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A FINITE-ELEMENT PROCEDURE TO DETERMINE BIOT'S COEFFICIENTS OF ANISOTROPIC POROELASTICITY

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Abstract. Numerical rock physics offers an alternative approach to laboratory measurements, being repeatable and essentially free from experimental errors. In this work, we present a set of oscillatory numerical experiments to determine the coefficients of the stress-strain relations of a poroelastic medium saturated by a single fluid. The experiments represent a set of low-frequency (quasi-static) compressibility tests applied to a representative sample of bulk material, where the fluid has enough time to reach pressure equilibration. Each test is associated with a boundary-value problem formulated in the space-frequency domain and solved with a finite-element procedure. The methodology is used to estimate the poroelasticity coefficients of a homogeneous sample of isotropic Utsira sandstone saturated with either brine or CO_2, and the eight coefficients corresponding to a periodic sequence of thin isotropic layers of brine-saturated mudstone and CO_2-saturated sandstone. This medium is transversely isotropic in the long-wavelength limit.

1 INTRODUCTION

Routine measurements in the laboratory are time-consuming, expensive and relatively informative, given the limited possibility to inspect the physical processes involved. Numerical experiments are inexpensive and informative since the physical process of wave propagation can be inspected during the experiment. Numerical rock physics offer in many cases an alternative procedure to laboratory measurements, being repeatable and essentially free from experimental errors. In particular, it is useful when the rock properties are well established, e.g., from dry-rock measurements or well logs.

The purpose of this paper is to present a collection of numerical experiments yielding the coefficients of the stress-strain relations in a fluid-saturated porous and anisotropic medium. A suitable model for fluid-saturated porous rocks is Biot's theory (Biot, 1956a,b, 1962; Carcione, 2007), since Biot's poroelasticity equations have shown to reproduce the main features of the static and dynamic behaviour of rocks (Müller et al., 2010). The wet-rock bulk modulus, also termed Gassmann modulus, K_c, can be written in terms of the dry-rock bulk modulus, K_m, effective-stress coefficient, α, and coupling modulus, B, as (Carcione, 2007)

$$K_c = K_m + \alpha B,$$

which depends on the properties of the single constituents of the porous medium.

The oscillatory experiments performed here are based on a finite-element (FE) numerical code that solves Biot's differential equation of motion at low frequencies. These experiments are completely controlled and may be an alternative to or precede the most costly real-field or laboratory experiments. Numerical experiments employing the finite-element method have been recently introduced to study the mesoscopic-loss mechanism in porous media (Santos et al., 2009; Carcione et al., 2011) and to obtain the stiffnesses of effective transversely isotropic media (Picotti et al., 2010; Santos et al., 2011).

In this work, we perform the simulation of compressibility tests in the space-frequency domain to compute the poroelasticity coefficients of an isotropic homogeneous medium, and the eight poroelasticity coefficients of a sequence of thin poroelastic layers (Gelinsky and Shapiro, 1997). These eight coefficients correspond to the quasi-static limit, where the fluid pressure is equilibrated across layer boundaries due to the diffusion of the Biot slow wave.

2 REVIEW OF BIOT'S THEORY.

We consider a porous solid saturated by a single phase, compressible viscous fluid and assume that the whole aggregate is isotropic. Let $u^s = (u_i^s)$ and $\tilde{u}^f = (\tilde{u}_i^f)$, $i = 1, \cdots, m$ denote the averaged displacement vectors of the solid and fluid phases, respectively, where m denotes the Euclidean dimension. Also let

$$u^f = \phi(\tilde{u}^f - u^s)$$

be the average relative fluid displacement per unit volume of bulk material, with ϕ denoting the effective porosity. Set $u = (u^s, u^f)$ and note that

$$\zeta = -\nabla \cdot u^f$$

represents the change in fluid content. Let $\varepsilon_{ij}(u^s)$ be the strain tensor of the solid and set

$$e = \nabla \cdot u^s = \varepsilon_{ii}.$$

Also, let $\sigma_{ij},\ i,j = 1,\cdots,m$, and p_f denote the stress tensor of the bulk material and the fluid pressure, respectively.

Following (Biot, 1962), the stress-strain relations of an isotropic medium can be written as

$$\sigma_{ij}(u) = 2\mu\,\varepsilon_{ij}(u^s) + \delta_{ij}(\lambda_c\, e - B\,\zeta), \tag{1a}$$
$$p_f(u) = -B\,e + M\zeta. \tag{1b}$$

The coefficient μ is equal to the shear modulus of the bulk material, considered to be equal to the shear modulus of the dry matrix (Gassmann, 1951). Also

$$\lambda_c = K_c - \frac{2}{m}\mu = E_c - 2\mu, \tag{2}$$

with K_c being the bulk modulus of the saturated material. The coefficients in (1) can be obtained from the relations (Gassmann, 1951), (Carcione, 2007),

$$\begin{aligned} \alpha &= 1 - \frac{K_m}{K_s}, \\ M &= \left(\frac{\alpha-\phi}{K_s} + \frac{\phi}{K_f}\right)^{-1}, \\ K_c &= K_m + \alpha^2 M, \\ B &= \alpha M, \end{aligned} \tag{3}$$

where K_s, K_m and K_f denote the bulk moduli of the grains composing the matrix, dry matrix and saturant fluid, respectively. The coefficient α is known as the effective stress coefficient of the bulk material.

