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THEORY OF AFFINE SHELLS: TOWARDS ADVANCED NUMERICAL APPROXIMATIONS

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Abstract. The well-known Theory of Shells, a topic of Geometry and Mathematical Physics, has been exposed, through the contributions of many authors, within the framework of Euclidean Geometry, i.e., based on the classical theory of surfaces in three-dimensional space, which is invariant under the Lie group generated by translations and rotations. On the other hand, we ourselves have already developed and presented an alternative foundation of the theory, invariant under the action of the Unimodular Affine group, i.e., dealing with Affine Surface Geometry. In this paper we analyze exclusively the behavior of physical objects of the shell in the interior, without reference to any boundary conditions at the edge. Our main goal here is to establish a comparison with the results already obtained in the Euclidean Theory of Shells, beginning with some geometrical objects and following, afterwards, through the exposition of some distinguished examples in further papers.

1 INTRODUCTION

Theory of Shells is a topic of Geometry and Mathematical Physics with a rich history and many, diverse applications to the real world: Engineering, Industry, Avionics, Face Surgery, and so on. The usual viewpoint of presentation makes use of the classical, Euclidean Geometry of Surfaces, i.e., invariant under the action of the *Euclidean Group* (John, 1965, 1971; Koiter, 1971). The authors have been working on an alternative foundation and development of the theory of shells which is invariant under the action of the *Unimodular Affine Group* (Blaschke, 1923; Gigena, 1993, 1996; Nomizu and Sasaki, 1994). The latter allows considering and developing the so called "*affine geometry of surfaces*".

All physical given data are included in the *reference configuration* (undeformed shell), while the goal of the theory is to approximately calculate the *current configuration* (deformed shell). For a given surface in three-dimensional space there are useful concepts such as "*affine normal*" and "*affine distance*", corresponding to the ones in Euclidean geometry. Under the influence of the forces the shell deforms and assumes a position of equilibrium, satisfying the appropriate non-linear equations for elastic solids.

In this paper we work under the assumption to be dealing with a perfectly elastic, homogeneous and isotropic material shell which, in its deformed state, is in equilibrium with forces. Our goal is to elaborate a model, and later on appropriate algorithms, in order to make a comparison between results previously published in the literature (all of them obtained in the framework of Euclidean Geometry (John, 1965, 1971; Koiter, 1971; Love, 1944; Möllmann, 1981; Reddy, 2008; Wagner and Gruttmann, 1994; Kinkel et al, 1999), and our own results in Affine Geometry (Gigena et al, 2002, 2003, 2004, 2005, 2010), by using some referential viewpoints and a few number of examples exposed in the literature. The approximation level, indicated to be reached in this paper, can be considerably improved by using a greater number of terms in the equations that govern the phenomena. In that case we could further minimize the other terms displayed as errors. The new affine Lamé parameters, λ and μ, together constitute a parametrization of the elastic *moduli* for homogeneous isotropic media, popular in mathematical literature, and are thus related to the other elastic *moduli*.

We are not interested in trying to find exact solutions for the shells, a goal somehow unattainable, because all of the expositions in the topic are made, in general, by using approximations, for example finite elements methods among others. Thus, we are simply opening our minds to a new geometry in the treatment of the subject.

2 AN ABBREVIATED VERSION OF AFFINE SHELLS

The middle surface of a (solid) shell in its original (undeformed) state, M_0, is parametrized locally by a vector function assumed to be enough smooth. Thus, it is usually understood, that X_0 is a topological immersion (embedding). for further details see (Gigena et al, 2002, 2003, 2004, 2005, 2010).

$$X_0 : U \to \mathbb{R}^3, \qquad \text{where} \quad U \subset \mathbb{R}^2 \tag{1}$$

Particles in the original state have curvilinear Lagrange coordinates $\left(U^1, U^2, U^3\right)$ that for our present purposes shall be chosen in a special way, by writing:

$$\left(U^1, U^2, U^3\right) = \left(u^1, u^2, u\right)$$

$$X\left(u^1,u^2,u\right)=X_0\left(u^1,u^2\right)+u\,\vec{\mathbf{n}}\,, \tag{2}$$

where we have obviously extended the previous function to

$$X:U\times\left(-h,h\right)\to\mathbb{R}^3\,, \tag{3}$$

and $\vec{\mathbf{n}}$ is the vector field normal to the middle surface. This normal can be the Euclidean normal, $\boldsymbol{N}_{\text{eu}}$, of the classical, Euclidean Theory of Surfaces, or the Unimodular Affine normal, $\boldsymbol{N}_{\text{ua}}$, of our own, current development. In each case, we shall clarify the situation when we deal with one or the other.

