-versa. They determine one from the other in a frame-dependent manner.

7 PRINCIPLE OF MATERIAL FRAME-INDIFFERENCE

It is obvious that *not* any proposed constitutive equations would physically make sense as material models. First of all, they may be frame-dependent. However, since the constitutive functions must characterize the intrinsic properties of the material body itself, it should be observer-independence in certain sense. Consequently, there must be some restrictions imposed on the constitutive functions so that they would be indifferent to the change of frame. This is the essential idea of the principle of material frame-indifference.

Remark 3. In the case that does not distinguish the Euclidean spaces relative to different observers, i.e., $\mathbb{E}_\phi = \mathbb{E}_{\phi^*} = \mathbb{E}$ and $\mathbb{V}_\phi = \mathbb{V}_{\phi^*} = \mathbb{V}$, as adopted usually in the literature (Truesdell and Noll (2004); Liu (2002, 2009)), the principle of material frame-indifference can simply be postulated as

$$\mathcal{F}_\phi(\,\bullet\,; p) = \mathcal{F}_{\phi^*}(\,\bullet\,; p), \qquad p \in \mathcal{B}. \tag{14}$$

where $\bullet$ represents the history of motion in $\mathfrak{H}(\mathcal{B}, \mathbb{E})$ which is the common domain of the two functionals, and their values are in the same tensor space $\mathbb{V} \otimes \mathbb{V}$.

This states that for different observers $\phi, \phi^* \in \mathfrak{C}$, they all have the same constitutive function, $\mathcal{F}_\phi = \mathcal{F}_{\phi^*}$. Note that in (14) the material point p is superfluously indicated to emphasize that it is valid only when the *material description* is used, because presumably the presence of reference configuration in the change of frame may change the properties of materials. □

Recall that a change of frame $* : \phi \to \phi^*$ is associated with an isometry between $\mathbb{E}_\phi$ and $\mathbb{E}_{\phi^*}$ and conversely, given an isometry

$$i : \mathbb{E}_\phi \to \mathbb{E}_{\phi^*}, \qquad i(\boldsymbol{x}) = \mathcal{I}(\boldsymbol{x} - \boldsymbol{x}_i) + \boldsymbol{x}_i^*,$$

there is a change of frame $i : \phi \to \phi_i$ (with the same notation for simplicity). For this change of frame $\mathcal{I} \in \mathcal{O}(\mathbb{V}_\phi, \mathbb{V}_{\phi^*})$ and $\mathbb{E}_{\phi_i} = \mathbb{E}_{\phi^*}$. Therefore, it must satisfy the following condition of Euclidean objectivity (13),

$$\mathcal{F}_{\phi_i}(i(\mathcal{X}^t); p) = \mathcal{I}(t)\, \mathcal{F}_\phi(\mathcal{X}^t; p)\, \mathcal{I}(t)^\top, \qquad \mathcal{X}^t \in \mathfrak{H}(\mathcal{B}, \mathbb{E}_\phi), \tag{15}$$

for which $i(\mathcal{X}^t) \in \mathfrak{H}(\mathcal{B}, \mathbb{E}_{\phi^*})$, and the values of $\mathcal{F}_{\phi_i} \in \mathbb{V}_{\phi^*} \otimes \mathbb{V}_{\phi^*}$.

Now, consider another isometry between $\mathbb{E}_\phi$ and $\mathbb{E}_{\phi^*}$,

$$j : \mathbb{E}_\phi \to \mathbb{E}_{\phi^*}, \qquad j(\boldsymbol{x}) = \mathcal{J}(\boldsymbol{x} - \boldsymbol{x}_j) + \boldsymbol{x}_j^*.$$