Let η denote the fluid viscosity and κ the absolute permeability. Then, Biot's equations in the diffusive range, stated in the space-frequency domain, are (Biot, 1956a,b, 1962; Carcione, 2007)

$$\begin{aligned} &\nabla\cdot\sigma(u) = 0, \\ &\left(\frac{\eta}{\kappa}\right) \mathrm{i}\omega u^f(x,\omega) + \nabla p_f(u) = 0, \end{aligned} \tag{4}$$

where $\omega = 2\pi f$ is the angular frequency, $\mathrm{i} = \sqrt{-1}$ and we have assumed no body forces.

3 OSCILLATORY TESTS. ISOTROPIC CASE

To determine the coefficients in the stress-strain relations, equation (4) is solved in the 2D case on a reference square $\Omega = (0,L)^2$ with boundary Γ in the (x_1,x_3)-plane containing a representative volume of homogeneous bulk material with boundary conditions representing compressibility tests. Since the sample is homogeneous and the fluid does not support shear deformations, the shear modulus μ will be assumed to be known and equal to the shear modulus of the dry sample. For samples with heterogeneities in the rock properties an specific numerical experiment can be designed to obtain μ (Santos et al., 2009).

Set $\Gamma = \Gamma^L \cup \Gamma^B \cup \Gamma^R \cup \Gamma^T$, where

$$\Gamma^L = \{(x_1,x_3)\in\Gamma : x_1 = 0\}, \quad \Gamma^R = \{(x_1,x_3)\in\Gamma : x_1 = L\},$$
$$\Gamma^B = \{(x_1,x_3)\in\Gamma : x_3 = 0\}, \quad \Gamma^T = \{(x_1,x_3)\in\Gamma : x_3 = L\}.$$

Denote by ν the unit outer normal on Γ and let χ be a unit tangent on Γ so that $\{\nu, \chi\}$ is an orthonormal system on Γ.

As a first step, we determine the complex plane-wave modulus for $\zeta = 0$, i.e., the fluid is not allowed to flow in or out of the sample ($\zeta = 0$). This experiment is associated with the solution of (4) with the following set of boundary conditions.

$$\sigma(u)\nu \cdot \nu = -\Delta P, \quad (x_1, x_3) \in \Gamma^T, \tag{5}$$
$$\sigma(u)\nu \cdot \chi = 0, \quad (x_1, x_3) \in \Gamma^T \cup \Gamma^L \cup \Gamma^R, \tag{6}$$
$$u^s \cdot \nu = 0, \quad (x_1, x_3) \in \Gamma^L \cup \Gamma^R, \tag{7}$$
$$u^s = 0, \quad (x_1, x_3) \in \Gamma^B, \tag{8}$$
$$u^f \cdot \nu = 0, \quad (x_1, x_3) \in \Gamma. \tag{9}$$

The solid is not allowed to move on the bottom boundary Γ^B, the fluid is not allowed to flow in or out of the sample, a uniform compression is applied on the boundary Γ^T and no tangential external forces are applied on the boundaries $\Gamma^L \cup \Gamma^R \cup \Gamma^T$. For a periodic sample obtained by a mirror reflection with respect to the x_1-axis of the domain Ω, this experiment mimics exactly the one described by White et al. (White et al., 1975). It was shown in (Santos et al., 2009) that uniqueness holds for the solution to (4) with the boundary conditions (5)-(9) for $\omega > 0$ sufficiently small.

Denoting by V^b the original volume of the sample, its (complex) oscillatory volume change, $\Delta V^b(\omega)$, we can define the *equivalent* complex plane-wave modulus $E_c(\omega)$, by using the relation

$$\frac{\Delta V^b(\omega)}{V^b} = -\frac{\Delta P}{E_c(\omega)}. \tag{10}$$

The use of a low enough frequency, compared to the location of the Biot relaxation peak (Carcione, 2007), implies $E_c(\omega) \approx E_c(0) = \lambda_c + 2\mu = K_c + \frac{4}{3}\mu$. After solving (4) with the boundary conditions (5)-(9), the vertical displacements $u_3^s(x_1, L, \omega)$ on Γ^T allow us to obtain an average vertical displacement $u_3^{s,T}(\omega)$ suffered by the boundary Γ^T. Then, for each frequency ω, the volume change produced by the compressibility test can be approximated by

$$\Delta V^b(\omega) \approx L u_3^{s,T}(\omega),$$

which enable us to compute $E_c(\omega)$ by using equation (10).

To determine the coefficients B and M, associated with the change in fluid content ζ, we solve equation (4) with the boundary conditions

$$\sigma(u)\nu \cdot \nu = -\Delta P^b, \quad (x_1, x_3) \in \Gamma^T, \tag{11}$$
$$\sigma(u)\nu \cdot \chi = 0, \quad (x_1, x_3) \in \Gamma^T, \tag{12}$$
$$p_f(u) = \Delta P^f \quad (x_1, x_3) \in \Gamma^T, \tag{13}$$
$$\sigma(u)\nu \cdot \chi = 0, \quad (x_1, x_3) \in \Gamma^L \cup \Gamma^R, \tag{14}$$
$$u^s \cdot \nu = 0, \quad (x_1, x_3) \in \Gamma^L \cup \Gamma^R, \tag{15}$$
$$u^s = 0, \quad (x_1, x_3) \in \Gamma^B, \tag{16}$$
$$u^f \cdot \nu = 0, \quad (x_1, x_3) \in \Gamma^L \cup \Gamma^R \cup \Gamma^B. \tag{17}$$

