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A GEOMETRICALLY EXACT COMPOSITE THIN-WALLED BEAM ELEMENT FOR FLEXIBLE MULTIBODY DYNAMICS

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Abstract. A geometrically exact thin-walled composite beam finite element for multibody applications is presented. In the proposed formulation the virtual work equations are written as a function of generalized strains, which are parametrized in terms of the director field and its derivatives. Finite rotations are parametrized with the total rotation vector. The derivatives of the director field are obtained via interpolation, thus simplifying the linearization of the virtual strains. The material constitutive relations are based on the mechanics of composite laminates. The formulation of typical joints is briefly presented. The finite element is implemented in a multibody algorithm that uses the generalized-alpha method to integrate in time the differential-algebraic system of equations.

1 INTRODUCTION

Thin-walled composite beams are widely used in different areas of engineering. In the last years, most research efforts regarding beam formulation have been directed to the analysis of mechanism analysis. With the recent advances in the development of flexible multibody algorithms, several multibody beam finite element formulations have been proposed. Most of these beam formulations assume an isotropic constitutive law. Often, this constitutive law is not capable of accurately describing the real behavior of some slender structures such as helicopter rotor blades, aircraft wings and wind turbine blades, which are built of composite materials. This situation motivated the present work, where we present a composite thin walled beam formulation for flexible multibody applications.

The analysis of flexible multibody beams generally involves a deep knowledge of tridimensional finite rotations. The non-vectorial nature of finite rotations introduces a great complexity to the finite element formulation; several approaches have been proposed in the literature to address this problem. The introduction of geometrically exact beam finite element formulations can be traced back to the works of Simo (Simo 1985) and Cardona (Cardona and Geradin 1988). After these pioneering works, several authors have addressed the problem of geometrically exact beams (Simo and Vu-Quoc 1988; Ibrahimbegovic 1995; Ibrahimbegović, Frey et al. 1995; Crisfield 1997; Ibrahimbegovic and Al Mikdad 1998; Jelenic and Crisfield 1999; Armero and Romero 2001; Betsch and Steinmann 2002; Ritto-Corrêa and Camotim 2002; Saravia, Machado et al. 2010).

The development of finite element algorithms for flexible multibody applications started in the early nineties with the work of Cardona et. al. (Cardona, Geradin et al. 1991). New approaches were quickly developed (Ibrahimbegovic and Mamouri 2000; Ibrahimbegovic, Taylor et al. 2003) and the different successful implementations pushed the subject from the scientific to the technological level. A few geometrically exact composite thin-walled beam formulations for multibody applications have been reported in the literature. Most of these formulations are based on the VAM (Variational Asymptotic Method) approach and are due to the group of Prof. Hodges (Hodges 1990; Cesnik and Hodges 1997; Yu, Liao et al. 2005; Hodges 2006).

We present in this work a geometrically exact finite element formulation for multibody applications. The present formulation is based on a composite thin-walled beam theory that includes transverse shear effects. In contrast to most geometrically exact beam finite elements, the virtual work equations are parametrized in terms of the director field and its derivatives. This greatly simplifies the expression of the Green-Lagrange strain tensor since no rotational variables appear in its expression. The total rotation vector is used to update the director field; the derivatives of the directors are obtained through interpolation.

2 THIN WALLED BEAM THEORY

2.1 Kinematics

The kinematic description of the thin-walled beam relates two states of a beam, an undeformed reference state $\mathcal{B}_0$, and a deformed state $\mathcal{B}$. We associate to $\mathcal{B}_0$ a material frame $\boldsymbol{E}_i$ and to $\mathcal{B}$ a spatial (floating) frame $\boldsymbol{e}_i$, both frames being orthonormal and coincident at time $t = 0$. The absolute displacements that occur during finite deformation are measured by a vector $\boldsymbol{U} = (u_1, u_2, u_3)$. The relation between the orthonormal frames is given by the linear transformation:

$$\boldsymbol{e}_i = \boldsymbol{\Lambda}\big(\boldsymbol{\theta}(x,t)\big)\boldsymbol{E}_i, \tag{1}$$

where $\boldsymbol{\Lambda}\big(\boldsymbol{\theta}(x)\big)$ is the total rotation tensor (a two-point tensor field $\in \mathrm{SO}(3)$; the special orthogonal Lie group) and $\boldsymbol{\theta}$ is the total Cartesian rotation vector.
Using Eq. (1) we can write the position vectors of a point in the beam in the reference and current configuration respectively as:

$$\boldsymbol{X}(x,\xi_2,\xi_3) = \boldsymbol{X}_0(x) + \sum_{i=2}^{3} \xi_i\, \boldsymbol{E}_i, \qquad \boldsymbol{x}(x,\xi_2,\xi_3,t) = \boldsymbol{x}_0(x,t) + \sum_{i=2}^{3} \xi_i\, \boldsymbol{e}_i. \tag{2}$$

In these equations the first term stands for the position a reference point and the second term stands for the position a point in the cross section relative to the reference point. In this work we set the centroid to be the reference point. We can also express the spatial position vector as:

$$\boldsymbol{x}(x,\xi_2,\xi_3,t) = \boldsymbol{x}_0(x,t) + \boldsymbol{\Lambda}\big(\boldsymbol{\theta}(x,t)\big)\, \boldsymbol{\xi} \tag{3}$$

where $\boldsymbol{\xi} = \sum_{i=2}^{3} \xi_i\, \boldsymbol{E}_i$ is the material position vector of a point with respect to the centroid**.** Note that, x is the running length coordinate and ξ_2 and ξ_3 are cross section coordinates. Also, the displacement field is:

$$\boldsymbol{U}(x,\xi_2,\xi_3,t) = \boldsymbol{x} - \boldsymbol{X} = \boldsymbol{u} + (\boldsymbol{\Lambda}(\boldsymbol{\theta}) \ - \mathbf{I})\, \boldsymbol{\xi}, \tag{4}$$

where $\boldsymbol{u}$ represents the displacement of the centroid. The nonlinear manifold of 3D rotation transformations $\boldsymbol{\Lambda}(\boldsymbol{\theta})$ (belonging to the special orthogonal Lie Group SO(3)) is described mathematically via the exponential map (Argyris 1982; Cardona and Geradin 1988). The set of kinematic variables is defined by three displacements and three rotations as:

$$\mathcal{V} \coloneqq \{\boldsymbol{\phi} = [\boldsymbol{u},\boldsymbol{\theta}]^T \colon [0,\ell] \to R^3\}, \qquad [\boldsymbol{u},\boldsymbol{\theta}]^T = [u_1,u_2,u_3,\theta_1,\theta_2,\theta_3]^T. \tag{5}$$

2.2 Strain Field

In order to obtain the expression of the Green-Lagrange strains we first obtain the derivatives of the position vectors of the undeformed and deformed configurations as:

$$\begin{gathered}\boldsymbol{X}_{,1} = \boldsymbol{X}_0' + \xi_2 \boldsymbol{E}_2' + \xi_3 \boldsymbol{E}_3', \quad \boldsymbol{X}_{,2} = \boldsymbol{E}_2, \quad \boldsymbol{X}_{,3} = \boldsymbol{E}_3, \\ \boldsymbol{x}_{,1} = \boldsymbol{x}_0' + \xi_2 \boldsymbol{e}_2' + \xi_3 \boldsymbol{e}_3', \quad \boldsymbol{x}_{,2} = \boldsymbol{e}_2, \quad \boldsymbol{x}_{,3} = \boldsymbol{e}_3.\end{gathered} \tag{6}$$

Injecting these vectors into the GL strain $\boldsymbol{E}_{GL} = \frac{1}{2}\big(\boldsymbol{x}_{,i}\cdot\boldsymbol{x}_{,j} - \boldsymbol{X}_{,i}\cdot\boldsymbol{X}_{,j}\big)$ (Bonet 1997) we obtain three non-vanishing components, in vector notation: $\boldsymbol{E}_{GL} = [E_{11} \;\; 2E_{12} \;\; 2E_{13}]^T$. Note that the existence of transverse shear strains implies $\boldsymbol{e}_1 \cdot \boldsymbol{x}_{,1} > 0$.

We can write the GL strain as:

$$\boldsymbol{E}_{GL} = \boldsymbol{D}\, \boldsymbol{\varepsilon}, \tag{7}$$

Where we have introduced a generalized strain vector such that:

$$\boldsymbol{D}=\begin{bmatrix}1 & \xi_3 & \xi_2 & 0 & 0 & 0 & \frac{1}{2}\xi_2{}^2 & \frac{1}{2}\xi_3{}^2 & \xi_2\xi_3\\ 0 & 0 & 0 & 1 & 0 & -\xi_3 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & \xi_2 & 0 & 0 & 0\end{bmatrix},\quad \boldsymbol{\varepsilon}=\begin{bmatrix}\epsilon\\ \kappa_2\\ \kappa_3\\ \gamma_2\\ \gamma_3\\ \kappa_1\\ \chi_2\\ \chi_3\\ \chi_{23}\end{bmatrix}=\begin{bmatrix}\frac{1}{2}(\boldsymbol{x}_0'\cdot\boldsymbol{x}_0'-\boldsymbol{X}_0'\cdot\boldsymbol{X}_0')\\ \boldsymbol{x}_0'\cdot\boldsymbol{e}_3'-\boldsymbol{X}_0'\cdot\boldsymbol{E}_3'\\ \boldsymbol{x}_0'\cdot\boldsymbol{e}_2'-\boldsymbol{X}_0'\cdot\boldsymbol{E}_2'\\ \boldsymbol{x}_0'\cdot\boldsymbol{e}_2-\boldsymbol{X}_0'\cdot\boldsymbol{E}_2\\ \boldsymbol{x}_0'\cdot\boldsymbol{e}_3-\boldsymbol{X}_0'\cdot\boldsymbol{E}_3\\ \boldsymbol{e}_2'\cdot\boldsymbol{e}_3-\boldsymbol{E}_2'\cdot\boldsymbol{E}_3\\ \boldsymbol{e}_2'\cdot\boldsymbol{e}_2'-\boldsymbol{E}_2'\cdot\boldsymbol{E}_2'\\ \boldsymbol{e}_3'\cdot\boldsymbol{e}_3'-\boldsymbol{E}_3'\cdot\boldsymbol{E}_3'\\ \boldsymbol{e}_2'\cdot\boldsymbol{e}_3'-\boldsymbol{E}_2'\cdot\boldsymbol{E}_3'\end{bmatrix} \tag{8}$$

The generalized beam strains belong to a material description and are expressed in a rectangular coordinate system.

