Asociación Argentina de Mecánica Computacional

Mecánica Computacional Vol XXX, págs. 99-115 (artículo completo)
Oscar Möller, Javier W. Signorelli, Mario A. Storti (Eds.)
Rosario, Argentina, 1-4 Noviembre 2011

HARMONIZATION OF THE CLASSICAL TECHNICAL THEORIES FOR THIN AND MODERATELY THICK PLATES

José M. Martínez Jiménez[a], José M. Martínez Valle[a], Alvaro Martínez Valle[a] and Pilar Martínez Jiménez[b]

[a]*Mechanics Department, EPS; Leonardo da Vinci Building, Rabanales Campus, Cordoba University, 14071, Cordoba, Spain, me1majij@uco.es, jmvalle@uco.es*

[b]*Applied Physics Department, EPS, Albert Einstein Building, Rabanales Campus, Cordoba University, 14071, Cordoba, Spain, fa1majip@uco.es*

Keywords: Elasticity, thick plates.

Abstract. This paper presents a new mathematical development, which allows to harmonize the technical calculation theories for moderately thick plates (first order shear deformation theory), in order to obtain a unique set of general governing equations, resulting in a great simplification in the theoretical development, while providing equally precise solutions as the classical theories. The obtained equations represent a major advantage for university level teaching since they simplify the theoretical development.

1 INTRODUCTION: CLASSICAL TECHNICAL THEORIES FOR PLATES

Love (1944) indicated in his original treatise *A Treatise on the Mathematical Theory of Elasticity*: "the Mathematical Theory of Elasticity is occupied with an attempt to reduce to calculation the state of strain, or relative displacement, within a solid body which is subject to the action of an equilibrating system of forces, or is in a state of slight internal relative motion, and with endeavours to obtain results which shall be practically important in applications to architecture, engineering, and all other useful arts in which the material of construction is solid".

The theory for bending of thin plates is summarized in the biharmonic equation

$$\Delta\Delta w = \frac{P}{D} \tag{1}$$

where D is a so-called cylindrical (flexural) rigidity of the plate, w is the function of the lateral displacements of the plate's points and P the intensity of a load distributed over the area of the plate's median surface acting in the direction of the Z-axis (see Fig. 1).

The theory for bending of moderately thick plates was studied by Michell and Love (1994), but only some particular problems were tackled with it.

Faithful to Love's thought, Bolle (1947), Reissner (1945), Mindlin (1951), B.F.Vlasov (1949,1958), Hencky (1947) and Reismann (1980) elaborated their technical calculation theories (first order shear deformation theory), with the objective of widening the application field of plates. The theory for bending of thick plates (second order plate theories) is studied by Donnell LL. H. (1976), Kromm (1953), Panc, V. (1975), Muhammad, A.K. (1990), Voyiadjis, G. Z. (1990), Kienzler, R. (2004) and Meenen J. (2004). Eisenberger (2004) studies the natural frequencies of moderately thick plates basing his work on the hypotheses made by B.F. Vlasov, though he does not mention this fact expressly.

Generally, all of those theories are characterised by the high level of mathematical complexity required to obtain solutions. Furthermore, the problems which are solved analytically only constitute several specific examples.

A rectangular plate is usually considered thin if its thickness is lower than a tenth of the minor dimension. When this limitation is not fulfilled one is entering the field of moderately thick plates (term introduced by Love (1944)), or thick plates. In the following sections of this paper the Analysis Theories for moderately thick plates will be associated with the theories by Bolle, Reissner, Mindlin, Vlasov and Reismann, which can be called first order shear theory, understanding that this is sufficient for practical purposes in a lot of cases.

Reissner (1945) suggests a correction of the previously mentioned equation in the following way:

$$D\cdot\Delta\Delta w = P - \frac{h^2(2-\mu)}{10(1-\mu)}\Delta P\,, \tag{2}$$

where h is the thickness of the plate and μ is a correction factor.

This equation is valid for thick plates. and was obtained through variational calculation introducing the terms corresponding to deformation by shear effort into the deformation energy, which obviously coincides with the first one if h (thickness) is sufficiently small.

Simultaneously, Bolle (1947) comes up with the two following equations for thick plates following a different approach, but taking into account similar hypotheses:

$$D\cdot\Delta\Delta w = P - \frac{2h^2}{10(1-\mu)}\Delta P \,,\quad \Delta\alpha = \frac{10}{h^2}\alpha \,,$$

where (3)

$$\alpha = -\frac{1}{2}\cdot\left(\frac{\partial \vartheta_y}{\partial y} + \frac{\partial \vartheta_x}{\partial x}\right)$$

and ϑ_x y ϑ_y are the rotations of the normal to the middle surface. Mindlin (1951) operates in a similar way to Bole, but whereas the former adopts a parabolic distribution of the shearing stresses depending on the thickness, the latter assumes that it is constant. The complete system of governing differential equations is the following.