For this experiment, $\varepsilon_{11} = \varepsilon_{22} = 0$, and the volume change of the sample is

$$e = \varepsilon_{33} = \frac{\Delta V^b}{V^b}. \tag{18}$$

Note that the applied fluid pressure on the boundary Γ^T induces fluid flow across such boundary. Then, measuring the average relative fluid displacement $u_3^f(x_1, L, \omega)$ on Γ^T allows us to obtain an average fluid displacement $u_3^{f,T}(\omega)$ across that boundary. Then, for each frequency ω, the change in fluid content ζ per unit bulk volume associated with this compressibility test, can be approximated by

$$\zeta = -\frac{\Delta V^f}{V^b} \approx -\frac{L u_3^{f,T}}{V^b}. \tag{19}$$

Hence, using the coefficient $E_c(\omega)$, already determined from the previous experiment, we get from (1), (11), (12) and (13):

$$\Delta P^b = E_c\, e - B\,\zeta, \tag{20}$$

$$\Delta P^f = -B\, e + M\,\zeta., \tag{21}$$

and from (20)-(21):

$$B = \frac{E_c\, e + \Delta P^b}{\zeta} \tag{22}$$

and

$$M = \frac{\Delta P^f + B\, e}{\zeta}. \tag{23}$$

From these equations we have

$$\alpha = \frac{B}{M}. \tag{24}$$

These equations hold at a very low frequency, such that the imaginary parts of the field variables and material coefficients are small enough to be neglected.

Other properties can be obtained by knowing these coefficients. For instance if K_s and K_f are known, the dry-rock modulus and the porosity can be obtained as

$$K_m = (1-\alpha)K_s \tag{25}$$

and

$$\phi = \left(\frac{1}{M} - \frac{\alpha}{K_s}\right)\left(\frac{1}{K_f} - \frac{1}{K_s}\right)^{-1}, \tag{26}$$

respectively. Since $E_c = K_c + 4\mu/3$, using (3), we may also obtain the shear modulus,

$$\mu = \frac{3}{4}(E_c - K_m - \alpha B). \tag{27}$$

4 OSCILLATORY TESTS. TRANSVERSELY ISOTROPIC CASE

Here, we consider the case in which the fluid-saturated poroelastic medium consists in an alternating sequence of thin isotropic plane horizontal layers. The frequency-domain stress-strain relations of a single layer n in a sequence of N layers, are:

$$\sigma_{kl}(u) = 2\mu^{(n)}\,\varepsilon_{kl}(u^s) + \delta_{kl}\left(\lambda_c^{(n)}\,\nabla\cdot u^s + \alpha^{(n)}M^{(n)}\nabla\cdot u^f\right), \tag{28}$$

$$p_f(u) = -\alpha^{(n)}M^{(n)}\nabla\cdot u^s - M^{(n)}\nabla\cdot u^f, \tag{29}$$

where for each layer n the coefficients $\mu^{(n)}, \lambda_c^{(n)}, \alpha^{(n)}$ and $M^{(n)}$ can be determined as indicated in (2)-(3).

As shown by Gelinsky & Shapiro (Gelinsky and Shapiro, 1997), a finely-layered medium behaves as a transversely isotropic (TI) equivalent medium at long wavelengths, with a vertical axis of azimuthal symmetry (the x_3-axis). They obtained the relaxed and unrelaxed limits, i.e., the low- and high-frequency limit real-valued stiffnesses, respectively. By high-frequency limit we mean a high enough frequency where the attenuation length of the Biot slow compressional wave is much larger than a mean characteristic, represented by the period of the stratification. At this limit, there is no flow and the layers behave like isolated media, since the slow wave is highly attenuated and the fluid pressure is no longer equilibrated. The transition frequency is given by

$$\omega_0 = \frac{\kappa M E_m}{\eta d^2}, \tag{30}$$

where $E_m = K_m + 4\mu/3$ and d is the period of the stratification.

The medium behaves as an equivalent (or effective) TI viscoelastic medium with complex and frequency-dependent stiffnesses, p_{IJ}, $I, J = 1, \ldots, 6$. Denoting by σ_{ij} the stress tensor of the equivalent TI medium, the corresponding stress-strain relations, stated in the space-frequency domain, are (Carcione, 2007)

$$\sigma_{11}(u) = p_{11}\,\epsilon_{11}(u^s) + p_{12}\,\epsilon_{22}(u^s) + p_{13}\,\epsilon_{33}(u^s) - B_6\zeta, \tag{31}$$
$$\sigma_{22}(u) = p_{12}\,\epsilon_{11}(u^s) + p_{11}\,\epsilon_{22}(u^s) + p_{13}\,\epsilon_{33}(u^s) - B_6\zeta, \tag{32}$$
$$\sigma_{33}(u) = p_{13}\,\epsilon_{11}(u^s) + p_{13}\,\epsilon_{22}(u^s) + p_{33}\,\epsilon_{33}(u^s) - B_7\zeta, \tag{33}$$
$$\sigma_{23}(u) = 2\,p_{55}\,\epsilon_{23}(u^s), \tag{34}$$
$$\sigma_{13}(u) = 2\,p_{55}\,\epsilon_{13}(u^s), \tag{35}$$
$$\sigma_{12}(u) = 2\,p_{66}\,\epsilon_{12}(u^s), \tag{36}$$
$$p_f(u) = -B_6\epsilon_{11}(u^s) - B_6\epsilon_{22}(u^s) - B_7\epsilon_{33}(u^s) + B_8\zeta. \tag{37}$$