Similarly, there is a change of frame $j : \phi \to \phi_j$ with $\mathcal{J} \in \mathcal{O}(\mathbb{V}_\phi, \mathbb{V}_{\phi^*})$, and $\mathbb{E}_{\phi_j} = \mathbb{E}_{\phi^*}$, and we have the condition of Euclidean objectivity,

$$\mathcal{F}_{\phi_j}(j(\mathcal{X}^t); p) = \mathcal{J}(t)\, \mathcal{F}_\phi(\mathcal{X}^t; p)\, \mathcal{J}(t)^\top, \qquad \mathcal{X}^t \in \mathfrak{H}(\mathcal{B}, \mathbb{E}_\phi), \tag{16}$$

for which $j(\mathcal{X}^t) \in \mathfrak{H}(\mathcal{B}, \mathbb{E}_{\phi^*})$, and the values of $\mathcal{F}_{\phi_j} \in \mathbb{V}_{\phi^*} \otimes \mathbb{V}_{\phi^*}$ as before.

Note that the two isometries induce two changes of frame from ϕ to two different frames ϕ_i and ϕ_j in the same Euclidean space $\mathbb{E}_{\phi^*}$. Moreover, the two constitutive functions $\mathcal{F}_{\phi_i}$ and $\mathcal{F}_{\phi_j}$ have the common domain $\mathfrak{H}(\mathcal{B}, \mathbb{E}_{\phi^*})$ and their values are in the same space $\mathbb{V}_{\phi^*} \otimes \mathbb{V}_{\phi^*}$. Therefore, we can postulate:

Principle of material frame-indifference. *Let ϕ_i and ϕ_j be two frames of reference induced by two isometries $i, j : \mathbb{E}_\phi \to \mathbb{E}_{\phi^*}$, then the corresponding constitutive function $\mathcal{F}_{\phi_i}$ and $\mathcal{F}_{\phi_j}$ must have the same form,*

$$\mathcal{F}_{\phi_i}(\,\bullet\,; p) = \mathcal{F}_{\phi_j}(\,\bullet\,; p), \qquad p \in \mathcal{B}, \tag{17}$$

where $\bullet$ *represents the history of motion in* $\mathfrak{H}(\mathcal{B}, \mathbb{E}_{\phi^*})$.

This simple relation renders mathematically the basic idea of frame-indifference of material behavior: the constitutive function, which models the intrinsic behavior of the material, is independent of observer – form invariance of constitutive functions, $\mathcal{F}_{\phi_i} = \mathcal{F}_{\phi_j}$ (equivalence of (14) when $\mathbb{E}_\phi = \mathbb{E}_{\phi^*}$).

We can now easily deduce the restriction on the constitutive function imposed by the principle of material frame-indifference. From the relation (15) and (16), we have

$$\begin{aligned}\mathcal{F}_\phi(\mathcal{X}^t; p) &= \mathcal{I}(t)^\top \mathcal{F}_{\phi_i}(i(\mathcal{X}^t); p)\, \mathcal{I}(t) = \mathcal{I}(t)^\top \mathcal{F}_{\phi_j}(i(\mathcal{X}^t); p)\, \mathcal{I}(t) \\ &= \mathcal{I}(t)^\top \mathcal{F}_{\phi_j}(j \circ (j^{-1} \circ i)(\mathcal{X}^t); p)\, \mathcal{I}(t) \\ &= \mathcal{I}(t)^\top \mathcal{J}(t)\, \mathcal{F}_\phi(\underline{(j^{-1} \circ i)}(\mathcal{X}^t); p)\, \mathcal{J}(t)^\top \mathcal{I}(t).\end{aligned}$$

The underlined composite mapping $q = (j^{-1} \circ i)$ is a Euclidean transformation (an isometry) from $\mathbb{E}_\phi$ to itself,

$$q : \mathbb{E}_\phi \to \mathbb{E}_\phi, \qquad q(\boldsymbol{x}) = Q(t)(\boldsymbol{x} - \boldsymbol{x}_0) + \boldsymbol{r}(t),$$

where $Q = \mathcal{J}^\top \mathcal{I} \in O(\mathbb{V}_\phi)$ is an orthogonal transformation on $\mathbb{V}_\phi$, and some $\boldsymbol{x}_0, \boldsymbol{r}(t) \in \mathbb{E}_\phi$. Therefore, from the above relations, we obtain the following consequence of the principle of material frame-indifference:

Condition of material objectivity. *In a frame of reference* ϕ, *the constitutive function* $\mathcal{F}_\phi$ *must satisfies the condition,*

$$\mathcal{F}_\phi(q(\mathcal{X}^t); p) = Q(t)\, \mathcal{F}_\phi(\mathcal{X}^t; p)\, Q(t)^\top, \qquad p \in \mathcal{B}, \tag{18}$$

for any history of motion $\mathcal{X}^t \in \mathfrak{H}(\mathcal{B}, \mathbb{E}_\phi)$ *and any Euclidean transformation*

$$q : \mathbb{E}_\phi \to \mathbb{E}_\phi, \qquad q(\boldsymbol{x}) = Q(t)(\boldsymbol{x} - \boldsymbol{x}_0) + \boldsymbol{r}(t),$$

with some orthogonal transformation $Q(t) \in O(\mathbb{V}_\phi)$, *and some* $\boldsymbol{x}_0, \boldsymbol{r}(t) \in \mathbb{E}_\phi$.

Since the condition (18) involves only one single frame of reference ϕ, it imposes a restriction on the constitutive function $\mathcal{F}_\phi$. Sometimes, the condition of material objectivity is referred to as the "principle of material objectivity", to impart its relevance in characterizing material property and Euclidean objectivity, as a more explicit form of the principle of material-frame indifference. Indeed, the original principle of material frame-indifference in the fundamental treatise (Truesdell and Noll (2004)) was formulated in the form (18) instead of more intuitive expression (17).

Remark 4. There is an apparent similarity between the two relations (13) and (18). We emphasize that in the condition of Euclidean objectivity (13), $\mathcal{Q}(t)$ is *the* orthogonal (isometric) transformation associated with the *change* of frame from ϕ to ϕ^*. While the condition of material objectivity (18) is valid in a *single* frame ϕ for an *arbitrary* orthogonal transformation $Q(t)$. Nevertheless, this apparent similarity still causes some confusions in the literature, in the occasional use of the "principle of frame-indifference", which may mean one condition for someone or the other condition for someone else (Gurtin et al. (2010); Ryskin (1985); Woods (1981)). From our discussions so far, as we understand, the frame-indifference should not be regarded as a physical principle, it merely concerns the transformation properties due to changes of frame for kinematic or non-kinematic quantities, while the principle of material frame-indifference concerns material properties relative to different observers. $\square$

8 CONSTITUTIVE EQUATIONS IN REFERENTIAL DESCRIPTION

For mathematical analysis, it is more convenient to use referential description so that motions can be defined on the Euclidean space instead of the set of material points. Therefore, for further discussions, we shall reinterpret the principle of material frame-indifference for constitutive equations, or equivalently the condition of material objectivity, relative to a reference configuration.

Let $\kappa : \mathcal{B} \to \mathcal{W}_{t_0}$ be a reference placement of the body at some instant t_0 (see Fig. 2), then $\kappa_\phi = \phi_{t_0} \circ \kappa : \mathcal{B} \to \mathbb{E}_\phi$ is the reference configurations of $\mathcal{B}$ in the frame ϕ, and

$$\boldsymbol{X} = \kappa_\phi(p) \in \mathbb{E}_\phi, \qquad p \in \mathcal{B}, \qquad \mathcal{B}_\kappa = \kappa_\phi(\mathcal{B}) \subset \mathbb{E}_\phi.$$

The motion $\chi : \mathcal{B} \times \mathbb{R} \to \mathbb{E}_\phi$ relative to the reference configuration κ_ϕ is given by

$$\chi_\kappa(\cdot, t) : \mathcal{B}_\kappa \to \mathbb{E}_\phi, \qquad \boldsymbol{x} = \chi(p,t) = \chi(\kappa_\phi^{-1}(\boldsymbol{X}), t) = \chi_\kappa(\boldsymbol{X}, t), \qquad \chi = \chi_\kappa \circ \kappa_\phi.$$