On the middle surface (***Euclidean Geometry***) the main geometrical objects, as treated mainly in (Gigena et al, 2002, 2003, 2004, 2005, 2010; John, 1965, 1971; Millman and Parker, 1977), are:

$$\left.\begin{array}{lll} I_{\text{eu}}=\sum\limits_{\alpha,\beta}a_{\alpha\beta}\,du^\alpha du^\beta & \text{with} & a_{\alpha\beta}=\dfrac{\partial X_0}{\partial u^\alpha}\cdot\dfrac{\partial X_0}{\partial u^\beta} \\ II_{\text{eu}}=\sum\limits_{\alpha,\beta}L_{\alpha\beta}\,du^\alpha du^\beta & \text{where} & L_{\alpha\beta}=N_{eu}\cdot\dfrac{\partial^2 X_0}{\partial u^\beta\partial u^\alpha} \\ III_{\text{eu}}=\sum\limits_{\alpha,\beta}M_{\alpha\beta}\,du^\alpha du^\beta & \text{where} & M_{\alpha\beta}=\sum\limits_{\gamma\lambda}a^{\gamma\lambda}\,L_{\alpha\lambda}\,L_{\beta\gamma} \end{array}\right\} \tag{4}$$

In the above display, and in what follows, the numerical coefficients $a_{\alpha\beta}$, $L_{\alpha\beta}$, $M_{\alpha\beta}$, are respectively the components of the first, second and third Euclidean fundamental forms of the middle surface. The Euclidean structure of the ambient space induces a Riemannian structure on the shell and we can obtain, by means of straightforward computations, the following expressions in normal coordinates:

$$\left.\begin{array}{l} A_{\alpha\beta}=\dfrac{\partial X}{\partial u^\alpha}\cdot\dfrac{\partial X}{\partial u^\beta}=a_{\alpha\beta}-2u\,L_{\alpha\beta}+u^2M_{\alpha\beta} \\ A_{\alpha 3}=A_{3\alpha}=\dfrac{\partial X}{\partial u^\alpha}\cdot\dfrac{\partial X}{\partial t}=\dfrac{\partial X}{\partial u^\alpha}\cdot N_{\text{eu}}=0 \\ A_{33}=\dfrac{\partial X}{\partial t}\cdot\dfrac{\partial X}{\partial t}=N_{\text{eu}}\cdot N_{\text{eu}}=1 \end{array}\right\} \tag{5}$$

On the middle surface (***Affine Geometry***) the main geometrical objects, in the ambient space $\mathbb{R}^3$, are obtained by using a previously chosen determinant function:

$$\left[\ ,\ ,\ \right]=\det \tag{6}$$

This choice allows introducing the following numerical coefficients:

$$h_{\alpha\beta}=\left[\left(X_0\right)_1,\left(X_0\right)_2,\left(X_0\right)_{\alpha\beta}\right], \tag{7}$$

with which we construct the Unimodular affine (pseudo)-metric tensor.

If we assume the surface to be everywhere non-degenerate, those coefficients can be normalized in order to obtain an affine invariant pseudometric and, with its appropriate use, the further affine normal vector field introduced, both of them, by means of the following equation:

$$\left.\begin{aligned} g_{\alpha\beta} &= |H|^{-1/4} h_{\alpha\beta}\,, \\ I_{ua} &= \sum_{\alpha,\beta} g_{\alpha\beta} du^\alpha du^\beta\,, \\ N_{ua} &= \tfrac{1}{2}\Delta(X_0)\,, \end{aligned}\right\} \tag{8}$$

with $\Delta X_0 = \frac{1}{\sqrt{|g|}} \sum_{\alpha=1}^{2} \frac{\partial}{\partial u^\alpha}\left(\sqrt{|g|}\sum_{\beta=1}^{2} g^{\alpha\beta}\left(X_0\right)_\beta\right)$ and $g = \det\left(g_{\alpha\beta}\right)$.

For further information see (Gigena et al, 2002, 2003, 2004, 2005, 2010).

So we may consider three connections:

1) The ***Levi-Civita connection*** with ***respect to the Euclidean metric*** I_{eu} : ∇_{eu}.

2) The ***Levi-Civita connection*** with ***respect to pseudometric*** I_{ua} : $\widetilde{\nabla}$.

3) The ***Affine Normal induced connection***: ∇ .

For further details see (Gigena et al, 2002, 2003, 2004, 2005, 2010).

We define next the ***Unimodular Affine Second Fundamental (Cubic) Form***, as previously introduced in (Gigena, 1993, 1996; Nomizu and Sasaki, 1994), in local coordinates, with $g_{\alpha\beta\gamma}$ totally symmetric in their indices.

$$\begin{gathered} \nabla\left(I_{\mathrm{ua}}\right) = II_{\mathrm{ua}} \\ II_{\mathrm{ua}} = \sum_{\alpha\beta\gamma} g_{\alpha\beta\gamma}\, du^\alpha du^\beta du^\gamma \end{gathered} \tag{9}$$

The ***Unimodular Affine Third Fundamental Form*** written

$$III_{\mathrm{ua}} = B_{\alpha\beta} du^\alpha du^\beta\,, \tag{10}$$

with $B_{\alpha\beta} = \sum_{\gamma} g_{\alpha\gamma} B^\gamma_\beta$ and $\left(N_{\mathrm{ua}}\right)_\alpha = -\sum_{\beta} B^\beta_\alpha \left(X_0\right)_\beta$.