In the following, we present a collection of experiments with $\zeta = 0$ and $\zeta \neq 0$, performed on a representative 2D sample of the layered medium to determine the relaxed stifnesses in (31)-(37). We employ the same notation defined for the isotropic case. Thus, we solve Biot's equation (4) as follows.

i) p_{33}: We solve eq (4) in Ω with the following boundary conditions

$$\sigma(u)\nu \cdot \nu = -\Delta P, \quad (x_1, x_3) \in \Gamma^T, \tag{38}$$
$$\sigma(u)\nu \cdot \chi = 0, \quad (x_1, x_3) \in \Gamma^T, \tag{39}$$
$$\sigma(u)\nu \cdot \chi = 0, \quad (x_1, x_3) \in \Gamma^L \cup \Gamma^R, \tag{40}$$
$$u^s \cdot \nu = 0, \quad (x_1, x_3) \in \Gamma^L \cup \Gamma^R, \tag{41}$$
$$u^s = 0, \quad (x_1, x_3) \in \Gamma^B, \tag{42}$$
$$u^f \cdot \nu = 0, \quad (x_1, x_3) \in \Gamma. \tag{43}$$

In this experiment, $\epsilon_{11}(u^s) = \epsilon_{22}(u^s) = \zeta = 0$ and from eq (33) we can determine p_{33} as follows. Denoting by V the original volume of the sample and its (complex) oscillatory volume change by $\Delta V(\omega)$, we have

$$\frac{\Delta V(\omega)}{V} = -\frac{\Delta P}{p_{33}(\omega)}. \tag{44}$$

Computing the average vertical displacement $u_3^{s,T}(\omega)$ at the boundary Γ^T, the volume change produced by the compressibility test can be approximated by $\Delta V(\omega) \approx L u_3^{s,T}(\omega)$, which enable us to compute $p_{33}(\omega)$ by using the relation (44).

ii) p_{11}: The boundary conditions are:

$$\sigma(u)\nu \cdot \nu = -\Delta P, \quad (x_1, x_3) \in \Gamma^R, \tag{45}$$
$$\sigma(u)\nu \cdot \chi = 0, \quad (x_1, x_3) \in \Gamma^R \cup \Gamma^B \cup \Gamma^T, \tag{46}$$
$$u^s \cdot \nu = 0, \quad (x_1, x_3) \in \Gamma^B \cup \Gamma^T, \tag{47}$$
$$u^s = 0, \quad (x_1, x_3) \in \Gamma^L, \tag{48}$$
$$u^f \cdot \nu = 0, \quad (x_1, x_3) \in \Gamma. \tag{49}$$

Now, $\epsilon_{33}(u^s) = \epsilon_{22}(u^s) = \zeta = 0$ and from eq (31) we determines p_{11} in the same way indicated for p_{33}.

iii) p_{55}: The boundary conditions are:

$$-\sigma(u)\nu = g, \quad (x_1, x_3) \in \Gamma^T \cup \Gamma^L \cup \Gamma^R, \tag{50}$$
$$u^s = 0, \quad (x_1, x_3) \in \Gamma^B, \tag{51}$$
$$u^f \cdot \nu = 0, \quad (x_1, x_3) \in \Gamma, \tag{52}$$

where

$$g = \begin{cases} (0, \Delta G), & (x_1, x_3) \in \Gamma^L, \\ (0, -\Delta G), & (x_1, x_3) \in \Gamma^R, \\ (-\Delta G, 0), & (x_1, x_3) \in \Gamma^T. \end{cases}$$

The change in shape of the rock sample allow us to compute $p_{55}(\omega)$ by using the relation

$$\tan[\theta(\omega)] = \frac{\Delta G}{p_{55}(\omega)}, \tag{53}$$

where $\theta(\omega)$ is the angle between the original positions of the lateral boundaries and the location after applying the shear stresses (Kolsky, 1963). The horizontal displacements $u_1^s(x_1, L, \omega)$ at the top boundary Γ^T are used to obtain, for each frequency, an average horizontal displacement $u_1^{s,T}(\omega)$ at the boundary Γ^T. Since $\tan[\theta(\omega)] \approx u_1^{s,T}(\omega)/L$, we obtain $p_{55}(\omega)$ from eq (53).

iv) p_{66}: Since this stiffness is associated with shear waves traveling in the (x_1, x_2)-plane, we rotate the sample 90° and apply the shear test indicated for p_{55}.

v) p_{13}: The boundary conditions are

$$\sigma(u)\nu \cdot \nu = -\Delta P, \quad (x_1, x_3) \in \Gamma^R \cup \Gamma^T, \tag{54}$$
$$\sigma(u)\nu \cdot \chi = 0, \quad (x_1, x_3) \in \Gamma, \tag{55}$$
$$u^s \cdot \nu = 0, \quad (x_1, x_3) \in \Gamma^L \cup \Gamma^B, \tag{56}$$
$$u^f \cdot \nu = 0, \quad (x_1, x_3) \in \Gamma. \tag{57}$$