Now we introduce a curvilinear coordinate system (x, n, s) and transform the GL strains to this coordinate system. The cross-section shape will be defined in this coordinate system by functions $\xi_i(n, s)$. The coordinate s is measured along the tangent to the middle line of the cross section, in clockwise direction and with origin conveniently chosen. Also, the thickness coordinate $n(-e/2 \leq e/2)$ is perpendicular to s and with origin in the middle line contour. To represent the GL strains in this curvilinear coordinate system we make use of a curvilinear transformation tensor $\boldsymbol{P}$ (Saravia, Machado et al. 2011). Hence, the GL strain vector in the curvilinear coordinate system is obtained by transforming the rectangular GL strains as:

$$\widehat{\boldsymbol{E}}_{GL}=[E_{xx}\quad 2E_{xs}\quad 2E_{xn}]^T=\boldsymbol{P}\,\boldsymbol{E}_{GL}=\boldsymbol{P}\,\boldsymbol{D}\,\boldsymbol{\varepsilon}, \tag{9}$$

The GL strain vector in curvilinear coordinates has a remarkably simple closed expression:

$$\widehat{\boldsymbol{E}}_{GL}=\begin{bmatrix}\epsilon+\xi_2\kappa_3+\xi_3\kappa_2+\frac{1}{2}\xi_2{}^2\chi_2+\frac{1}{2}\xi_3{}^2\chi_3+\xi_2\xi_3\chi_{23}\\ \bar{\xi}_2'\gamma_2+\bar{\xi}_3'\gamma_3+\left(\xi_2\bar{\xi}_3'-\xi_3\bar{\xi}_2'\right)\kappa_1\\ -\bar{\xi}_3'\gamma_2+\bar{\xi}_2'\gamma_3+\left(\xi_2\bar{\xi}_2'+\xi_3\bar{\xi}_3'\right)\kappa_1\end{bmatrix}, \tag{10}$$

where the prime symbol has been used to denote derivation with respect to the *s* coordinate.

The location of a point anywhere in the cross-section can be expressed as:

$$\xi_2(n,s)=\bar{\xi}_2(s)-n\frac{d\bar{\xi}_3}{ds},\qquad \xi_3(n,s)=\bar{\xi}_3(s)+n\frac{d\bar{\xi}_2}{ds}, \tag{11}$$

where ξ_i locates a point anywhere in the cross section and $\bar{\xi}_i$ locates the points lying in the middle-line contour.

The strain state of the composite laminate (see (Barbero 2008)) will be described by a shell strain vector:

$$\boldsymbol{\epsilon}_s=[\varepsilon_{xx}\quad \gamma_{xs}\quad \gamma_{xn}\quad \varkappa_{xx}\quad \varkappa_{xs}]^T. \tag{12}$$

We now introduce Eq. (11) into Eq. (10) to express the GL strains as a function of the mid-surface coordinates $\bar{\xi}_i$ and its derivatives, we find that a matrix $\boldsymbol{\mathcal{T}}$ establish the relationship between the GL curvilinear strains and the generalized strains as:

$$\boldsymbol{\epsilon}_s=\boldsymbol{\mathcal{T}}\,\boldsymbol{\varepsilon}. \tag{13}$$

Substituting Eq. (11) into Eq. (10) and neglecting higher order terms in the thickness (terms in n^2) we obtain:

$$\boldsymbol{\mathcal{T}}(s) = \begin{bmatrix} 1 & \bar{\xi}_3 & \bar{\xi}_2 & 0 & 0 & 0 & \frac{1}{2}\bar{\xi}_2^{\;2} & \frac{1}{2}\bar{\xi}_3^{\;2} & \bar{\xi}_2\bar{\xi}_3 \\ 0 & 0 & 0 & \bar{\xi}_2' & \bar{\xi}_3' & \bar{\xi}_2\bar{\xi}_3' - \bar{\xi}_3\bar{\xi}_2' & 0 & 0 & 0 \\ 0 & 0 & 0 & -\bar{\xi}_3' & \bar{\xi}_2' & \bar{\xi}_2\bar{\xi}_2' + \bar{\xi}_3\bar{\xi}_3' & 0 & 0 & 0 \\ 0 & \bar{\xi}_2' & -\bar{\xi}_3' & 0 & 0 & 0 & -\bar{\xi}_2\bar{\xi}_3' & \bar{\xi}_3\bar{\xi}_2' & \left(\bar{\xi}_2\bar{\xi}_2' - \bar{\xi}_3\bar{\xi}_3'\right) \\ 0 & 0 & 0 & 0 & 0 & -\left(\bar{\xi}_2'^{\,2} + \bar{\xi}_3'^{\,2}\right) & 0 & 0 & 0 \end{bmatrix}. \tag{14}$$

It's interesting to note that the matrix $\boldsymbol{\mathcal{T}}$ plays the role of a double transformation matrix that directly maps the generalized strains $\boldsymbol{\varepsilon}$ into the curvilinear GL strain $\boldsymbol{\epsilon}_s$ without the need of an intermediate transformation.

2.3 Constitutive Relations

As mentioned before, the present formulation can handle composite materials in a geometrically exact framework without modifying the classical thin-walled beam approach. We have chosen the GL strain as a measure of strain; this implies that we must use a material stress tensor, the second Piola-Kirchhoff stress tensor $\boldsymbol{\sigma}$, as work conjugate variable. For an orthotropic lamina, the relationship between $\boldsymbol{\sigma}$ and the GL strain tensor, can be expressed in curvilinear coordinates as a matrix of stiffness coefficients Q_{ij} (Jones 1999; Barbero 2008):

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{ss} \\ \sigma_{nn} \\ \sigma_{sn} \\ \sigma_{xn} \\ \sigma_{xs} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & Q_{16} \\ Q_{12} & Q_{22} & Q_{23} & 0 & 0 & Q_{26} \\ Q_{13} & Q_{23} & Q_{33} & 0 & 0 & Q_{36} \\ 0 & 0 & 0 & Q_{44} & Q_{45} & 0 \\ 0 & 0 & 0 & Q_{45} & Q_{55} & 0 \\ Q_{16} & Q_{26} & Q_{36} & 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{ss} \\ \epsilon_{nn} \\ \gamma_{sn} \\ \gamma_{xn} \\ \gamma_{xs} \end{bmatrix}. \tag{15}$$

In matrix form:

$$\boldsymbol{\sigma} = \boldsymbol{Q}\,\boldsymbol{\epsilon}_s. \tag{16}$$

In the above equation Q_{ij} are components of the *transformed constitutive (or stiffness) matrix* defined in terms of the elastic properties (elasticity moduli and Poisson coefficients) and fiber orientation of the ply (Barbero 2008).
The shell stress resultants in a lamina result from the integration of stresses in the thickness, and are thus defined as:

$$N_{ij} = \int_{-e/2}^{e/2} \sigma_{ij}\, dn\ , \qquad M_{ij} = \int_{-e/2}^{e/2} \sigma_{ij}\, n\, dn. \tag{17}$$

Employing Eqs. (15) and (17) and neglecting the normal stress in the thickness (i.e. $\sigma_{nn} = 0$) it is possible to obtain a constitutive relation between the shell forces and strains as:

$$\begin{bmatrix} N_{xx} \\ N_{ss} \\ N_{xs} \\ N_{sn} \\ N_{xn} \\ M_{xx} \\ M_{ss} \\ M_{xs} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & 0 & 0 & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{23} & 0 & 0 & B_{12} & B_{22} & B_{26} \\ A_{13} & A_{23} & A_{33} & 0 & 0 & B_{16} & B_{26} & B_{66} \\ 0 & 0 & 0 & A_{44}^H & A_{45}^H & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{45}^H & A_{55}^H & 0 & 0 & 0 \\ B_{11} & B_{12} & B_{16} & 0 & 0 & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & 0 & 0 & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & 0 & 0 & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{ss} \\ \gamma_{xs} \\ \gamma_{sn} \\ \gamma_{xn} \\ \kappa_{xx} \\ \kappa_{ss} \\ \kappa_{xs} \end{bmatrix}, \tag{18}$$

where: N_{xx}, N_{ss}, and N_{xs} are axial, hoop and shear-membrane shell forces and N_{xn} and N_{sn} are transverse shear shell forces. Also; M_{xx}, M_{ss} and M_{xs} are axial bending, hoop bending and twisting shell moments, respectively. The same nomenclature is extended to the shell strain resultants, thus: ε_{xx} and ε_{ss} are axial and hoop normal strains, γ_{xs}, γ_{sn} and γ_{xn} are shear shell strains and $\varkappa_{xx}$, $\varkappa_{ss}$ and $\varkappa_{xs}$ are axial, hoop and twisting curvatures respectively. The coefficients A_{ij}, A_{ij}^H , B_{ij} and D_{ij} in the constitutive matrix are shell stiffness-coefficients that result from the integration of Q_{ij} in the thickness (Barbero 2008).