We can define the corresponding constitutive functions with respect to the reference configuration,

$$\mathcal{F}_\phi(\chi^t; p) = \mathcal{F}_\phi(\chi_\kappa^t \circ \kappa_\phi;\, \kappa_\phi^{-1}(\boldsymbol{X})) := \mathcal{F}_\kappa(\chi_\kappa^t; \boldsymbol{X}),$$

and from (18), the condition of material objectivity for the constitutive function in the reference configuration can be restated as

$$\mathcal{F}_\kappa(q(\chi_\kappa^t); \boldsymbol{X}) = Q(t)\, \mathcal{F}_\kappa(\chi_\kappa^t; \boldsymbol{X})\, Q(t)^\top, \qquad \boldsymbol{X} \in \mathcal{B}_\kappa, \tag{19}$$

for any history of motion $\chi_\kappa^t \in \mathfrak{H}(\mathcal{B}_\kappa, \mathbb{E}_\phi)$ and any Euclidean transformation

$$q : \mathbb{E}_\phi \to \mathbb{E}_\phi, \qquad q(\chi_\kappa(\boldsymbol{X}, t)) = Q(t)(\chi_\kappa(\boldsymbol{X}, t) - \boldsymbol{x}_0) + \boldsymbol{r}(t),$$

for some orthogonal transformation $Q(t) \in O(\mathbb{V}_\phi)$, and some $\boldsymbol{x}_0, \boldsymbol{r}(t) \in \mathbb{E}_\phi$.

Remark 5. Note that the condition (19) is valid for any Euclidean transformation $q : \mathbb{E}_\phi \to \mathbb{E}_\phi$, which can also be interpreted as a time-dependent rigid deformation of the body in the Euclidean space $\mathbb{E}_\phi$. This interpretation is sometimes viewed as an alternative version of the principle of material frame-indifference and is called the "principle of invariance under superimposed rigid body motions". □

Simple materials

According to the principle of determinism (11), thermomechanical histories of any part of the body can affect the response at any point of the body. In most applications, such a non-local property is irrelevant. Therefore, it is usually assumed that only thermomechanical histories in an arbitrary small neighborhood of $\boldsymbol{X}$ affects the material response at the point $\boldsymbol{X}$, and hence the global history functions can be approximated at $\boldsymbol{X}$ by Taylor series up to certain order in a small neighborhood of $\boldsymbol{X}$. In particular, when only linear approximation is concerned, the constitutive function is restricted to a special class of materials,

$$\mathcal{F}_\kappa(\chi_\kappa^t(\cdot), \boldsymbol{X}) = \mathcal{H}_\kappa(\nabla_{\boldsymbol{X}} \chi_\kappa^t(\boldsymbol{X}), \boldsymbol{X}),$$

so that we can write the constitutive equation for the stress as

$$T(\boldsymbol{X}, t) = \mathcal{H}_\kappa(F_\kappa^t; \boldsymbol{X}), \qquad F_\kappa^t \in \mathfrak{H}(\{\boldsymbol{X}\}, L(\mathbb{V}_\phi)), \qquad \boldsymbol{X} \in \mathcal{B}_\kappa, \tag{20}$$

where $F_\kappa^t = \nabla_{\boldsymbol{X}} \chi_\kappa^t$ is the deformation gradient and the domain of the history is a single point $\{\boldsymbol{X}\}$. Note that although the constitutive function depends only on local values at the position $\boldsymbol{X}$, it is still general enough to define a material with memory of local deformation in the past. A material with constitutive equation (20) is called a *simple material*. The class of simple materials is general enough to include most of the materials of practical interests, such as: elastic solids, viscoelastic solids, as well as elastic fluids and Navier-Stokes fluids.