In this experiment, $\epsilon_{22} = \zeta = 0$, and from eqs (31) and (33) we get

$$\tau_{11} = p_{11}\epsilon_{11} + p_{13}\epsilon_{33}, \tag{58}$$
$$\tau_{33} = p_{13}\epsilon_{11} + p_{33}\epsilon_{33},$$

where ϵ_{11} and ϵ_{33} are the strain components at the right lateral side and top side of the sample, respectively. Then from eq (58) and using $\tau_{11} = \tau_{33} = -\Delta P$ [c.f. eq (54)], we obtain $p_{13}(\omega)$ as

$$p_{13}(\omega) = \frac{p_{11}\epsilon_{11} - p_{33}\epsilon_{33}}{\epsilon_{11} - \epsilon_{33}}. \tag{59}$$

vi) B_7-B_8: We solve (4) with the boundary conditions

$$\sigma(u)\nu\cdot\nu = -\Delta P^b, \quad (x_1,x_3)\in\Gamma^T, \tag{60}$$
$$\sigma(u)\nu\cdot\chi = 0, \quad (x_1,x_3)\in\Gamma^T, \tag{61}$$
$$\sigma(u)\nu\cdot\chi = 0, \quad (x_1,x_3)\in\Gamma^L\cup\Gamma^R, \tag{62}$$
$$u^s\cdot\nu = 0, \quad (x_1,x_3)\in\Gamma^L\cup\Gamma^R, \tag{63}$$
$$u^s = 0, \quad (x_1,x_3)\in\Gamma^B, \tag{64}$$
$$u^f\cdot\nu = 0, \quad (x_1,x_3)\in\Gamma\setminus\Gamma^T \tag{65}$$
$$p_f(u) = \Delta P^f \quad (x_1,x_3)\in\Gamma^T. \tag{66}$$

Then, $\varepsilon_{11}=\varepsilon_{22}=0$, and the volume change of the sample is

$$e = \varepsilon_{33} = \frac{\Delta V^b}{V^b}. \tag{67}$$

As in the isotropic case, we can obtain an average fluid displacement $u_3^{f,T}(\omega)$ across the boundary Γ^T. Then, the change in fluid content ζ per unit bulk volume associated with this compressibility test can be approximated by

$$\zeta = -\frac{\Delta V^f}{V^b} \approx -\frac{L u_3^{f,T}}{V^b}. \tag{68}$$

Hence, using the coefficient $p_{33}(\omega)$ already computed from (31), (31), (60) and (66) we have

$$-\Delta P^b = p_{33}\, e - B_7\,\zeta, \tag{69}$$
$$\Delta P^f = -B_7\, e + B_8\,\zeta., \tag{70}$$

where B_7 and B_8 can be determined.

vii) B_6: The boundary conditions are

$$\sigma(u)\nu\cdot\nu = -\Delta P^b, \quad (x_1,x_3)\in\Gamma^R, \tag{71}$$
$$\sigma(u)\nu\cdot\chi = 0, \quad (x_1,x_3)\in\Gamma^R\cup\Gamma^B\cup\Gamma^T, \tag{72}$$
$$u^s\cdot\nu = 0, \quad (x_1,x_3)\in\Gamma^B\cup\Gamma^T, \tag{73}$$
$$u^s = 0, \quad (x_1,x_3)\in\Gamma^L, \tag{74}$$
$$u^f\cdot\nu = 0, \quad (x_1,x_3)\in\Gamma\setminus\Gamma^R, \tag{75}$$
$$p_f(u) = \Delta P^f \quad (x_1,x_3)\in\Gamma^R. \tag{76}$$

In this experiment, $\epsilon_{33}(u^s)=\epsilon_{22}(u^s)=0$. Since B_8 has already be determined from the previous experiment, we use $e=\varepsilon_{11}$ and compute ζ by measuring the average fluid displacement $u_1^{f,R}(\omega)$ across the boundary Γ^R as indicated in (19). Then, from (37) and (75) we have

$$B_6 = \frac{B_8\zeta - \Delta P^f}{e}. \tag{77}$$

5 A WEAK FORMULATION OF THE EQUATION OF MOTION

Let us analyze in detail the isotropic case. The transversely isotropic case can be treated in a similar fashion. In order to state a variational formulation for (4) and either (5)-(9) or

(11)-(17), we need to introduce some notation. For $X \subset \mathbb{R}^d$ with boundary ∂X, let $(\cdot,\cdot)_X$ and $\langle\cdot,\cdot\rangle_{\partial X}$ denote the complex $L^2(X)$ and $L^2(\partial X)$ inner products for scalar, vector, or matrix valued functions. Also, for $s \in \mathbb{R}$, $\|\cdot\|_{s,X}$ and $|\cdot|_{s,X}$ will denote the usual norm and seminorm for the Sobolev space $H^s(X)$ (Adams, 1975). In addition, if $X = \Omega$ or $X = \Gamma$, the subscript X may be omitted such that $(\cdot,\cdot) = (\cdot,\cdot)_\Omega$ or $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_\Gamma$. Let us introduce the spaces

$$H^1_{0,B}(\Omega) = \{v \in [H^1(\Omega)]^2 : v\cdot\nu = 0 \ \text{ on } \Gamma^L \cup \Gamma^R, v = 0 \ \text{ on } \Gamma^B\},$$

$$H^C(\mathrm{div};\Omega) = \{v \in [L^2(\Omega)]^2 : \ \nabla\cdot v \in L^2(\Omega), v\cdot\nu = 0 \ \text{ on } \Gamma\},$$

and

$$H^O(\mathrm{div};\Omega) = \{v \in [L^2(\Omega)]^2 : \ \nabla\cdot v \in L^2(\Omega), v\cdot\nu = 0 \ \text{ on } \Gamma\setminus\Gamma^T\},$$