The affine invariant pseudometric I_{ua}, on the middle surface, allows to define on the whole shell a pseudo-metric, which is a Unimodular Affine invariant:

$$\begin{gathered} M_0 :\; g_{\alpha\beta} = I_{\mathrm{ua}}\left(\left(X_0\right)_\alpha, \left(X_0\right)_\beta\right) \\ G = \sum G_{ij}\, du^i du^j \qquad \text{with} \qquad G_{ij} := G\left(\left(X\right)_i, \left(X\right)_j\right) \end{gathered} \tag{11}$$

By definition,

$$\begin{gathered} G_{\alpha\beta} := g_{\alpha\beta} - 2uB_{\alpha\beta} + u^2 \sum_{\lambda} B^\lambda_\alpha B_{\beta\lambda} \\ G_{3\alpha} = G_{\alpha 3} = G\left(X_\alpha, N_{ua}\right) := 0 \qquad ; \qquad G_{33} = G\left(N_{ua}, N_{ua}\right) := 1 \end{gathered} \tag{12}$$

(Greek indices run from 1 to 2)

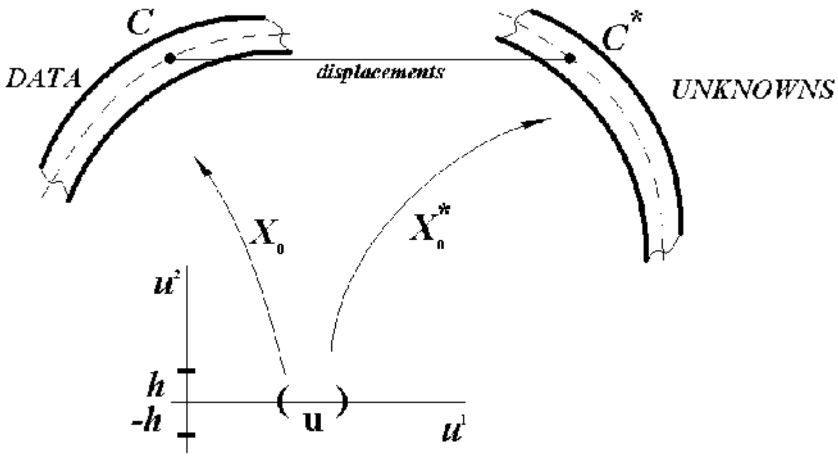
For $u = u^3$ enough small, it holds:

$$\det\left(G_{ij}\right) \neq 0\,, \tag{13}$$

showing that, indeed, G is a pseudo-Riemannian, Unimodular affine invariant metric defined on the shell. For more information see (Gigena et al, 2002, 2003, 2004, 2005, 2010).

3 COMPARING GEOMETRICAL OBJECTS

Our goal here is to create a model, and suitable algorithms, in order to make a comparison between our results and those of other people already obtained in the rest of the literature.



Thus, first of all, we point out the relationship between the second fundamental form in Euclidean Geometry and the first fundamental form in Unimodular Affine Geometry.

$$I_{\mathrm{ua}} \leftrightarrow II_{\mathrm{eu}} \tag{14}$$

In fact, if we consider a system constituted by the ambient space and the two basic invariants in both geometries, and define an immersion in a suitable open subset of the plane, what some authors refer to as a parameter space, we may write by definition

$$\left(\mathbb{R}^3, \underbrace{\cdot}_{euclidian}, \underbrace{[\ ,\ ,\ ]}_{affine}\right); \qquad \left(X_0\right): U \rightarrow \mathbb{R}^3 \qquad U \subset \mathbb{R}^2 \tag{15}$$

and from Gauss Euclidean equation:

$$\left(X_0\right)_{\alpha\beta} = L_{\alpha\beta} N_{eu} + \sum_{\lambda=1}^{2} \Gamma^{\lambda}_{\alpha\beta} \left(X_0\right)_{\lambda}\,, \tag{16}$$

we may express, from the equality

$$h_{\alpha\beta} = \left[\left(X_0\right)_1, \left(X_0\right)_2, L_{\alpha\beta} N_{eu} + \sum_{\lambda=1}^{2} \Gamma^{\lambda}_{\alpha\beta} \left(X_0\right)_{\lambda}\right] \tag{17}$$

by using the well known properties of the determinant function, that

$$h_{\alpha\beta} = \left[(X_0)_1, (X_0)_2, N_{eu} \right] L_{\alpha\beta} \tag{18}$$

and

$$0 \neq \det\left(h_{\alpha\beta}\right) = \left[(X_0)_1, (X_0)_2, N_{eu} \right]^2 \det\left(L_{\alpha\beta}\right). \tag{19}$$

It follows that the immersed surfaces apt for treatment of a meaningful theory in affine geometry are those which, in its Euclidean version, have identically non-vanishing Gaussian curvature K, in the sense defined in (Millman and Parker, 1977). To treat, similarly, other kinds of surfaces one would need to use the Affine Theory of Plane Curves, as developed for example in (Calabi et al, 1996, 1998), and to further make reduction of codimension.

Among the quantities that can be used for an approximate two-dimensional description of the state of the shell some refer to the stress system and others to the geometry (deformation system), with "constituent equations" relating the two types.