Although the last relationships were derived for a single lamina, we can obtain the constitutive relations for a laminate by spanning the integrals in the thickness of the lamina over the different layers of the laminate (each layer being a single lamina). Therefore, using the hypotheses of plane stress in the laminate and rigid cross section (it can be seen that according to this hypothesis $\varepsilon_{ss} = \gamma_{ns} = 0$, but in order to avoid overstiffening effects we set $N_{ss} = \gamma_{ns} = 0$ (Barbero 2008), thus generating a mild inconsistency typical of thin-walled beam formulations), the relations (18) simplify to:

$$\begin{bmatrix} N_{xx} \\ N_{xs} \\ N_{xn} \\ M_{xx} \\ M_{xs} \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{16} & 0 & \bar{B}_{11} & \bar{B}_{16} \\ \bar{A}_{16} & \bar{A}_{66} & 0 & \bar{B}_{16} & \bar{B}_{66} \\ 0 & 0 & \bar{A}_{55}^H & 0 & 0 \\ \bar{B}_{11} & \bar{B}_{16} & 0 & \bar{D}_{11} & \bar{D}_{16} \\ \bar{B}_{16} & \bar{B}_{66} & 0 & \bar{D}_{16} & \bar{D}_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \gamma_{xs} \\ \gamma_{xn} \\ \varkappa_{xx} \\ \varkappa_{xs} \end{bmatrix}, \tag{19}$$

where $\bar{A}_{\mathrm{ij}}$ are components of the laminate reduced in-plane stiffness matrix, $\bar{B}_{\mathrm{ij}}$ are components of the reduced bending-extension coupling matrix, $\bar{D}_{\mathrm{ij}}$ are components of the reduced bending stiffness matrix and $\bar{A}_{55}^H$ is the component of the reduced transverse shear stiffness matrix.

We can express the above relation in matrix form as:

$$\boldsymbol{N}_s = \boldsymbol{C}\, \boldsymbol{\epsilon}_s, \tag{20}$$

where $\boldsymbol{C}$ is the composite shell constitutive matrix and $\boldsymbol{\epsilon}_s$ is the curvilinear shell strain vector defined in Eq. (13).

2.4 Beam Forces

In order to reduce the 2D formulation to a 1D formulation we need to express the shell forces as a function of the generalized strains. Replacing Eq. (13) into Eq. (20) we obtain;

$$\boldsymbol{N}_s = \boldsymbol{C}\, \boldsymbol{\mathcal{T}}\, \boldsymbol{\varepsilon}. \tag{21}$$

Now, we transform the shell forces in Eq. (21) back to the "generalized space" by using the double transformation matrix $\boldsymbol{\mathcal{T}}$. Hence, we obtain the transformed back shell strain as:

$$\boldsymbol{N}_s^G = \boldsymbol{\mathcal{T}}^T \boldsymbol{N}_s = \boldsymbol{\mathcal{T}}^T \boldsymbol{C}\, \boldsymbol{\mathcal{T}}\, \boldsymbol{\varepsilon}. \tag{22}$$

We see that $\boldsymbol{N}_s^G$ is a vector of generalized shell stresses defined in the global coordinate system. It is a function of the cross section mid-contour and thus integration over the contour gives the vector of *generalized beam forces* (work conjugate with the generalized strains) as:

$$\boldsymbol{S}(x) = \int_S \boldsymbol{N}_s^G ds \;= \left(\int_S \boldsymbol{\mathcal{T}}^T \boldsymbol{C}\, \boldsymbol{\mathcal{T}}\, ds \right) \boldsymbol{\varepsilon}(x) = \mathbb{D}\, \boldsymbol{\varepsilon}(x), \tag{23}$$

$$\mathbb{D} = \int_S \boldsymbol{\mathcal{T}}^T \boldsymbol{C}\, \boldsymbol{\mathcal{T}}\, ds. \tag{24}$$

It's interesting to note that $\mathbb{D}$ contains the functions $\bar{\xi}_i$ that define the cross section mid-contour and also all the anisotropic material constants. Besides, it contains not only all geometrical couplings but also all material couplings.

The beam constitutive matrix $\mathbb{D}$ is obtained in a closed form and thus it does not involve a 2D finite element analysis of the cross section (as, for example, in the VABS approach (Cesnik and Hodges 1997)). Although the constitutive constants are not as accurate that the ones obtained with the latter method, the present approach is simpler, faster and it also opens the possibility of addressing optimization problems of large deformation of thin-walled composite beams.

3 VARIATIONAL FORMULATION

The weak form of equilibrium of a three dimensional body $\mathcal{B}$ is given by (Washizu 1968; Zienkiewicz 2000):

$$G(\boldsymbol{\phi}, \delta\boldsymbol{\phi}) = \int_{\mathcal{B}_0} \boldsymbol{\sigma} \cdot \delta\boldsymbol{\epsilon}\, dV - \int_{\mathcal{B}_0} \boldsymbol{\rho_0 b} \cdot \delta\boldsymbol{\phi}\, dV - \int_{\partial\mathcal{B}_0} (\boldsymbol{p} \cdot \delta\boldsymbol{u} + \boldsymbol{m} \cdot \delta\boldsymbol{\theta})\, d\Omega, \tag{25}$$

where $\boldsymbol{b}$**,** $\boldsymbol{p}$ and $\boldsymbol{m}$ are: body forces, prescribed external forces and prescribed external moments per unit length respectively. $\boldsymbol{\epsilon}$ is the GL strain tensor, work conjugate to the second Piola-Kirchhoff stress tensor $\boldsymbol{\sigma}$. Where $\boldsymbol{\sigma}$ could be defined in either a rectangular or a curvilinear coordinate system (such a distinction is, at least here, unnecessary).

3.1 Variations of the director field

The admissible variation of the director field is required to obtain the variation of the generalized strains. From Eq. (1), we can write:

$$\delta\boldsymbol{e}_i = \delta(\boldsymbol{\Lambda}(x)\boldsymbol{E}_i) = \delta\boldsymbol{\Lambda}(x)\boldsymbol{E}_i. \tag{26}$$

The admissible variation of the rotation tensor (Lie variation) is obtained by superposing an infinitesimal virtual rotation onto the existing finite rotation, see e.g. (Betsch 1998; Ritto-Corrêa and Camotim 2002). This virtual rotation can belong to a material vector space or a spatial vector space, they will be called $\delta\boldsymbol{\Theta}$ and $\delta\boldsymbol{w}$ respectively. It's interesting to note that both virtual rotations are elements of the tangent space at $\boldsymbol{\Lambda}$, i.e. $T_{\boldsymbol{\Lambda}}SO(3)$, $\delta\boldsymbol{\Theta} \in T_{\boldsymbol{\Lambda}}^{mat}$ and $\delta\boldsymbol{w} \in T_{\boldsymbol{\Lambda}}^{spat}$. Both virtual rotation vectors are often called *spins*.

Considering the latter we can construct a perturbed rotation tensor by using either the spatial or the material form of compound rotation as:

$$\boldsymbol{\Lambda}_\epsilon = exp(\epsilon\, \delta\widetilde{\boldsymbol{w}})\boldsymbol{\Lambda} = \boldsymbol{\Lambda}\, exp\big(\epsilon\, \delta\widetilde{\boldsymbol{\Theta}}\big). \tag{27}$$

where $\sim$ indicates the skew symmetric matrix of a vector $\boldsymbol{b}$ such that $\widetilde{\boldsymbol{b}}\,\boldsymbol{a} = \boldsymbol{b} \times \boldsymbol{a}$. Now, by making use of the cartesian rotation vector, we can propose:

$$\boldsymbol{\Lambda}_\epsilon = exp\left(\widetilde{\boldsymbol{\theta}} + \epsilon\, \delta\widetilde{\boldsymbol{\theta}}\right), \tag{28}$$

and try to find an incremental rotation tensor $\delta\widetilde{\boldsymbol{\theta}}$ such that it belongs to the same tangent space as the rotation tensor $\widetilde{\boldsymbol{\theta}}$, i.e. $T_{\boldsymbol{I}}SO(3)$. Recalling Eq. (27) for the material virtual rotation tensor and recalling that $\boldsymbol{\Lambda} = exp\left(\widetilde{\boldsymbol{\theta}}\right)$ we have:

$$exp\left(\widetilde{\boldsymbol{\theta}} + \epsilon\, \delta\widetilde{\boldsymbol{\theta}}\right) = exp\left(\epsilon\, \delta\widetilde{\boldsymbol{\Theta}}\right) exp\left(\widetilde{\boldsymbol{\theta}}\right). \tag{29}$$

By taking derivatives with respect to the parameter ϵ at $\epsilon = 0$ we can obtain (see e.g. (Ibrahimbegović, Frey et al. 1995; Mäkinen 2007)):

$$\delta\boldsymbol{\Theta} = \boldsymbol{T}\,\delta\boldsymbol{\theta}, \quad \delta\boldsymbol{w} = \boldsymbol{T}^T\delta\boldsymbol{\theta} \tag{30}$$

where $\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{\theta})$ is a linear mapping between the tangent spaces $T_{\boldsymbol{I}}^{spat}SO(3) \rightarrow T_{\boldsymbol{\Lambda}}^{spat}SO(3)$ (Cardona and Geradin 1988). Note that, unlike $\boldsymbol{\Lambda}$, $\boldsymbol{T}$ changes the base point $\boldsymbol{I}$ into $\boldsymbol{\Lambda}$.