For the Euclidean transformation $q : \mathbb{E}_\phi \to \mathbb{E}_\phi$ and $\boldsymbol{x} = \chi_\kappa(\boldsymbol{X}, t)$ from (19), we have

$$\nabla_{\boldsymbol{X}} q(\chi_\kappa^t(\boldsymbol{X})) = \nabla_{\boldsymbol{x}} q(\chi_\kappa^t(\boldsymbol{X})) \nabla_{\boldsymbol{X}} \chi_\kappa^t(\boldsymbol{X}) = Q^t F_\kappa^t(\boldsymbol{X}).$$

Therefore, we obtain the following main result for simple materials:

Condition of material objectivity. *For simple materials relative to a reference configuration, the constitutive equation* $T(\boldsymbol{X}, t) = \mathcal{H}_\kappa(F_\kappa^t; \boldsymbol{X})$ *satisfies*

$$\mathcal{H}_\kappa(Q^t F_\kappa^t; \boldsymbol{X}) = Q(t)\, \mathcal{H}_\kappa(F_\kappa^t; \boldsymbol{X})\, Q(t)^\top, \tag{21}$$

for any history of deformation gradient $F_\kappa^t \in \mathfrak{H}(\{\boldsymbol{X}\}, L(\mathbb{V}_\phi))$ *and any orthogonal transformation* $Q(t) \in O(\mathbb{V}_\phi)$.

Remark 6. The condition (21) is the most well-known result in constitutive theories of continuum mechanics. It is the ultimate goal to obtain this result regardless of whoever agree or disagree with each other on frame-indifference and the principle of material frame-indifference controversy. □

Form invariance

It is also interesting to see how the principle of material frame-indifference in the form invariance of (17) takes in referential description. Let $\kappa : \mathcal{B} \to \mathcal{W}_{t_0}$ be a reference placement of the body at some instant t_0, and $i : \mathbb{E}_\phi \to \mathbb{E}_{\phi^*}$ be an isometry, then

$$\kappa_i : \mathcal{B} \to \mathbb{E}_{\phi^*}, \qquad \kappa_i = i \circ \kappa_\phi = i \circ \phi_{t_0} \circ \kappa,$$

is the reference configurations of $\mathcal{B}$ in the frame $\phi_i = i \circ \phi_{t_0}$ under the isometry i, and

$$\boldsymbol{X}_i = \kappa_i(p) \in \mathbb{E}_{\phi^*}, \qquad p \in \mathcal{B}, \qquad \mathcal{B}_{\kappa_i} = \kappa_i(\mathcal{B}) \subset \mathbb{E}_{\phi^*}.$$

The motion $\chi : \mathcal{B} \times \mathbb{R} \to \mathbb{E}_\phi$ relative to the reference configuration κ_i is given by

$$\chi_{\kappa_i}(\cdot, t) : \mathcal{B}_{\kappa_i} \to \mathbb{E}_{\phi^*}, \qquad \boldsymbol{x}_i = \chi_{\phi_i}(p, t) = i(\chi(p, t)) = \chi_{\kappa_i}(\kappa_i(p), t),$$

so that we have

$$\chi_{\phi_i} = i \circ \chi = \chi_{\kappa_i} \circ \kappa_i, \qquad \chi_{\kappa_i} = i \circ \chi \circ \kappa_i^{-1}. \tag{22}$$

We can define the constitutive functions with respect to the reference configuration κ_i,

$$\mathcal{F}_{\phi_i}(i(\chi^t); p) = \mathcal{F}_{\phi_i}(\chi_{\kappa_i}^t \circ \kappa_i;\ \kappa_i^{-1}(\boldsymbol{X}_i)) := \mathcal{F}_{\kappa_i}(\chi_{\kappa_i}^t; \boldsymbol{X}_i),$$

and we can do similar things for another isometry $j : \mathbb{E}_\phi \to \mathbb{E}_{\phi^*}$. Then from the form invariance (17), after some calculations, we obtain

$$\mathcal{F}_{\kappa_i}(\chi_{\kappa_j}^t \circ q^*;\ \boldsymbol{X}_i) = \mathcal{F}_{\kappa_j}(\chi_{\kappa_j}^t; \boldsymbol{X}_j), \tag{23}$$

for any deformation history $\chi^t_{\kappa_j} \in \mathfrak{H}(\mathcal{B}_{\kappa_j}, \mathbb{E}_{\phi^*})$ and the isometry $q^* = j \circ i^{-1} : \mathbb{E}_{\phi^*} \to \mathbb{E}_{\phi^*}$ with its associated orthogonal transformation $Q^* = \mathcal{J}\,\mathcal{I}^\top \in O(\mathbb{V}_{\phi^*})$.