4 RECOVERING THE DEFORMED MIDDLE SURFACE

The main objective in the Theory of Shells, whether Euclidean or Affine, is to determined the extent of deformation of the shell once a solicitation is applied on it. And the best way of analyzing this is to see what happens to the corresponding middle surfaces. Thus, when the solicitation is produced on the shell in its original, undeformed state, it produces a three-dimensional stress tensor, (Gigena et al, 2002, 2003, 2004, 2005, 2010; John, 1965, 1971),

$$\tau = \sum_{i,j} \tau^{ij} \frac{\partial}{\partial u^i} \otimes \frac{\partial}{\partial u^j}, \tag{20}$$

which is well-known to be symmetric, i.e.,

$$\tau^{ij} = \tau^{ji} \tag{21}$$

That first action produces, next, the deformation whose extent is contained as information in the corresponding strain tensor

$$\varepsilon_{ij} := \tfrac{1}{2}\left(G^*_{ij} - G_{ij} \right), \tag{22}$$

Let us recall that the relation between both of them is established through the stress-strain relation (John, 1965, 1971)

$$\left(\tau^{mk}\right) := \sqrt{\frac{G}{G^t}} \frac{\partial W}{\partial\left(\varepsilon_{mk}\right)} \tag{23}$$

which, in terms of tensors with mixed components, (John, 1965, 1971), may also be written

$$\tau_i^m = \sqrt{\frac{G}{G^*}} \frac{\partial W}{\partial \varepsilon_m^i}, \tag{24}$$

as already analyzed in previous articles (Gigena et al, 2002, 2003, 2004, 2005, 2010).

In this context, in terms of the components of the ***"pseudo-stress tensor",*** defined by

$$T_j^m := \sqrt{\frac{G^*}{G}}\; t_j^m - \delta_j^m W \tag{25}$$

we may also write

$$\begin{gathered} T_i^m = \left(W_{s_1} - W\right)\delta_i^m + \left(2W_{s_1} + 2W_{s_2}\right)\varepsilon_i^m + \\ + \left(4W_{s_2} + 3W_{s_3}\right)\sum_k \varepsilon_i^k \varepsilon_k^m + 6\; W_{s_3} \sum_{s,k} \varepsilon_i^s \varepsilon_s^k \varepsilon_k^m \end{gathered} \tag{26}$$

where, as seen in (Gigena et al, 2002, 2003, 2004, 2005, 2010; John, 1965, 1971),

$$s_1 = \sum_i \varepsilon_i^i \qquad\qquad s_2 = \sum_{i,j} \varepsilon_j^i \varepsilon_i^j \qquad\qquad s_3 = \sum_{i,j,k} \varepsilon_j^i \varepsilon_k^j \varepsilon_i^k \tag{27}$$

It turns out that the equations of equilibrium can be written

$$t^{ij}{}_{,,j} + c_{hj}^i\, t^{hj} + c_{hj}^h\, t^{ij} = 0\,, \tag{28}$$

where the double comma indicates covariant derivative with respect to the Levi-Civita connection on the undeformed shell $\left(\mathrm{C},\, \mathrm{G}\right)$, with

$$c_{jk}^i = \tfrac{1}{2}\bar{G}^{*ir}\left(G^*_{rj,,k} + G^*_{rk,,j} - G^*_{jk,,r}\right) \tag{29}$$

and where we also have, as a consequence,

$$\sum_m T^m_{i\,;\,m} = \left(\sqrt{\frac{G^*}{G}}\right)\sum_{m,r,s}\left(c_{mr}^r t^{ms} G^*_{si} - t^{rs} c_{rm}^m G^*_{si} - t^{mr} c_{rm}^s G^*_{si} + t^{ms} G^*_{si;m} - \tfrac{1}{2} t^{ms} G^*_{sm;i}\right) = 0 \tag{30}$$

In order to compare components of stress and strain tensors we need additional notations. So, following the one used by Fritz John in (John, 1965, 1971), the so-called "general form of an expression"

$$F\left(p,q\right)\left(u+v+w\right) \tag{31}$$

representing a vector, in a suitable space, where u, v, w, p, q are vectors themselves. The notation indicates that each of the components is a sum of a linear form in the components of u, a linear form in the components of v, and a linear form in the components of w. The coefficients of these linear forms are functions of the components of the vectors p and q defined and differentiable as often as needed for all sufficiently small "lengths" $|p|$ and $|q|$. The letter F stands for a different expression in every equation to be considered. For more information, see (Gigena et al, 2002, 2003, 2004, 2005, 2010; John, 1965, 1971).

The components of the stress and strain tensors, of type (1,1) $t = \left(t_k^i\right)$ and $\varepsilon = \left(\varepsilon_k^i\right)$ are related, in terms of the Lamé coefficients or parameters $\lambda,\ \mu$, by the following equation

$$t_i^m = \lambda \sum_j \varepsilon_j^j \delta_i^m + 2\mu\varepsilon_i^m + F\left(\varepsilon\right)\varepsilon^2 \tag{32}$$

since such coefficients are defined by the relation

$$W := \frac{\lambda}{2}\left(s_1\right)^2 + \mu s_2 + F\left(\varepsilon\right)\varepsilon^3 . \tag{33}$$

where, s_1, s_2, s_3 were defined in (27) and where we observe that the first two terms, on the right-hand side, are quadratic in terms of the strain tensor (operator) $\varepsilon = \left(\varepsilon_k^i\right)$, while the third term involves all of those components of order higher than two, representing otherwise the "remainder", of paramount importance when coming to the corresponding numerical estimates.