Now, recalling Eq. (27) we obtain the kinematically admissible variation of the rotation tensor as:

$$\delta\boldsymbol{\Lambda} = \frac{d}{d\epsilon}\left[\boldsymbol{\Lambda}\, exp\left(\epsilon\, \delta\widetilde{\boldsymbol{\Theta}}\right)\right]\Big|_{\epsilon=0} = \boldsymbol{\Lambda}\,\delta\widetilde{\boldsymbol{\Theta}} = \delta\widetilde{\boldsymbol{w}}\,\boldsymbol{\Lambda}. \tag{31}$$

From the last equation it's straightforward to verify that $\delta\widetilde{\boldsymbol{\Theta}} = \boldsymbol{\Lambda}^T\,\delta\widetilde{\boldsymbol{w}}\,\boldsymbol{\Lambda}$. Therefore, we can recall Eq. (26) to write:

$$\delta\boldsymbol{e}_i = \boldsymbol{\Lambda}\,(\delta\boldsymbol{\Theta} \times \boldsymbol{E}_i) = \delta\boldsymbol{w} \times \boldsymbol{e}_i. \tag{32}$$

Now, recalling Eq. (30), we can write the last equation as a function of the total rotation vector like:

$$\delta\boldsymbol{e}_i = (\boldsymbol{T}^T\,\delta\boldsymbol{\theta}) \times \boldsymbol{e}_i. \tag{33}$$

Noting that $\boldsymbol{e}' = \widetilde{\boldsymbol{T}^T\,\boldsymbol{\theta}'}\,\boldsymbol{e}$ we can find the variation of the director's derivative as:

$$\delta\boldsymbol{e}_i' = (\delta\boldsymbol{T}^T\,\boldsymbol{\theta}' + \boldsymbol{T}^T\,\delta\boldsymbol{\theta}') \times \boldsymbol{e}_i + (\boldsymbol{T}^T\,\boldsymbol{\theta}') \times [(\boldsymbol{T}^T\,\delta\boldsymbol{\theta}) \times \boldsymbol{e}_i]. \tag{34}$$

The set of kinematically admissible variations can now be defined as:

$$\delta\mathcal{V} \coloneqq \{\delta\boldsymbol{\phi} = [\delta\boldsymbol{u}, \delta\boldsymbol{\theta}]^T\colon [0,\ell] \rightarrow R^3 | \delta\boldsymbol{\phi} = 0 \text{ on } \mathcal{S}\}, \tag{35}$$

where $\mathcal{S}$ describes de boundaries with prescribed displacements and rotations.

3.2 Virtual Generalized Strains

The variations of the directors and its derivatives are now used to obtain the virtual generalized strains. Considering that $\delta\boldsymbol{E_i} = 0$ and that $\delta\boldsymbol{X}_0' = 0$, and performing the variation to Eq. (8) we obtain:

$$\delta\boldsymbol{\varepsilon} = \begin{bmatrix} \boldsymbol{x}_0' \cdot \delta\boldsymbol{u}' \\ \boldsymbol{e}_3' \cdot \delta\boldsymbol{u}' + \boldsymbol{x}_0' \cdot \delta\boldsymbol{e}_3' \\ \boldsymbol{e}_2' \cdot \delta\boldsymbol{u}' + \boldsymbol{x}_0' \cdot \delta\boldsymbol{e}_2' \\ \boldsymbol{e}_2 \cdot \delta\boldsymbol{u}' + \boldsymbol{x}_0' \cdot \delta\boldsymbol{e}_2 \\ \boldsymbol{e}_3 \cdot \delta\boldsymbol{u}' + \boldsymbol{x}_0' \cdot \delta\boldsymbol{e}_3 \\ \delta\boldsymbol{e}_2' \cdot \boldsymbol{e}_3 + \boldsymbol{e}_2' \cdot \delta\boldsymbol{e}_3 \\ 2\left(\delta\boldsymbol{e}_2' \cdot \boldsymbol{e}_2'\right) \\ 2\left(\delta\boldsymbol{e}_3' \cdot \boldsymbol{e}_3'\right) \\ \delta\boldsymbol{e}_2' \cdot \boldsymbol{e}_3' + \boldsymbol{e}_2' \cdot \delta\boldsymbol{e}_3' \end{bmatrix}. \tag{36}$$

To maintain the compactness of the formulation, it will be useful to write the last expression as a function of a new set of kinematic variables $\delta\boldsymbol{\varphi}$ as:

$$\delta\boldsymbol{\varepsilon} = \mathbb{H}\ \delta\boldsymbol{\varphi}. \tag{37}$$

Where:

$$\mathbb{H} = \begin{bmatrix} \boldsymbol{x}_0'^{\,T} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{e}_3'^{\,T} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{x}_0'^{\,T} \\ \boldsymbol{e}_2'^{\,T} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{x}_0'^{\,T} & \boldsymbol{0} \\ \boldsymbol{e}_2^T & \boldsymbol{0} & \boldsymbol{x}_0'^{\,T} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{e}_3^T & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{x}_0'^{\,T} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{e}_2'^{\,T} & \boldsymbol{e}_3^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{e}_2'^{\,T} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{e}_3'^{\,T} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{e}_3'^{\,T} & \boldsymbol{e}_2'^{\,T} \end{bmatrix}, \qquad \delta\boldsymbol{\varphi} = \begin{bmatrix} \delta\boldsymbol{u}' \\ \delta\boldsymbol{\theta} \\ \delta\boldsymbol{e}_2 \\ \delta\boldsymbol{e}_3 \\ \delta\boldsymbol{e}_2' \\ \delta\boldsymbol{e}_3' \end{bmatrix}. \tag{38}$$

3.3 Internal Virtual Work

Recalling Eq. (25), the first term can be written in its shell form as:

$$G_{int}(\boldsymbol{\phi}, \delta\boldsymbol{\phi}) = \int_{\ell} \int_{S} \delta\boldsymbol{\epsilon}_s^T\ \boldsymbol{N}_s\ ds\ \ dx. \tag{39}$$

The reduction to a one dimensional formulation is now aided by the deduction of 1D beam forces presented in Eq. (23). Transforming the virtual curvilinear shell strains into virtual generalized strains we can rewrite the last expression as:

$$G_{int}(\boldsymbol{\phi}, \delta\boldsymbol{\phi}) = \int_{\ell} \delta\boldsymbol{\varepsilon}^T \left(\int_{S} \boldsymbol{\mathcal{T}}^T\ \boldsymbol{N}_s\ ds \right) dx \tag{40}$$

In which the term in parentheses is the generalized beam forces vector $\boldsymbol{S}(x)$ (see Eq. (23)). Lastly, we write the one dimensional version of the virtual work principle in terms of the generalized strains and the generalized beam forces:

$$G_{int}(\boldsymbol{\phi}, \delta\boldsymbol{\phi}) = \int_{\ell} \delta\boldsymbol{\varepsilon}^T \boldsymbol{S}\, dx. \tag{41}$$

3.4 External Virtual Work

The virtual work of external forces can be written as:

$$G_{ext}(\boldsymbol{\phi}, \delta\boldsymbol{\phi}) = \int_{l} (\boldsymbol{n} \cdot \delta\boldsymbol{u} + \boldsymbol{m} \cdot \delta\boldsymbol{\theta})\, dx, \tag{42}$$

where $\boldsymbol{n}$ is the external forces vector and $\boldsymbol{m}$ the external moments vector. These vectors are defined according to:

$$\begin{aligned} \boldsymbol{n} &= \int_S \int_e \boldsymbol{b}\, dn\, ds + \int_S \boldsymbol{t}\, ds, \\ \boldsymbol{m} &= \int_S \int_e \boldsymbol{X} \times \boldsymbol{b}\, dn\, ds + \int_S \boldsymbol{X} \times \boldsymbol{t}\, ds, \end{aligned} \tag{43}$$

where $\boldsymbol{b}$ is the distributed body force vector and $\boldsymbol{t}$ is external stress vector.

3.5 Virtual work of the inertia forces

In the derivation of the virtual work of the inertia forces we use a material approach in order to avoid the Lie derivative in the linearization process. The inertial virtual work is:

$$G_{iner}(\boldsymbol{\phi}, \delta\boldsymbol{\phi}) = \int_{\mathcal{B}_0} \rho_0\, \delta\boldsymbol{x}^T \ddot{\boldsymbol{x}}\ dx \tag{44}$$

Using a material description we have:

$$\begin{aligned} \delta\boldsymbol{x} &= \delta\boldsymbol{x}_0 + \delta\boldsymbol{\Lambda}\,\boldsymbol{\xi} = \delta\boldsymbol{x}_0 + \boldsymbol{\Lambda}\,\delta\widetilde{\boldsymbol{\Theta}}\,\boldsymbol{\xi} \\ \ddot{\boldsymbol{x}} &= \ddot{\boldsymbol{x}}_0 + \ddot{\boldsymbol{\Lambda}}\,\boldsymbol{\xi} = \ddot{\boldsymbol{x}}_0 + \left(\boldsymbol{\Lambda}\,\widetilde{\boldsymbol{\Omega}}\,\widetilde{\boldsymbol{\Omega}} + \boldsymbol{\Lambda}\,\dot{\widetilde{\boldsymbol{\Omega}}}\right)\boldsymbol{\xi} \end{aligned} \tag{45}$$