Let

$$\mathcal{V}^{(C)} = \left[H^1_{0,B}(\Omega)\right]^2 \times H^C_0(\mathrm{div};\Omega), \quad \mathcal{V}^{(O)} = \left[H^1_{0,B}(\Omega)\right]^2 \times H^O_0(\mathrm{div};\Omega).$$

Then multiply equation (4) by $v = \left(v^s, v^f\right) \in \mathcal{V}^{(C)}$, use integration by parts and apply the boundary conditions (5)-(6) to see that the solution $u^{(C)} = (u^{(s,C)}, u^{(f,C)}) \in \mathcal{V}^{(C)}$ of (4) and (5)-(9) satisfies *the weak form*:

$$\Lambda(u^{(C)}, v) = -\left\langle \Delta P, v^s\cdot\nu\right\rangle_{\Gamma^T}, \qquad \forall v = \left(v^s, v^f\right) \in \mathcal{V}^{(C)}, \tag{78}$$

where for $u = (u^s, u^f), v = (v^s, v^f) \in [H^1(\Omega)]^2 \times H(\mathrm{div};\Omega)$, the bilinear form $\Lambda(u,v)$ is defined by

$$\begin{aligned}\Lambda(u,v) &= \mathrm{i}\omega\left(\frac{\eta}{\kappa}u^f, v^f\right) + \sum_{l,m}\left(\tau_{lm}(u), \varepsilon_{lm}(v^s)\right) - \left(p_f(u), \nabla\cdot v^f\right)\\ &= \mathrm{i}\omega\left(\frac{\eta}{\kappa}u^f, v^f\right) + \left(\mathbf{D}\,\tilde{\epsilon}(u), \tilde{\epsilon}(v)\right).\end{aligned} \tag{79}$$

In (79), the matrix $\mathbf{D}$ and the column vector $\tilde{\varepsilon}((u))$ are defined by

$$\mathbf{D} = \begin{pmatrix} \lambda_c + 2\mu & \lambda_c & B & 0\\ \lambda_c & \lambda_c + 2\mu & B & 0\\ B & B & M & 0\\ 0 & 0 & 0 & 4\mu\end{pmatrix}, \quad \tilde{\varepsilon}(u) = \begin{pmatrix}\varepsilon_{11}(u^s)\\ \varepsilon_{22}(u^s)\\ \nabla\cdot u^f\\ \varepsilon_{12}(u^s)\end{pmatrix}.$$

The term $(\mathbf{D}\,\tilde{\epsilon}(u), \tilde{\epsilon}(v))$ in (79) is associated with the strain energy of our system, so that the matrix $\mathbf{D}$ must be positive definite, with entries satisfying the conditions

$$\mu > 0, \tag{80a}$$

$$K_c - \alpha^2 M = K_m > 0, \tag{80b}$$

$$M > 0. \tag{80c}$$

Proceeding in a similar fashion, the solution $u^{(O)} = (u^{(s,O)}, u^{(f,O)}) \in \mathcal{V}^{(O)}$ of (4) and (11)-(17) satisfies *the weak form*:

$$\Lambda(u^{(O)}, v) = -\left\langle \Delta P^s, v^s\cdot\nu\right\rangle_{\Gamma^T} - \left\langle \Delta P^f, v^f\cdot\nu\right\rangle_{\Gamma^T}, \qquad \forall v = \left(v^s, v^f\right) \in \mathcal{V}^{(O)}. \tag{81}$$

Existence of the solution of the boundary value problem (4) and either (5)-(9) or (11)-(17) will be assumed.

Uniqueness for the solution $u^{(C)}$ of (78) was demonstrated in (Santos et al., 2009); uniqueness of the solution $u^{(O)}$ of (81) follows with a similar argument.

6 THE FINITE-ELEMENT PROCEDURE

Let $\mathcal{T}^h(\Omega)$ be a non-overlapping partition of Ω into rectangles Ω_j of diameter bounded by h such that $\overline{\Omega} = \cup_{j=1}^{J}\overline{\Omega}_j$. To approximate the solid displacement vector, we employ the finite-element (FE) space

$$\mathcal{N}^h_{0,B} = \{v : v|_{\Omega_j} \in P_{1,1} \times P_{1,1},\ v \cdot \nu = 0 \text{ on } \Gamma^L \cup \Gamma^R,\\ v = 0 \text{ on } \Gamma^B\} \cap [C^0(\overline{\Omega})]^2,$$

where $P_{1,1}$ denotes the polynomials of degree not greater than 1 on each variable. The local degreees of freedom (DOF) for $\mathcal{N}^{h,P}_{0,B}$ are the values at the four corners of Ω_j.