From now on we establish that in the same sense have to be interpreted all of the expressions to follow. Hence, by taking partial derivatives, we can write

$$W_{s_1} = \partial_{s_1} W = \lambda\ s_1 \quad , \qquad W_{s_2} = \partial_{s_2} W = \mu . \tag{34}$$

From the latter we obtain, successively:

$$\left.\begin{aligned} 2W_{s_1} + 2W_{s_2} &= 2\mu + 2\lambda\ s_1 \\ 4W_{s_2} + 3W_{s_3} &= 2\mu + F\left(t\right)t^3 , \\ W_{s_1} - W &= \lambda\ s_1 - \frac{1}{2}\lambda\left(s_1\right)^2 - \frac{1}{2}\mu s_2 + F\left(\varepsilon\right)\varepsilon^3 \end{aligned}\right\} \tag{35}$$

Then, by using the Taylor´s series development

$$\left(1+x\right)^{-1/2} = 1 + \left(-\frac{1}{2}\right)x + \frac{3/4}{2}x^2 + \ldots,$$

we can express

$$\sqrt{\frac{G}{G^*}} = \left(\frac{G^*}{G}\right)^{-1/2} = 1 - \frac{1}{2}s_1 + \frac{3}{8}\left(s_1\right)^2 + \ldots \tag{36}$$

and

$$t_i^m = \sqrt{\frac{G}{G^*}}\left(W_{s_1}\delta_i^m + 2W_{s_2}\varepsilon_i^m + 3W_{s_3}\sum_j \varepsilon_j^m \varepsilon_i^j\right) \tag{37}$$

becomes, first

$$t_i^m = \left(1 - \frac{1}{2}s_1 + \frac{3}{8}\left(s_1\right)^2 + \ldots\right)\left(W_{s_1}\delta_i^m + 2W_{s_2}\varepsilon_i^m + 3W_{s_3}\sum_j \varepsilon_j^m \varepsilon_i^j\right) \tag{38}$$

and, afterwards

$$t_i^m = \frac{\lambda}{2}\sum_j \varepsilon_j^j \delta_i^m + 2\mu\varepsilon_i^m + F\left(\varepsilon\right)\varepsilon^2 \tag{39}$$

The trace of the ***stress tensor*** (operator) can be expressed then by

$$\sum_j t_j^j = \left(\frac{3}{2}\lambda + 2\mu\right)\sum_j \varepsilon_j^j + F\left(\varepsilon\right)\varepsilon^2 \tag{40}$$

where $\sum_j \varepsilon_j^j$ itself represents the trace of the ***strain tensor*** (operator).

Hence, from the above we can also write $\varepsilon_i^m = \frac{1}{2\mu} t_i^m - \frac{\lambda}{2\mu} s_1 \delta_i^m + F(t)t^2$ or, also,

$$\varepsilon_i^m = \frac{1}{2\mu} t_i^m - \frac{1-2\mu}{2\mu} \sum_j t_j^j \, \delta_i^m + F(t)t^2 \,. \tag{41}$$

The pseudo-stress tensor may be expressed, then, as

$$\begin{aligned} T_i^m = t_i^m + \frac{2}{\mu} \sum_i t_i^m t_i^s + \left(\frac{5\mu - 2}{\mu} \right) \sum_j t_j^j \, t_i^s - \\ - \tfrac{1}{2} \left(\frac{1}{2\mu} \sum_{r,s} t_s^r t_r^s - \left(\frac{1-2\mu}{2\mu} \right) \sum_r t_r^r \sum_s t_s^s \right) \delta_i^m + F(t)t^3 \end{aligned} \tag{42}$$

Following with this kind of setting, we introduce next the *"vector"* $\eta = \left(\eta_k^i \right)$ by means of the relation:

$$G_{ik} = \delta_k^i + \eta_k^i \,, \tag{43}$$

which measures the difference between the metric matrix and that corresponding to the identity. Then, we obtain the following estimate for the components of the corresponding inverse matrix $\left(G^{ik} \right) := \left(G_{ik} \right)^{-1}$:

$$G^{ik} = \delta_k^i + \eta_k^i + F(\eta)\left(\eta^2\right). \tag{44}$$

It follows, too, that the Christoffel symbols satisfy the following estimate

$$\Gamma_{kr}^i = F(\eta)(\eta') \,, \tag{45}$$

and it also holds the next estimate

$$t_{ik} = t_k^i + F(\eta, t)(\eta \; t) \,. \tag{46}$$

For the metric tensor in the deformed "strained" state we have, by definition,

$$G_{ik}^* = G_{ik} + 2\varepsilon_{ik} \,. \tag{47}$$

Hence, we can also estimate here that

$$G_{ik}^* = G_{ik} + \frac{2}{\mu} t_{ik} - 2\left(\frac{1-2\mu}{2\mu} \right) \sum_j t_j^j \delta_k^i + F(t,\eta)\left(t^2 + \eta \, t\right), \tag{48}$$