The virtual work can now be expressed as:

$$G_{iner}(\boldsymbol{\phi}, \delta\boldsymbol{\phi}) = \int_{\mathcal{B}_0} \rho_0\left(\delta\boldsymbol{x}_0 + \boldsymbol{\Lambda}\,\delta\widetilde{\boldsymbol{\Theta}}\,\boldsymbol{\xi}\right)^T \left[\ddot{\boldsymbol{x}}_0 + \left(\boldsymbol{\Lambda}\,\widetilde{\boldsymbol{\Omega}}\,\widetilde{\boldsymbol{\Omega}} + \boldsymbol{\Lambda}\,\dot{\widetilde{\boldsymbol{\Omega}}}\right)\boldsymbol{\xi}\right] dx \tag{46}$$

Integrating over the cross section we obtain:

$$G_{iner}(\boldsymbol{\phi}, \delta\boldsymbol{\phi}) = \int_{l} m\left(\delta\boldsymbol{x}_0{}^T \ddot{\boldsymbol{x}}_0\right) + \delta\boldsymbol{\Theta}^T\left(\boldsymbol{J}\,\dot{\boldsymbol{\Omega}} + \widetilde{\boldsymbol{\Omega}}\,\boldsymbol{J}\,\boldsymbol{\Omega}\right) dx, \tag{47}$$

where we have assumed that pole (reference point) of the cross section is coincident with the center of mass, then $\int_A \boldsymbol{\xi}\, dA = 0$. Also, the cross sectional mass and constant inertia tensors are given by:

$$m = \int_A \rho_0\, dA\,, \qquad \boldsymbol{J} = \int_A \rho_0\, \tilde{\boldsymbol{\xi}}^T \tilde{\boldsymbol{\xi}}\, dA \tag{48}$$

where ρ is the material density. It's interesting to note that the constant inertia tensor is characteristic of material descriptions.

4 LINEARIZED EQUILIBRIUM EQUATIONS

4.1 Weak Form of 1D Equilibrium

The variational equilibrium statement can now be presented in terms of generalized components of 1D forces and strains. Recalling Eqs. (41) and (42) the virtual work of a composite beam is written in its one dimensional form as:

$$G(\boldsymbol{\phi}, \delta\boldsymbol{\phi}) = \int_{\ell} \delta\boldsymbol{\varepsilon}^T \boldsymbol{S}\, dx - \int_{l} (\boldsymbol{n}\cdot\delta\boldsymbol{u} + \boldsymbol{m}\cdot\delta\boldsymbol{\theta})\, dx. \tag{49}$$

Using Eq. (37) it's possible to re-write the last expression as:

$$G(\boldsymbol{\phi}, \delta\boldsymbol{\phi}) = \int_{\ell} [\mathbb{H}\, \delta\boldsymbol{\varphi}]^T\, \boldsymbol{S}\, dx - \int_{l} (\boldsymbol{n}\cdot\delta\boldsymbol{u} + \boldsymbol{m}\cdot\delta\boldsymbol{\theta})\, dx. \tag{50}$$

4.2 Linearization of the internal virtual work

The linearization of the variational equilibrium equations is obtained through the directional derivative and, assuming conservative loading, its application gives two tangent terms; the material and the geometric stiffness matrices. Applying the directional derivative in the direction $\Delta\boldsymbol{\phi}$ to the internal virtual work and recalling Eqs. (41) and (36), we obtain the tangent stiffness as:

$$D\; G_{int}(\boldsymbol{\phi}, \delta\boldsymbol{\phi})\cdot\Delta\boldsymbol{\phi} = \int_{\ell} (\delta\boldsymbol{\varepsilon}^T\, \mathbb{D}\, \Delta\boldsymbol{\varepsilon} + \Delta\delta\boldsymbol{\varepsilon}^T\, \boldsymbol{S}) dx, \tag{51}$$

where ℓ is the length of the undeformed beam.

Using Eq. (37) the first term of the right hand side of the above equation gives de material stiffness terms as:

$$D_1 G_{int}(\boldsymbol{\phi}, \delta\boldsymbol{\phi})\cdot\Delta\boldsymbol{\phi} = \int_{\ell} \delta\boldsymbol{\varphi}^T\, \mathbb{H}^T\, \mathbb{D}\, \mathbb{H}\, \Delta\boldsymbol{\varphi}\, dx. \tag{52}$$

On the other hand, from the second term, the general expression of the geometric stiffness operator gives:

$$D_2\; G_{int}(\boldsymbol{\phi}, \delta\boldsymbol{\phi})\cdot\Delta\boldsymbol{\phi} = \int_{\ell} \Delta\delta\boldsymbol{\varepsilon}^T \boldsymbol{S}\, dx. \tag{53}$$

The linearization of the virtual generalized strains gives:

$$\Delta\delta\boldsymbol{\varepsilon} = \begin{bmatrix} \delta\boldsymbol{u}'\cdot\Delta\boldsymbol{u}' \\ \delta\boldsymbol{u}'\cdot\Delta\boldsymbol{e}_3' + \delta\boldsymbol{e}_3'\cdot\Delta\boldsymbol{u}' + \boldsymbol{x}_0'\cdot\Delta\delta\boldsymbol{e}_3' \\ \delta\boldsymbol{u}'\cdot\Delta\boldsymbol{e}_2' + \delta\boldsymbol{e}_2'\cdot\Delta\boldsymbol{u}' + \boldsymbol{x}_0'\cdot\Delta\delta\boldsymbol{e}_2' \\ \delta\boldsymbol{u}'\cdot\Delta\boldsymbol{e}_2 + \delta\boldsymbol{e}_2\cdot\Delta\boldsymbol{u}' + \boldsymbol{x}_0'\cdot\Delta\delta\boldsymbol{e}_2 \\ \delta\boldsymbol{u}'\cdot\Delta\boldsymbol{e}_3 + \delta\boldsymbol{e}_3\cdot\Delta\boldsymbol{u}' + \boldsymbol{x}_0'\cdot\Delta\delta\boldsymbol{e}_3 \\ \delta\boldsymbol{e}_2'\cdot\Delta\boldsymbol{e}_3 + \delta\boldsymbol{e}_3\cdot\Delta\boldsymbol{e}_2' + \boldsymbol{e}_3\cdot\Delta\delta\boldsymbol{e}_2' + \boldsymbol{e}_2'\cdot\Delta\delta\boldsymbol{e}_3 \\ 2\left(\boldsymbol{e}_2'\cdot\Delta\delta\boldsymbol{e}_2' + \delta\boldsymbol{e}_2'\cdot\Delta\boldsymbol{e}_2'\right) \\ 2\left(\boldsymbol{e}_3'\cdot\Delta\delta\boldsymbol{e}_3' + \delta\boldsymbol{e}_3'\cdot\Delta\boldsymbol{e}_3'\right) \\ \delta\boldsymbol{e}_2'\cdot\Delta\boldsymbol{e}_3' + \delta\boldsymbol{e}_3'\cdot\Delta\boldsymbol{e}_2' + \boldsymbol{e}_3'\cdot\Delta\delta\boldsymbol{e}_2' + \boldsymbol{e}_2'\cdot\Delta\delta\boldsymbol{e}_3' \end{bmatrix}. \tag{54}$$

To complete de development of the geometric stiffness matrix, we need to find the linearization of the virtual generalized strains, i.e. $\Delta\delta\boldsymbol{\varepsilon}^T$, for what we first need to obtain the linearized virtual directors. Using Eq. (33), the linearization of the virtual directors can be obtained as:

$$\Delta\delta\boldsymbol{e}_i = (\Delta\boldsymbol{T}^T\,\delta\boldsymbol{\theta}) \times \boldsymbol{e}_i + (\boldsymbol{T}^T\,\delta\boldsymbol{\theta}) \times [(\boldsymbol{T}^T\,\Delta\boldsymbol{\theta}) \times \boldsymbol{e}_i]. \tag{55}$$

4.3 Linearization of the work of the inertia forces

Now we need to linearize the virtual work of the inertia forces. Using Eq. (30) we can obtain the angular velocity and angular acceleration spin vectors as:

$$\boldsymbol{\Omega} = \boldsymbol{T}\,\dot{\boldsymbol{\theta}}, \qquad \boldsymbol{\Omega} = \dot{\boldsymbol{T}}\,\dot{\boldsymbol{\theta}} + \boldsymbol{T}\,\ddot{\boldsymbol{\theta}} \tag{56}$$

Replacing the above expression into the expression (47) we obtain:

$$\delta W_{iner}(\boldsymbol{\phi}, \delta\boldsymbol{\phi}) = \int_l \boldsymbol{m}\left(\delta\boldsymbol{x}_0{}^T\ddot{\boldsymbol{x}}_0\right) + \delta\boldsymbol{\theta}^T\left(\boldsymbol{T}^T\boldsymbol{J}\,\dot{\boldsymbol{T}}\,\dot{\boldsymbol{\theta}} + \boldsymbol{T}^T\boldsymbol{J}\,\boldsymbol{T}\,\ddot{\boldsymbol{\theta}} + \boldsymbol{T}^T\widetilde{\left(\boldsymbol{T}\,\dot{\boldsymbol{\theta}}\right)}\boldsymbol{J}\,\boldsymbol{T}\,\dot{\boldsymbol{\theta}}\right)\,dx, \tag{57}$$

The last expression is already in linear form with respect to the acceleration field $\ddot{\boldsymbol{\phi}}$, so its linearization involves only the linearization with respect to a change in configuration and velocities:

$$\begin{aligned} &D\delta W_{iner}(\boldsymbol{\phi}, \delta\boldsymbol{\phi}) \cdot \left[\Delta\left(\boldsymbol{\phi}, \dot{\boldsymbol{\phi}}, \ddot{\boldsymbol{\phi}}\right)\right] \\ &= D\delta W_{iner} \cdot [\Delta\boldsymbol{\phi}] + D\delta W_{iner} \cdot \left[\Delta\dot{\boldsymbol{\phi}}\right] + \int_l \delta\boldsymbol{\phi}^T\,\boldsymbol{M}\,\Delta\ddot{\boldsymbol{\phi}} \end{aligned} \tag{58}$$

In the above equation, the mass matrix is:

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{m} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{T}^T\boldsymbol{J}\,\boldsymbol{T} \end{bmatrix}, \tag{59}$$

where $\boldsymbol{m} = m\,\boldsymbol{I}$, being $\boldsymbol{I}$ is the 3×3 identity matrix.