Moreover, from (15) and (16), we obtain by the use of verifiable relations, $j \circ i^{-1} = \kappa_j \circ \kappa_i^{-1}$ and $\chi_{\kappa_j} = q^* \chi_{\kappa_i} q^{*-1}$ (from (22)),

$$\mathcal{F}_{\kappa_i}(q^*(\chi^t_{\kappa_i}); \boldsymbol{X}_i) = Q^*(t)\mathcal{F}_{\kappa_i}(\chi^t_{\kappa_i}; \boldsymbol{X}_i)Q^*(t)^\top, \qquad \boldsymbol{X}_i \in \mathcal{B}_{\kappa_i}, \tag{24}$$

for any deformation history $\chi_{\kappa_i} \in \mathfrak{H}(\mathcal{B}_{\kappa_i}, \mathbb{E}_{\phi^*})$ and any Euclidean transformation $q^* : \mathbb{E}_{\phi^*} \to \mathbb{E}_{\phi^*}$ with some orthogonal transformation $Q^*(t) \in O(\mathbb{V}_{\phi^*})$.

Of course, we can see that the above condition (24) formulated in the Euclidean space $\mathbb{E}_{\phi^*}$ is equivalent to the condition of material objectivity (19) derived directly from (18) in $\mathbb{E}_\phi$.

Remark 7. In the principle of material frame-indifference, we have postulated that constitutive functions are independent of observers stated as $\mathcal{F}_{\phi_i} = \mathcal{F}_{\phi_j}$ in (17). We emphasize that the form invariance is valid only when it is formulated in material description. Indeed, from the above discussion, we have $\mathcal{F}_{\kappa_i} \neq \mathcal{F}_{\kappa_j}$ in referential description, instead, they must satisfy the relation (23). □

Remark 8. It is interesting to give the following example, which shows typically why some misconception persisted. Let F_κ, F^*_κ and $T = \mathcal{T}_\kappa(F_\kappa)$, $T^* = \mathcal{T}^*_\kappa(F^*_\kappa)$, be the deformation gradients and the constitutive equations for the stress in two different frames relative to some reference configuration κ. One finds, in most textbooks, that the objectivity conditions are given by

$$F^*_\kappa = QF_\kappa, \qquad \mathcal{T}^*_\kappa(F^*_\kappa) = Q\,\mathcal{T}_\kappa(F_\kappa)\,Q^\top,$$

and the principle of material frame-indifference by

$$\mathcal{T}^*_\kappa(\bullet) = \mathcal{T}_\kappa(\bullet),$$

which combine to give the well-known condition of material objectivity,

$$\mathcal{T}_\kappa(QF_\kappa) = Q\,\mathcal{T}_\kappa(F_\kappa)\,Q^\top.$$

This is the correct result equivalent to the condition (21). However, we have already shown that $F^*_\kappa = QF_\kappa$ is valid only when the reference configuration is unaffected by the change of frame, and the principle of material frame-indifference does not imply the form invariance $\mathcal{T}^*_\kappa(\bullet) = \mathcal{T}_\kappa(\bullet)$ in referential description. Nevertheless, the lucky incident that the two inadequate assumptions would lead to the correct general result is quite striking and might have contributed to some misconception over the decades. □

9 CONCLUSIONS

This paper discusses some misconceptions related with MFI within the framework of Continuum Mechanics. This means that space is distinguished from time and relativistic space-time is not included in the discussion.

Some misconceptions are discussed throughout the paper in the remarks. Unfortunately, since the publication of Truesdell and Noll's treatise, controversies about the MFI seems to be persistent recently in some research papers and books. The authors hope that this paper will help to clarify the basic concepts and to exert caution about some erroneous interpretations.

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