while, for the tensor with components c_{kr}^i measuring the change in the Levi-Civita connections, from the "unstrained" natural state to the deformed "strained" state, we estimate

$$c_{kr}^i = F(\eta, t)\left(t' + \eta' t\right). \tag{49}$$

These, and the other previous estimates, shall be used in the recovery of the strained middle surface, together with the geometrical information available so far. We see, for example, from equation (15) that on the middle surface, characterized by the equation $u^3 := u = 0$, we have

$$G_{\alpha\beta}\left(u^1,u^2,0\right) = g_{\alpha\beta}\left(u^1,u^2\right). \tag{50}$$

Hence, and correspondingly, we may also write that

$$G^*_{\alpha\beta}\left(u^1,u^2,0\right) = g^*_{\alpha\beta}\left(u^1,u^2\right). \tag{51}$$

The main geometrical information is being contained in the middle surfaces M_0, M_0^* through the corresponding difference tensors: $\varepsilon_{\alpha\beta} := \frac{1}{2}\left(g^*_{\alpha\beta} - g_{\alpha\beta}\right)$, $\sigma_{\alpha\beta\gamma} := g^*_{\alpha\beta\gamma} - g_{\alpha\beta\gamma}$ and $w_{\alpha\lambda} := B^*_{\alpha\lambda} - B_{\alpha\lambda}$, defined on the parameter space and relating the coefficients of the first, second and third fundamental forms, treated in (Gigena et al, 2002).

Thus, the known data are those corresponding to the undeformed middle surface M_0, i.e., $g_{\alpha\beta}$, $g_{\alpha\beta\gamma}$, $B_{\alpha\lambda}$, and the problem is to determined the deformed ones, $g^*_{\alpha\beta}$, $g^*_{\alpha\beta\gamma}$, $B^*_{\alpha\lambda}$, because these furnish a description of the deformed middle surface M_0^* up to rigid motions in the Unimodular Affine Geometry of surfaces. In case we want actually to obtain, or recover, the actual deformed middle surface we can do so by applying one, or more, of the several Existence Fundamental Theorems, as treated for example in (Gigena, 1996; Nomizu and Sasaki, 1994). The corresponding method for the recovery of the deformed middle surface have been previously treated, in the Euclidean case, by F. John in its papers (John, 1965, 1971), while several other authors treat the recovery by means of approximated methods (Arciniega and Reddy, 2007; Reddy, 2008; Wagner and Gruttmann, 1994; Kinkel et al, 1999).

The affine pseudo-Riemannian structure of the deformed shell may be induced by writing

$$X^*\left(u_1,u_2,v\right) = X_0^*\left(u_1,u_2\right) + v\, N^*_{\mathrm{ua}}. \tag{52}$$

So, from (48), we may express the affine objects components: $g^*_{\alpha\beta}$, $g^*_{\alpha\beta\gamma}$, and $B^*_{\alpha\beta}$, which are going to play a role comparable to the ones in the Euclidean case: $a^*_{\alpha\beta}$, $L^*_{\alpha\beta}$ and $M^*_{0_{\mathrm{eu}}}$, developed previously in (John, 1965, 1971). The two latter are treated by assuming suitable integrability conditions exposed, for example, in (Millman and Parker, 1977), where the Fundamental Theorem of Existence and Uniqueness (Congruence) is stated. The corresponding material for the affine case may be found fully developed in (Gigena, 1996) and, with a different kind of notation, in (Nomizu and Sasaki, 1994).

If we consider that shell deformation is the same, whatever is the geometry, but treated in a different way for each case, we have the following relations.

Euclidean Gauss Equation:

$$\left(X_0\right)^*_{\alpha\beta} = L^*_{\alpha\beta} N^*_{\mathrm{eu}} + \Gamma^{*1}_{\alpha\beta}\left(X_0\right)^*_1 + \Gamma^{*2}_{\alpha\beta}\left(X_0\right)^*_2 \tag{53}$$

where, in the middle surface $(u=0)$, we have

$$a^*_{\alpha\beta} = \left(X_0\right)^*_1 \cdot \left(X_0\right)^*_2 = X^*_1\left(u_1,u_2,0\right) \cdot X^*_2\left(u_1,u_2,0\right) \tag{54}$$

Affine Gauss Equation:

$$\left(X_0\right)^*_{\alpha\beta} = g^*_{\alpha\beta} N^*_{\text{ua}} + \Gamma^{*1}_{\alpha\beta}\left(X_0\right)^*_1 + \Gamma^{*2}_{\alpha\beta}\left(X_0\right)^*_2 . \tag{55}$$

Notice that the meaning for the coefficients $\Gamma^{*\lambda}_{\alpha\beta}$ is different in the above equations (53) and (55), since we have preferred to use the same in order to avoid unnecessary complications in notation.