The linearization of the virtual work of the inertia forces in the directions $\Delta\boldsymbol{\phi}$ and $\Delta\dot{\boldsymbol{\phi}}$ give raise to centrifugal and gyroscopic inertia matrices. The centrifugal and gyroscopic inertia effects are, for most applications, negligible (Geradin and Cardona 2001); because it is costly to evaluate the complex matrices that evolve from the treatment of these effects, they will be disregarded. Thus, the linearized version of the virtual work of the inertia forces reduces to:

$$D\delta W_{iner}(\boldsymbol{\phi}, \delta\boldsymbol{\phi}) \cdot \left[\Delta\left(\boldsymbol{\phi}, \dot{\boldsymbol{\phi}}, \ddot{\boldsymbol{\phi}}\right)\right] = \int_l \delta\boldsymbol{\phi}^T\,\boldsymbol{M}\,\Delta\ddot{\boldsymbol{\phi}} \tag{60}$$

5 FINITE ELEMENT FORMULATION

The implementation of the proposed finite element is based on linear interpolation and one point reduced integration (thus avoiding shear locking). A relevant procedure of the finite element implementation is the use of interpolation to obtain the derivatives of the director field, this greatly simplifies the expression of the tangent stiffness matrix.

5.1 Interpolations and Directors Update

We interpolate the position vectors in the undeformed and deformed configuration as:

$$\boldsymbol{X} = \sum_{j=1}^{nn} N_j\, \widehat{\boldsymbol{X}}_j, \qquad \boldsymbol{x} = \sum_{j=1}^{nn} N_j \left(\widehat{\boldsymbol{X}}_j + \widehat{\boldsymbol{u}}_j\right), \tag{61}$$

where ^ indicates a nodal value, j is the node index and nn is the number of nodes per element. Also, we define:

$$\boldsymbol{N}_j = \begin{bmatrix} N_j & 0 & 0 \\ 0 & N_j & 0 \\ 0 & 0 & N_j \end{bmatrix}, \tag{62}$$

The same interpolation is also applied to the configuration and its variation, so:

$$\boldsymbol{\phi} = \sum_{j=1}^{nn} N_j\, \widehat{\boldsymbol{\phi}}_j\,, \quad \boldsymbol{\phi}' = \sum_{j=1}^{nn} N_j'\, \widehat{\boldsymbol{\phi}}_j\,, \quad \delta\boldsymbol{\phi} = \sum_{j=1}^{nn} N_j\, \delta\widehat{\boldsymbol{\phi}}_j\,, \quad \delta\boldsymbol{\phi}' = \sum_{j=1}^{nn} N_j'\, \delta\widehat{\boldsymbol{\phi}}_j\,. \tag{63}$$

Using Eq. (1) the director at the iteration $n+1$ is found as: ${}^{n+1}\boldsymbol{e}_i = \boldsymbol{\Lambda}({}^{n}\boldsymbol{\theta})\, \boldsymbol{E}_i$, where $\boldsymbol{\Lambda}$ is the *total rotation tensor*.

A simple way to obtain the derivatives of the director field is to use interpolation. So, being N_j linear Lagrangian shape function coefficients, it will be assumed that:

$$\boldsymbol{e}_i' \cong \sum_{j=1}^{nn} N_j'\, \hat{\boldsymbol{e}}_i^j \tag{64}$$

Where $\hat{\boldsymbol{e}}_i^j$ stands for the director i at the node j and nn is the number of nodes per element, which in the present case is 2.

5.2 Discrete Virtual Directors

Assuming holonomic constraints we may interchange variations and derivatives, i.e. $\delta(\boldsymbol{e}') = (\delta\boldsymbol{e})'$. Using this property, we can use Eq. (64) to obtain the variation of the directors and its derivatives as:

$$\delta\boldsymbol{e}_i \cong \sum_{j=1}^{nn} N_j\, \delta\hat{\boldsymbol{e}}_i^j, \qquad \delta\boldsymbol{e}_i' \cong \sum_{j=1}^{nn} N_j'\, \delta\hat{\boldsymbol{e}}_i^j, \tag{65}$$

The obtention of the linearization of the directors and its derivatives is more involved and requires the linearization of the tangential transformation. Observing the linearization of the variation of the directors appears in the virtual strains (and also in its linearization) always pre multiplied by some constant vector $\boldsymbol{a}$, for simplicity in the arranging of terms, it's preferable to obtain the expression for this product and not only for the second variation. Thus, recalling Eq. (55) we find that:

$$\boldsymbol{a} \cdot \Delta\delta\boldsymbol{e}_i = \boldsymbol{a} \cdot \{(\Delta\boldsymbol{T}^T\, \delta\boldsymbol{\theta}) \times \boldsymbol{e}_i + (\boldsymbol{T}^T\, \delta\boldsymbol{\theta}) \times [(\boldsymbol{T}^T\, \Delta\boldsymbol{\theta}) \times \boldsymbol{e}_i]\} \tag{66}$$

Switching to matrix notation, using spinors in place of cross products and reordering some terms we can re-write the above equation as:

$$\boldsymbol{a}\cdot\Delta\delta\boldsymbol{e}_i = \delta\boldsymbol{\theta}^T\Delta\boldsymbol{T}(\tilde{\boldsymbol{e}}_i\boldsymbol{a}) + \delta\boldsymbol{w}^T(\tilde{\boldsymbol{a}}\,\tilde{\boldsymbol{e}}_i)\Delta\boldsymbol{w}, \tag{67}$$

where $\tilde{\hat{\boldsymbol{e}}}_i^j$ is the spinor of the director i at node j and:

$$\Delta\boldsymbol{T}(\tilde{\boldsymbol{e}}_i\boldsymbol{a}) = D\,[\boldsymbol{T}(\tilde{\boldsymbol{e}}_i\boldsymbol{a})]\cdot\Delta\boldsymbol{\theta}. \tag{68}$$

The linearization of the term $\boldsymbol{T}(\tilde{\boldsymbol{e}}_i\boldsymbol{a})$ can be found in (Ritto-Corrêa and Camotim 2002; Saravia, Machado et al. 2011). Now, recalling Eq. (30) it's possible to rewrite the discrete form of Eq. (67) as:

$$\boldsymbol{a}\cdot\Delta\delta\boldsymbol{e}_i \cong \delta\widehat{\boldsymbol{\theta}}^T\left[\sum_{j=1}^{nn} N_j\left[\boldsymbol{\Xi}\left(\boldsymbol{a},\hat{\boldsymbol{e}}_i^j\right) + \boldsymbol{T}\,\tilde{\boldsymbol{a}}\,\tilde{\hat{\boldsymbol{e}}}_i^j\boldsymbol{T}^T\right]\right]\Delta\widehat{\boldsymbol{\theta}}, \tag{69}$$

Where:

$$\boldsymbol{\Xi}\left(\boldsymbol{a},\hat{\boldsymbol{e}}_i^j\right) = D[\boldsymbol{T}(\tilde{\boldsymbol{e}}_i\boldsymbol{a})]\cdot\Delta\boldsymbol{\theta}, \tag{70}$$

In the same form, the expression for the second variation of the director's derivatives can be found in its discrete form by making use of Eq. (65) :

$$\boldsymbol{a}\cdot\Delta\delta\boldsymbol{e}_i' = \delta\widehat{\boldsymbol{\theta}}^T\left[\sum_{j=1}^{nn} N_j'\left[\boldsymbol{\Xi}\left(\boldsymbol{a},\hat{\boldsymbol{e}}_i^j\right) + \boldsymbol{T}\,\tilde{\boldsymbol{a}}\,\tilde{\hat{\boldsymbol{e}}}_i^j\boldsymbol{T}^T\right]\right]\Delta\widehat{\boldsymbol{\theta}} \tag{71}$$

5.3 Discrete Virtual Strains

Having derived the expressions for the discrete virtual directors, its derivatives and its corresponding linearization, it's now possible to find a discrete expression for the discrete virtual generalized strain and its linearization.

We can relate the two kinematic vectors $\delta\boldsymbol{\varphi}$ and $\delta\boldsymbol{\phi}$ by means of a matrix $\mathbb{B}$ as:

$$\delta\boldsymbol{\varphi} \cong \sum_{j=1}^{nn}\mathbb{B}_j\,\delta\widehat{\boldsymbol{\phi}}_j, \tag{72}$$

Where:

$$\mathbb{B}_j = \begin{bmatrix} \boldsymbol{N}_j' & \boldsymbol{0} \\ \boldsymbol{0} & N_j\boldsymbol{T}_j^T \\ \boldsymbol{0} & N_j\tilde{\boldsymbol{e}}_2^{j\,T}\boldsymbol{T}_j^T \\ \boldsymbol{0} & N_j\tilde{\boldsymbol{e}}_3^{j\,T}\boldsymbol{T}_j^T \\ \boldsymbol{0} & N_j'\tilde{\boldsymbol{e}}_2^{j\,T}\boldsymbol{T}_j^T \\ \boldsymbol{0} & N_j'\tilde{\boldsymbol{e}}_3^{j\,T}\boldsymbol{T}_j^T \end{bmatrix}, \qquad \delta\widehat{\boldsymbol{\phi}}_j = \begin{bmatrix}\delta\widehat{\boldsymbol{u}}_j \\ \delta\widehat{\boldsymbol{\theta}}_j\end{bmatrix}. \tag{73}$$

Where $\sim$ indicates the skew symmetric matrix of a vector and $\wedge$ indicates a nodal variable. Thus $\tilde{\boldsymbol{e}}_j^i$ is a skew director in the direction j of the node i and $\boldsymbol{T}_j^T$ is the transpose of the tangential transformation at the node j. Henceforth summation over index j will be implicitly defined, so we will omit the summation symbol and the node index j.