To approximate the fluid displacement we employ the following closed subspaces of the vector part of the Raviart-Thomas-Nedelec space of zero order (Raviart and Thomas, 1975; Nedelec, 1980). When $\zeta = 0$, we used the space

$$\mathcal{W}^{h,C} = \{v :\ v|_{\Omega_j} \in P_{1,0} \times P_{0,1},\ v \cdot \nu = 0 \text{ on } \Gamma\},$$

while for $\zeta \neq 0$

$$\mathcal{W}^{h,O} = \{v :\ v|_{\Omega_j} \in P_{1,0} \times P_{0,1},\ v \cdot \nu = 0 \text{ on } \Gamma \setminus \Gamma^T\}.$$

The corresponding local DOF are the values of the normal component of the fluid displacement u^f at the four sides of the rectangles Ω_j.

Let us introduce the FE spaces

$$\mathcal{V}^{(h,C)} = \mathcal{N}^h_{0,B} \times \mathcal{W}^{h,C}, \qquad \mathcal{V}^{(h,O)} = \mathcal{N}^h_{0,B} \times \mathcal{W}^{h,O}.$$

Then the FE procedure to compute the approximate solution of (78) is defined as follows: find $u^{(h,C)} = \left(u^{(s,h,C)}, u^{(f,h,C)}\right) \in \mathcal{V}^{h,C}$ such that

$$\Lambda(u^{(h,C)}, v) = \langle \Delta P, v^s \cdot \nu \rangle_{\Gamma^T}, \qquad v = \left(v^s, v^f\right) \in \mathcal{V}^{(h,C)}. \tag{82}$$

Similarly, the FE procedure to compute the approximate solution of (81) is: find $u^{(h,O)} = \left(u^{(s,h,O)}, u^{(f,h,O)}\right) \in \mathcal{V}^{(h,O)}$ such that

$$\Lambda(u^{(h,O)}, v) = -\left\langle \Delta P^b, v^s \cdot \nu \right\rangle_{\Gamma^T} - \left\langle \Delta P^f, v^f \cdot \nu \right\rangle_{\Gamma^T},\\ v = \left(v^s, v^f\right) \in \mathcal{V}^{(h,O)}. \tag{83}$$

Since $u^{(h,C)} \in \mathcal{V}^{(C)}, u^{(h,O)} \in \mathcal{V}^{(O)}$, uniqueness for the discrete problems (82) and (83) follows with the same argument than for the continuous case provided the frequency ω is sufficiently samll. Existence follows from finite dimensionality.

Assuming that the solution $u = (u^s, u^f)$ of our boundary value problems are such that $u^s \in [H^{3/2}(\Omega)]^2$ and $u^f, \in [H^1(\Omega)]^2, \nabla \cdot u^f \in H^1(\Omega)$, the argument given in (Santos et al., 2009) can be applied here to show that for sufficiently small $h > 0$ the error associated with the FE procedures (82) and (83) is of order h. More specifically, it can be shown that the following apriori error estimate holds

$$\|u^{(s,h,m)} - u^{(s,m)}\|_1 + \|u^{(f,h,m)} - u^{(f,m)}\|_0 + \|\nabla \cdot \left(u^{(f,h,m)} - u^{(f,m)}\right)\|_0\\ \leq C(\omega) h \left(\|u^{(s,m)}\|_{3/2} + \|u^{(f,m)}\|_1 + \|\nabla \cdot u^{(f,j)}\|_1\right), \quad m = C, O. \tag{84}$$

The solution of the eight boundary problems associated with the transversely isotropic case were solved using the FE method employing the same spaces than for the isotropic case and using the corresponding boundary conditions.

7 NUMERICAL EXAMPLES

Let us consider a sample of homogeneous and isotropic sandstone from the North-Sea Utsira formation. Carbon dioxide (CO_2) has been injected in this formation during the last 15 years, displacing brine from the pore space (Carcione et al., 2006). Table 1 gives the properties of the single constituents of the sandstone, corresponding to a depth of 850 m, a pore pressure of 10.7 MPa, a confining pressure of 18 MPa and 37 °C.

Table 1. Properties of the Utsira sandstone.

Grain bulk modulus, K_s (GPa)	40
Frame bulk modulus, K_m (GPa)	1.37
shear modulus, μ_m (GPa)	0.82
porosity, ϕ	0.36
permeability, κ (D)	1.6
Brine viscosity, η_w (Pa s)	0.0012
bulk modulus, K_w (GPa)	2.6
CO_2 viscosity, η_g (Pa s)	0.00015
bulk modulus, K_g (MPa)	25

We show results of the numerical experiments at frequencies $f = 10^{-4}$ Hz and $f = 10^{-6}$ Hz on a square sample of side length 50 cm using a mesh size of 50 $\times$ 50. Identical results are obtained with finer meshes. Experiments with $\zeta = 0$ use $\Delta P = 1$, otherwise $\Delta P^b = \Delta P^f = 1$ and for this experiment we also run tests by setting $\Delta P^b = 1, \Delta P^f = 0$ and $\Delta P^b = 0, \Delta P^f = 1$ with identical estimates for the coefficients. Table 2 shows the real part (the corresponding imaginary parts are negligible) of the computed double-precision numerical results for brine and CO_2 saturating the Utsira sandstone, compared to the actual poroelasticity values.