Thus, by the very definition of the Second Fundamental (Cubic) Form

$$\nabla\left(I_{\text{ua}}\right) = II_{\text{ua}} \tag{56}$$

we may write that

$$g^*_{\alpha\beta\gamma} = g^*_{\alpha\beta;\gamma} = \partial_\gamma g^*_{\alpha\beta} - \Gamma^{*1}_{\alpha\gamma} g^*_{1\beta} - \Gamma^{*2}_{\beta\gamma} g^*_{2\alpha} \tag{57}$$

It follows, by always reducing the situation to the middle surface $(u=0)$ at the end of the process, that we may successively write

$$\begin{aligned} \partial_\beta G^*_{\alpha s} &:= \frac{\partial}{\partial u^\beta} G^*\left(X^*_\alpha, X^*_s\right) \\ &= G^*\left(X^*_{\alpha\beta}, X^*_s\right) + G^*\left(X^*_\alpha, X^*_{s\beta}\right) \end{aligned} \tag{58}$$

and, by rotating indexes, we also have

$$\begin{aligned} \partial_\alpha G^*_{\beta s} &= \partial_\alpha G^*\left(X^*_\beta, X^*_s\right) \\ &= G^*\left(X^*_{\beta\alpha}, X^*_s\right) + G^*\left(X^*_\beta, X^*_{s\alpha}\right) \end{aligned} \tag{59}$$

$$\begin{aligned} \partial_s G^*_{\alpha\beta} &= \partial_s G^*\left(X^*_\alpha, X^*_\beta\right) \\ &= G^*\left(X^*_{\alpha s}, X^*_\beta\right) + G^*\left(X^*_\alpha, X^*_{\beta s}\right) \end{aligned} \tag{60}$$

Then, we readily get that

$$\frac{1}{2}\left(\partial_\beta G^*_{\alpha s} + \partial_\alpha G^*_{\beta s} - \partial_s G^*_{\alpha\beta}\right) = G^*\left(X^*_{\alpha\beta}, X^*_s\right) \tag{61}$$

Now, from the latter we get successively that

$$\frac{1}{2} G^{*3s}\left(\partial_\beta G^*_{\alpha s} + \partial_\alpha G^*_{\beta s} - \partial_s G^*_{\alpha\beta}\right) = G^*\left(X^*_{\alpha\beta}, G^{*3s} X^*_s\right), \tag{62}$$

$$\begin{aligned} G^*\left(X^*_{\alpha\beta}, G^{*3s} X^*_s\right) = G^{*33} g^*_{\alpha\beta} G^*\left(N^*_{\text{ua}}, X^*_3\right) + \\ + \Gamma^{*1}_{\alpha\beta}\left(G^{*31} G^*_{11} + G^{*32} G^*_{21} + G^{*33} G^*_{31}\right) + \\ + \Gamma^{*2}_{\alpha\beta}\left(G^{*21} G^*_{12} + G^{*22} G^*_{22} + G^{*23} G^*_{32}\right), \end{aligned} \tag{63}$$

where we further observe that the last two term vanish since

$$G^{*3s} G^*_{s1} = \delta^3_1 = 0 \tag{64}$$

and

$$G^{*3s}G^*_{s2} = \delta^3_2 = 0\,. \tag{65}$$

On the other hand we may also write similarly

$$\begin{aligned} G^*\left(X^*_{\alpha\beta}, G^{*2s}X^*_s\right) &= G^{*23}g^*_{\alpha\beta}G^*\left(N^*_{\text{ua}}, X^*_3\right) + \\ &+\Gamma^{*1}_{\alpha\beta}\underbrace{\left(G^{*21}G^*_{11} + G^{*22}G^*_{21} + G^{*23}G^*_{31}\right)}_{\delta^2_1=0} + \\ &+\Gamma^{*2}_{\alpha\beta}\underbrace{\left(G^{*21}G^*_{12} + G^{*22}G^*_{22} + G^{*23}G^*_{32}\right)}_{\delta^2_2=1} \end{aligned} \tag{66}$$

$$\begin{aligned} G^*\left(X^*_{\alpha\beta}, G^{*1s}X^*_s\right) &= G^{*13}g^*_{\alpha\beta}G^*\left(N^*_{\text{ua}}, X^*_3\right) + \\ &+\Gamma^{*1}_{\alpha\beta}\underbrace{\left(G^{*11}G^*_{11} + G^{*12}G^*_{21} + G^{*13}G^*_{31}\right)}_{\delta^1_1=1} + \\ &+\Gamma^{*2}_{\alpha\beta}\underbrace{\left(G^{*11}G^*_{12} + G^{*12}G^*_{22} + G^{*13}G^*_{32}\right)}_{\delta^1_2=0} \end{aligned} \tag{67}$$