Finally, recalling Eq. (37) we can write the virtual generalized strains as:

$$\delta\boldsymbol{\varepsilon} \cong \mathbb{H}\ \mathbb{B}\ \delta\widehat{\boldsymbol{\phi}}, \tag{74}$$

The discrete form of the incremental virtual strains, i.e. $\Delta\delta\boldsymbol{\varepsilon}$, is more difficult to obtain. Using the structure of the geometric stiffness operator of Eq. (53) we can obtain a matrix $\mathbb{G}$ as to satisfy the equality $\Delta\delta\boldsymbol{\varepsilon}^T\boldsymbol{S} = \delta\boldsymbol{\varphi}^T\mathbb{G}\ \Delta\boldsymbol{\varphi}$, a lengthy manipulation gives:

$$\mathbb{G} = \begin{bmatrix} \bar{S}_1 & \mathbf{0} & \bar{Q}_2 & \bar{Q}_3 & \bar{M}_3 & \bar{M}_2 \\ & \boldsymbol{A} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ & & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ & & & \mathbf{0} & \bar{M}_1 & \mathbf{0} \\ & Sym & & & 2\bar{P}_2 & \bar{P}_{23} \\ & & & & & 2\bar{P}_3 \end{bmatrix}. \tag{75}$$

where:

$$\begin{aligned} \boldsymbol{A} = \sum_{j=1}^{2} \Big\{ & \left(M_2 N_j' + Q_3 N_j\right) \left[\Xi\left(\boldsymbol{x}_0', \hat{\boldsymbol{e}}_3^j\right) + \boldsymbol{T}\,\tilde{\boldsymbol{x}}_0'\,\tilde{\hat{\boldsymbol{e}}}_3^j \boldsymbol{T}^T\right] \\ & + \left(M_3 N_j' + Q_2 N_j\right) \left[\Xi\left(\boldsymbol{x}_0', \hat{\boldsymbol{e}}_2^j\right) + \boldsymbol{T}\,\tilde{\boldsymbol{x}}_0'\,\tilde{\hat{\boldsymbol{e}}}_2^j \boldsymbol{T}^T\right] \\ & + T\left[N_j' \left[\Xi\left(\boldsymbol{e}_3, \hat{\boldsymbol{e}}_2^j\right) + \boldsymbol{T}\,\tilde{\boldsymbol{e}}_3\,\tilde{\hat{\boldsymbol{e}}}_2^j \boldsymbol{T}^T\right] + N_j \left[\Xi\left(\boldsymbol{e}_2', \hat{\boldsymbol{e}}_3^j\right) + \boldsymbol{T}\,\tilde{\boldsymbol{e}}_2'\,\tilde{\hat{\boldsymbol{e}}}_3^j \boldsymbol{T}^T\right]\right] \\ & + P_2 N_j' \left[\Xi\left(\boldsymbol{e}_2', \hat{\boldsymbol{e}}_2^j\right) + \boldsymbol{T}\,\tilde{\boldsymbol{e}}_2'\,\tilde{\hat{\boldsymbol{e}}}_2^j \boldsymbol{T}^T\right] + P_3 N_j' \left[\Xi\left(\boldsymbol{e}_3', \hat{\boldsymbol{e}}_3^j\right) + \boldsymbol{T}\,\tilde{\boldsymbol{e}}_3'\,\tilde{\hat{\boldsymbol{e}}}_3^j \boldsymbol{T}^T\right] \\ & + P_{23}\left[N_j' \left[\Xi\left(\boldsymbol{e}_3', \hat{\boldsymbol{e}}_2^j\right) + \boldsymbol{T}\,\tilde{\boldsymbol{e}}_3'\,\tilde{\hat{\boldsymbol{e}}}_2^j \boldsymbol{T}^T\right] + N_j' \left[\Xi\left(\boldsymbol{e}_2', \hat{\boldsymbol{e}}_3^j\right) + \boldsymbol{T}\,\tilde{\boldsymbol{e}}_2'\,\tilde{\hat{\boldsymbol{e}}}_3^j \boldsymbol{T}^T\right]\right]\Big\} \end{aligned} \tag{76}$$

It's interesting to note that $\boldsymbol{A}$ result to be symmetric and as a consequence $\mathbb{G}$ is also symmetric. Although it's strictly not a necessary condition, the fact that the matrix $\mathbb{G}$ is symmetric, guarantees the symmetry of the tangent stiffness matrix.

5.4 Tangent Stiffness Matrix

Introducing Eq. (72) into Eq. (52) we can obtain the discrete form of the material virtual work as:

$$D_1 G_{int}\left(\widehat{\boldsymbol{\phi}}, \delta\widehat{\boldsymbol{\phi}}\right) \cdot \Delta\widehat{\boldsymbol{\phi}} = \int_\ell \left(\mathbb{B}\ \delta\widehat{\boldsymbol{\phi}}\right)^T \mathbb{H}^T\ \mathbb{D}\ \mathbb{H}\ \left(\mathbb{B}\ \Delta\widehat{\boldsymbol{\phi}}\right) dx. \tag{77}$$

Then, the element material stiffness matrix is:

$$\boldsymbol{k}_M = \int_\ell \mathbb{B}^T \mathbb{H}^T \mathbb{D}\ \mathbb{H}\ \mathbb{B}\ dx. \tag{78}$$

Proceeding in a similar way, we use Eqs. (75) and (53) to obtain the discrete geometric stiffness terms as:

$$D_2 G_{int}\left(\widehat{\boldsymbol{\phi}}, \delta\widehat{\boldsymbol{\phi}}\right) \cdot \Delta\widehat{\boldsymbol{\phi}} = \int_\ell \left(\mathbb{B}\ \delta\widehat{\boldsymbol{\phi}}\right)^T \mathbb{G}\ \left(\mathbb{B}\ \Delta\widehat{\boldsymbol{\phi}}\right) dx. \tag{79}$$

Therefore, the element geometric stiffness matrix becomes:

$$\boldsymbol{k}_G = \int_\ell \mathbb{B}^T \mathbb{G}\ \mathbb{B}\ dx. \tag{80}$$

Following the standard steps of the finite element method, the element and global tangent stiffness matrices are:

$$\boldsymbol{k}_T = \int_\ell \mathbb{B}^T(\mathbb{H}^T\mathbb{D}\,\mathbb{H} + \mathbb{G})\,\mathbb{B}\,dx,$$
$$\mathbb{K} = \sum_{e=1}^{els} \boldsymbol{k}_T, \tag{81}$$

where the summation operator is used to represent the finite element assembly process.

5.5 Tangent Mass Matrix

Using linear interpolation for the acceleration field, i.e. $\ddot{\boldsymbol{\phi}} = \sum_{j=1}^{nn} \boldsymbol{N}_n\,\hat{\ddot{\boldsymbol{\phi}}}_n$, we can obtain the discrete version of the mass matrix in expression (59). First, the discrete version of the linearized virtual work of the acceleration forces is written as:

$$D\delta W_{iner}(\boldsymbol{\phi},\delta\boldsymbol{\phi})\cdot\left[\Delta\left(\boldsymbol{\phi},\dot{\boldsymbol{\phi}},\ddot{\boldsymbol{\phi}}\right)\right] \cong \int_l \delta\hat{\boldsymbol{\phi}}^T\mathbb{N}^T\boldsymbol{M}\,\mathbb{N}\,\Delta\hat{\ddot{\boldsymbol{\phi}}}, \tag{82}$$

where $\widehat{}$ indicates nodal values and we have defined $\mathbb{N} = \sum_{j=1}^{nn}\mathbb{N}_j$ and

$$\delta\boldsymbol{\phi} \cong \sum_{j=1}^{nn}\mathbb{N}_j\,\delta\hat{\boldsymbol{\phi}}^j, \qquad \mathbb{N}_j = \begin{bmatrix}\boldsymbol{N}_j & \mathbf{0}\\ \mathbf{0} & \boldsymbol{N}_j\end{bmatrix}. \tag{83}$$

Implicitly assuming summation over index j we can write the discrete form for the tangent mass matrix (59) as:

$$\mathbb{M} = \mathbb{N}_j^T\boldsymbol{M}\,\mathbb{N}_j = \begin{bmatrix}\boldsymbol{N}_j\boldsymbol{m}\,\boldsymbol{N}_j & \mathbf{0}\\ \mathbf{0} & \boldsymbol{N}_j\boldsymbol{T}_j^T\boldsymbol{J}\,\boldsymbol{T}_j\boldsymbol{N}_j\end{bmatrix}, \tag{84}$$

6 MULTIBODY DYNAMICS

6.1 Equation of motion of the constrained system

The formulation of the dynamic behavior of multibody systems gives a set of differential-algebraic system of equations if Lagrange multipliers are used to impose the constraints. In the present work, the numerical solution of the constrained algebraic problem is found through the augmented Lagrangian method.