Table 2. Poroelasticity coefficients. Homogeneous isotropic case

Brine	Exact	Numerical (10^{-4} Hz)	Numerical (10^{-6} Hz)
E_c (GPa)	8.53521452857964	8.53521452857994	8.53521452857968
B (GPa)	6.28721842634875	6.28721842634815	6.28721842634840
α	0.96575000000000	0.96574999994000	0.96574999999999
M (GPa)	6.51019252016376	6.51019252032828	6.51019252016439
Carbon dioxide			
E_c (GPa)	2.52803425332936	2.52803425332967	2.52803425332936
B (GPa)	0.0669955164338892	0.0669955164338744	0.06699551643388
α	0.965750000000	0.965749994174	0.96574999999954
M (GPa)	0.0693714899651972	0.0693714899651819	0.06937148996519

Now, we consider a periodic sequence of thin layers of CO_2-saturated Utsira sandstone and brine-saturated mudstone. The mudstone has K_s = 20 GPa, ρ_s = 2600 K_g/m^3, K_m = 7 GPa, μ_m= 6 GPa and κ = 0.01 D. The numerical experiments were applied on a square sample containing ten periods of the sandstone-mudstone sequence using a mesh size of 60 $\times$ 60. The frequency is $f = 10^{-8}$ Hz. The comparison to the coefficients predicted in (Gelinsky and Shapiro, 1997) (see appendix) are shown in Table 3.

Table 3. Stiffnesses coefficients (GPa). TI case

	Exact	Numerical (10^{-8} Hz)
p_{33}	8.38837046932193	8.38837046932529
p_{11}	12.9764285908037	12.9407760304821
p_{13}	2.04418544762456	2.03235989198574
p_{55}	2.92277227722772	2.9227722772189
p_{66}	5.13666666666667	5.19651777935068
B_6	0.290783756251698	0.290809523149004
B_7	0.330886200120806	0.330886200449567
B_8	0.401861752837481	0.401861752835741

There are several advantages of the presented procedure. One main advantage is that allows for the first time to determine the four coefficients in BiotÅŻ constitutive relations that so far were determined using the so call *gedanken* closed, jacketed and unjacketed compressibility experiments (Gassmann, 1951; Biot and Willis, 1957) on homogeneous samples. Another advantage is that this numerical experiments can be applied to highly heterogeneous poroelastic materials, in which case and considering only the closed-pore experiments we would get coefficients associated with an homogeneous isotropic or transversely isotropic medium equivalent to the original fluid-saturated material (Santos et al., 2009). Besides, from these constants, other properties of the medium, such as the porosity and the bulk modulus of the rock matrix, can be obtained.

8 CONCLUSIONS

We have defined a set of numerical experiments representing quasi-static compressibility tests to determine the coefficients of poroelastic media saturated by a single-phase fluid. The experiments are based on a finite-element solution of Biot's equation in the diffusive range, with suitable boundary conditions. The technique has been applied to homogeneous isotropic media and a sequence of thin poroelastic layers, which is described by an equivalent transversely isotropic medium at long wavelengths. The poroelasticity coefficients were determined with high accuracy for the North-Sea Utsira sandstone saturated with either brine and CO_2, and a periodic sequence of brine-saturated mudstone and CO_2-saturated sandstone thin layers. The methodology can be extended to more general situations, such as anisotropic and/or fractured Biot media, that will be the subject of forthcoming research.

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10 APPENDIX

According to (Gelinsky and Shapiro, 1997)[their eq. (14)], the quasi-static (relaxed) effective constants of a stack of thin poroelastic layers are

$$
\begin{aligned}
p_{66} &= B_1 = \langle \mu \rangle, \\
p_{11} - 2p_{66} &= p_{12} = B_2 = 2\left\langle \frac{\lambda_m \mu}{E_m} \right\rangle + \left\langle \frac{\lambda_m}{E_m} \right\rangle^2 \left\langle \frac{1}{E_m} \right\rangle^{-1} \\
&\qquad + \frac{{B_6}^2}{B_8}, \\
p_{13} &= B_3 = \left\langle \frac{\lambda_m}{E_m} \right\rangle \left\langle \frac{1}{E_m} \right\rangle^{-1} + \frac{B_6 B_7}{B_8}, \\
p_{33} &= B_4 = \left\langle \frac{1}{E_m} \right\rangle^{-1} + \frac{{B_7}^2}{B_8} \\
&= \left[\left\langle \frac{1}{E_m} \right\rangle - \left\langle \frac{\alpha}{E_m} \right\rangle^2 \left\langle \frac{E_G}{M E_m} \right\rangle^{-1} \right]^{-1}, \\
p_{55} &= B_5 = \langle \mu^{-1} \rangle^{-1}, \\
B_6 &= -B_8 \left(2\left\langle \frac{\alpha\mu}{E_m} \right\rangle + \left\langle \frac{\alpha}{E_m} \right\rangle \left\langle \frac{\lambda_m}{E_m} \right\rangle \left\langle \frac{1}{E_m} \right\rangle^{-1} \right), \\
B_7 &= -B_8 \left\langle \frac{\alpha}{E_m} \right\rangle \left\langle \frac{1}{E_m} \right\rangle^{-1}, \\
B_8 &= \left[\left\langle \frac{1}{M} \right\rangle + \left\langle \frac{\alpha^2}{E_m} \right\rangle - \left\langle \frac{\alpha}{E_m} \right\rangle^2 \left\langle \frac{1}{E_m} \right\rangle^{-1} \right]^{-1},
\end{aligned}
\tag{85}
$$

where

$$
\lambda_m = K_m - \frac{2}{3}\mu, \quad E_m = K_m + \frac{4}{3}\mu, \quad E_G = E_m + \alpha^2 M \tag{86}
$$

and $\langle \cdot \rangle$ denotes the spatial weighted average.

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