Now, in order to properly express the term $G^*\left(N^*_{\text{ua}}, X^*_3\right)$, appearing in all of these equations, we may proceed to use the very well known Gram determinant identities:

$$\left[X^*_1, X^*_2, N^*_{\text{ua}}\right]\left[X^*_1, X^*_2, X^*_3\right] = \pm\det\begin{bmatrix} G^*_{11} & G^*_{12} & G^*_{13} \\ G^*_{12} & G^*_{22} & G^*_{23} \\ 0 & 0 & G^*\left(N^*_{\text{ua}}, X^*_3\right) \end{bmatrix}, \tag{68}$$

so that

$$\left[X^*_1, X^*_2, N^*_{\text{ua}}\right]\left[X^*_1, X^*_2, X^*_3\right] = \pm G^*\left(N^*_{\text{ua}}, X^*_3\right)\det\left[G^*_{\alpha\beta}\right]. \tag{69}$$

Following the same reasoning, we also have

$$\left[X^*_1, X^*_2, X^*_3\right]^2 = \pm\det\left[G^*_{ij}\right] \tag{70}$$

and

$$\left[X^*_1, X^*_2, N^*_{\text{ua}}\right]^2 = \pm\det\begin{bmatrix} G^*_{11} & G^*_{12} & 0 \\ G^*_{12} & G^*_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \pm\det\begin{bmatrix} G^*_{11} & G^*_{12} \\ G^*_{12} & G^*_{22} \end{bmatrix} \tag{71}$$

Thus, we obtain

$$G^*\left(N^*_{\text{ua}}, X^*_3\right) = \frac{\left[X^*_1, X^*_2, X^*_3\right]}{\left[X^*_1, X^*_2, N^*_{\text{ua}}\right]} = \sqrt{\frac{\det\left[G^*_{ij}\right]}{\det\left[G^*_{\alpha\beta}\right]}} = \left(G^{*33}\right)^{-\frac{1}{2}}. \tag{72}$$

So far, from the above it follows that the coefficients of the Affine Gauss equation (55) may be expressed in terms of the components of the deformed metric of shell, since $g^*_{\alpha\beta}$ is obtained from (63), $\Gamma^{*1}_{\alpha\beta}$ from (67) and $\Gamma^{*2}_{\alpha\beta}$ from (66), in the three cases having into account the latter equation (72).

It follows that the rest of components of the First, Second and Third Affine Fundamental Forms may be expressed, too, in terms of the components of the deformed metric shell G^*_{ij}, in every case reducing those terms to the middle surface $(u=0)$. See, for example, (Gigena, 1996; Nomizu and Sasaki, 1994).

From the information recorded in this way, and by also using equation (48), we can properly approximate the First, Second and Third Fundamental Forms of Affine Geometry and, by the further use of the Fundamental Existence Theorem (Gigena, 1996; Nomizu and Sasaki, 1994), recover the deformed middle surface.

The latter is a two-step procedure, whereas we consider first the above expression involving the Affine Gauss equation, recorded in (55), together with the expression of the local derivatives of the Affine normal vector field, obtained in a similar fashion to that recorded as part of equation (10):

$$\left.\begin{array}{c}\left(X_0\right)^*_{\alpha\beta} = g^*_{\alpha\beta} N^*_{\mathrm{ua}} + \Gamma^{*1}_{\alpha\beta}\left(X_0\right)^*_1 + \Gamma^{*2}_{\alpha\beta}\left(X_0\right)^*_2 \\ \left(N^*_{\mathrm{ua}}\right)_\alpha = -\sum_\beta B^{*\beta}_{\alpha}\left(X_0\right)^*_\beta\end{array}\right\} \tag{73}$$

This is a system of five partial differential equations, when considering that involves the vector fields $\left(X_0\right)^*_{11}$, $\left(X_0\right)^*_{12}$, $\left(X_0\right)^*_{22}$, $\left(N^*_{\mathrm{ua}}\right)_1$, $\left(N^*_{\mathrm{ua}}\right)_2$. Correspondingly, we have a system of fifteen scalar partial differential equations, when expressing each one of the latter in terms of the ambient space coordinates. That the system is integrable, and allows to recover firstly the vector fields $\left(X_0\right)^*_1$, $\left(X_0\right)^*_2$ and N^*_{ua}, follows precisely from the integrability conditions, already exposed in (Gigena et al, 2002).

Thus, with this first step of the procedure we obtain those three vector fields in terms of the components of the deformed metric shell G^*_{ij}. Having always in mind their approximate values expressed by equation (48), and also considering those values reduced to the deformed middle surface $(u=0)$.

In the next, second step of the procedure, we use those expressions for $\left(X_0\right)^*_1$ and $\left(X_0\right)^*_2$ in order to recover the vector field representing the deformed middle surface $\left(X_0\right)^*$. Here the integrability conditions are obviously satisfied since $\left(X_0\right)^*_{12} = \left(X_0\right)^*_{21}$.

5 CONCLUSIONS

What we are trying to show is that, even though deformation may have the same aspect in one or the other geometry, our different point of view furnishes the theory an open mind scope when working in shell structures. This is precisely the philosophy exposed by John in his papers by using Euclidean Geometry. On the other hand, approximate shell equations are offered having in mind that their solution, for appropriate boundary conditions, will furnish an approximation to the solution of the three-dimensional problem. This shall be the object of our next paper dedicated to the subject, where various kinds of examples shall be worked out and exposed.

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