The equations of motion of the multibody system are (Geradin and Cardona 2001):

$$\begin{cases}\boldsymbol{M}\ddot{\boldsymbol{\phi}} + \boldsymbol{B}^T(p\boldsymbol{\Phi} + k\boldsymbol{\lambda}) = \boldsymbol{g}\left(\boldsymbol{\phi},\dot{\boldsymbol{\phi}},t\right)\\ k\boldsymbol{\Phi}(\boldsymbol{\phi},t) = \mathbf{0}\end{cases} \tag{85}$$

where $\boldsymbol{B}^T$ is the constraints gradient matrix, $\boldsymbol{\Phi}$ is the constraints vector, $\boldsymbol{\lambda}$ is the Lagrange multipliers vector and $\boldsymbol{g}$ is the apparent forces vector (sum of internal, external and complementary inertia forces). Also, p and k are the penalty and scaling factors.

The linearized discrete equations of motion are obtained using Eqs. (81) and (84) as:

$$\begin{bmatrix}\mathbb{M} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\Delta\hat{\ddot{\boldsymbol{\phi}}}\\ \Delta\boldsymbol{\lambda}\end{bmatrix} + \begin{bmatrix}\mathbb{K} + p\boldsymbol{B}^T\boldsymbol{B} & k\boldsymbol{B}^T\\ k\boldsymbol{B} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\Delta\hat{\boldsymbol{\phi}}\\ \Delta\boldsymbol{\lambda}\end{bmatrix} = \begin{bmatrix}\boldsymbol{r}\\ -\boldsymbol{\Phi}\end{bmatrix} \tag{86}$$

where $\boldsymbol{r}$ is the vector of residual forces:

$$\boldsymbol{r} = \boldsymbol{g}(\boldsymbol{\phi}, \dot{\boldsymbol{\phi}}, t) - \mathbb{M}\,\ddot{\widehat{\boldsymbol{\phi}}} - \boldsymbol{B}^T(p\boldsymbol{\Phi} + k\boldsymbol{\lambda}) \tag{87}$$

It's interesting note that we have neglected dependence of inertia forces with the configuration, which is consistent with the presented derivation of the inertial virtual work. Also, since the penalty factor was chosen to be sufficiently large, we assumed that the effect of the geometric stiffness associated to the Lagrange multipliers is negligible compared to the effect of the penalty term, i.e. $p\boldsymbol{B}^T\boldsymbol{B} \gg \partial_{\boldsymbol{\phi}}(k\,\boldsymbol{B}^T\boldsymbol{\lambda})$.

6.2 Formulation of Joints

The formulation of joints is generally based on kinematic relations between the configuration variables of two nodes. Often, the treatment of rotation kinematic constraints is aided by the definition of nodal triads that are not part of the beam finite element formulation. In the present formulation, the treatment of rotational constraints is greatly simplified by that fact that nodal triads are part of the finite element model, and thus no additional triad definitions are needed.

Following the idea of Cardona et. al. (1991), each joint will be formulated as an element. Hence, an element stiffness matrix and an element internal force vector is provided by the joint formulation and assembled into the global system in a conventional manner and the Lagrange multipliers associated with the imposed constraints are treated as additional degrees of freedom.

For the sake of shortness we only present the formulation of a hinge joint, other typical joint can be formulated similarly. The hinge imposes three vectorial constraints between two nodes, a displacement vector constraint and two director constraints.We express them as:

$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{u}_B - \boldsymbol{u}_A \\ \boldsymbol{e}_1^A \cdot \boldsymbol{e}_2^B \\ \boldsymbol{e}_1^A \cdot \boldsymbol{e}_3^B \end{bmatrix} = \boldsymbol{0}\,. \tag{88}$$

The variation of the constraints give:

$$\delta\boldsymbol{\Phi} = \begin{bmatrix} \delta\boldsymbol{u}_B - \delta\boldsymbol{u}_A \\ \delta\boldsymbol{e}_1^A \cdot \boldsymbol{e}_2^B + \boldsymbol{e}_1^A \cdot \delta\boldsymbol{e}_2^B \\ \delta\boldsymbol{e}_1^A \cdot \boldsymbol{e}_3^B + \boldsymbol{e}_1^A \cdot \delta\boldsymbol{e}_3^B \end{bmatrix} = \begin{bmatrix} \delta\boldsymbol{u}_B - \delta\boldsymbol{u}_A \\ (\delta\widetilde{\boldsymbol{w}}_A\boldsymbol{e}_1^A) \cdot \boldsymbol{e}_2^B + \boldsymbol{e}_1^A \cdot (\delta\widetilde{\boldsymbol{w}}_B\boldsymbol{e}_2^B) \\ (\delta\widetilde{\boldsymbol{w}}_A\boldsymbol{e}_1^A) \cdot \boldsymbol{e}_3^B + \boldsymbol{e}_1^A \cdot (\delta\widetilde{\boldsymbol{w}}_B\boldsymbol{e}_3^B) \end{bmatrix}. \tag{89}$$

Reordering some terms and invoking Eq. (30) we can re-write the last expression as:

$$\delta\boldsymbol{\Phi} = \boldsymbol{B}\delta\widehat{\boldsymbol{\phi}}\,, \qquad \text{where} \qquad \boldsymbol{B} = \begin{bmatrix} -\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & (\tilde{\boldsymbol{e}}_1^A\,\boldsymbol{e}_2^B)^T\,\boldsymbol{T}_1^T & \boldsymbol{0} & (\tilde{\boldsymbol{e}}_2^B\,\boldsymbol{e}_1^A)^T\,\boldsymbol{T}_2^T \\ \boldsymbol{0} & (\tilde{\boldsymbol{e}}_1^A\,\boldsymbol{e}_3^B)^T\,\boldsymbol{T}_1^T & \boldsymbol{0} & (\tilde{\boldsymbol{e}}_3^B\,\boldsymbol{e}_1^A)^T\,\boldsymbol{T}_2^T \end{bmatrix}. \tag{90}$$

In the above expression, $\boldsymbol{B}$ is the 5×12 constraints gradient matrix. As it can be seen, the expression of the constraints gradient matrix is very simple and does not contain the rotation tensor.

Now, the discrete equation of motion for a rigid and massless hinge element can be written as:

$$\begin{bmatrix} p\boldsymbol{B}^T\boldsymbol{B} & k\boldsymbol{B}^T \\ k\boldsymbol{B} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Delta}\widehat{\boldsymbol{\phi}} \\ \boldsymbol{\Delta\lambda} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{B}^T(p\boldsymbol{\Phi} + k\boldsymbol{\lambda}) \\ -\boldsymbol{\Phi} \end{bmatrix}. \tag{91}$$

Both the pseudo stiffness matrix and the pseudo internal forces vector are assembled into the global system in a conventional fashion.

7 AN EXAMPLE

The present finite element is under development, because of that we present only one study case. We analyze an EGlass-Epoxy bi-pendulum that falls under the effect of gravity. The pendulum has a square cross section with b=1, h=1, e=0.1, laminated in a {45,-45,-45,45} configuration, see Figure 1.

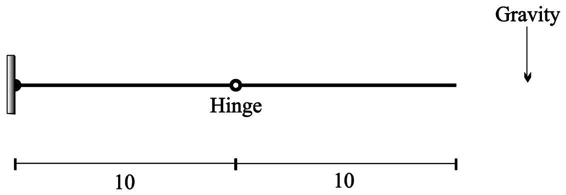

Figure 1 – Bi-pendulum

We compare the proposed pendulum against an Abaqus 3D shell model, the Figure 2 presents the evolution of displacements with time.

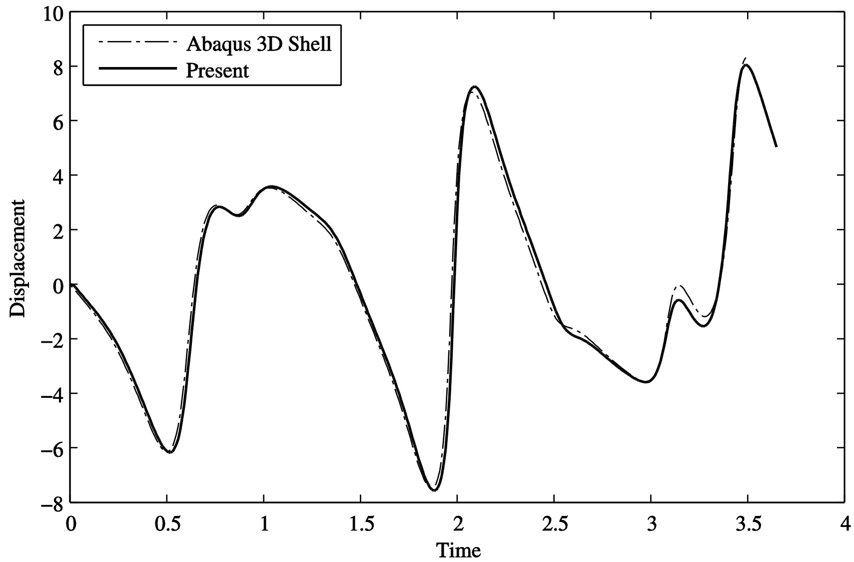

Figure 2 – Tip displacements of the composite bi-pendulum.

8 CONCLUSIONS

A geometrically exact composite thin-walled beam element for multibody applications has been presented. In the proposed formulation the virtual work equations was written as a function of generalized strains, which are parametrized in terms of the director field and its derivatives. Also, the material constitutive relation was based on the mechanics of composite laminates. The formulation of typical joints was briefly presented.

Although the present formulation is still under development, the finite element is showing good agreement with shell 3D elements.

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