$$\begin{gathered}
\frac{\partial \vartheta_y}{\partial x} - \frac{\partial \vartheta_x}{\partial y} + \Delta w = -\frac{12\cdot(1+\mu)}{5\cdot E\cdot h}\cdot P \\
\Delta\vartheta_x - \frac{(1+\mu)}{2}\frac{\partial}{\partial x}\left(\frac{\partial \vartheta_y}{\partial y} + \frac{\partial \vartheta_x}{\partial x}\right) = \frac{5\cdot(1-\mu)}{h^2}\cdot\left(\vartheta_x - \frac{\partial w}{\partial y}\right) \quad (4) \\
\Delta\vartheta_y - \frac{(1+\mu)}{2}\frac{\partial}{\partial y}\left(\frac{\partial \vartheta_x}{\partial x} + \frac{\partial \vartheta_y}{\partial y}\right) = \frac{5\cdot(1-\mu)}{h^2}\cdot\left(\vartheta_y + \frac{\partial w}{\partial x}\right)
\end{gathered}$$

Furthermore, B.F. Vlasov (1949,1958) proposes also another major correction assuming in addition that the normal element of the plate is bent in such a way that the shearing strains in the thickness of the plate vary according to parabolic law. This author proposes the following system of differential equation in order to solve the problem.

$$\begin{gathered}
\frac{\partial \vartheta_y}{\partial x} - \frac{\partial \vartheta_x}{\partial y} + \Delta w = -\frac{3}{2\cdot G\cdot h}\cdot P \\
\Delta\vartheta_x - \frac{(1+\mu)}{2}\frac{\partial}{\partial x}\left(\frac{\partial \vartheta_y}{\partial y} + \frac{\partial \vartheta_x}{\partial x}\right) + \frac{1}{4}\cdot\frac{\partial}{\partial y}(\Delta w) = \frac{5\cdot(1-\mu)}{h^2}\cdot\left(\vartheta_x - \frac{\partial w}{\partial y}\right) \\
\quad (5) \\
\Delta\vartheta_y - \frac{(1+\mu)}{2}\frac{\partial}{\partial y}\left(\frac{\partial \vartheta_x}{\partial x} + \frac{\partial \vartheta_y}{\partial y}\right) - \frac{1}{4}\cdot\frac{\partial}{\partial x}(\Delta w) = \frac{5\cdot(1-\mu)}{h^2}\cdot\left(\vartheta_y + \frac{\partial w}{\partial x}\right)
\end{gathered}$$

This system corresponds to the one expressed by Zienkiewicz (1994), which is the basis for the formulation of elements such as Quadratic Serendipity elements (QS) and Langrangian elements (QL).

Reismann H.and Pawlik (1980) in his work *Elasticity: Theory and applications,* also assuming the inclusion of the terms corresponding to the deformation by shear effort in the deformation energy, arrives at

$$D\cdot\Delta M = -P \,,\quad \Delta w = -M - \frac{P}{\kappa^2\cdot G\cdot h}$$

where (6)

$$M = M_x + M_y$$

and the coefficient $\frac{1}{\kappa^2}$, which is determined by kinematic considerations, is $\frac{1}{\kappa^2} = 1.16$ for $\mu = 0.3$.

With respect to the dynamic analysis of plates, Eisenberger (2004) compares the natural frequencies obtained in the case of a plate in which h=a/10 in his article *Dynamic Stiffness vibration analysis for higher order plate models*. The results can be seen in the following table:

m	n	CPT	FSDT	HDT
1	1	0.0963	0.0930	0.0931
3	1	0.4816	0.4149	0.4158
2	2	0.3853	0.3406	0.3411
3	3	0.8669	0.6834	0.6862

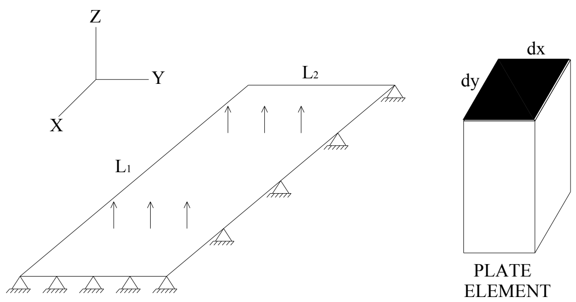
CPT: classical plate theory; FSDT: first order shear deformation theory (Mindlin´s formulation); HDT: higher order plate model (Vlasov´s formulation)

Table 1: Four normalized natural frequencies for example plate

2 HARMONIZATION HYPOTHESIS OF THE TECHNICAL THEORIES OF THIN-THICK PLATES WITH A CONSTANT THICKNESS

As it will be seen in the following section the technical calculation equations rise in complexity as the hypotheses tend to consider the implications due to the shear phenomenon. Nevertheless, the value of the rotation around oz-axis in the different theories should be appealed, (see fig. 1). In the one of thin plates it is zero, and in those used by Bolle-Reissner (1947) and B.F. Vlasov (1949,1958) that can be seen for the lower upper side of the plate element $\left(z = \pm h/2\right)$, one obtains equal and opposite rotations, which would imply an angular distortion throughout the thickness which is not in accordance with the deformation of the plate.

Under the term harmonization it is understood pooling the technical theories of the first order shear deformation, i.e..; to complement the respective hypotheses of these theories in such form that an unique system of governing equations may be obtained by transforming the equations of each case.



$$w_{xy} = 0$$
Kirchhoff

$$w_{xy} = -\frac{1}{2}\cdot\left(\frac{\partial\vartheta_y}{\partial y} + \frac{\partial\vartheta_x}{\partial x}\right)\cdot z$$
Bolle−Reissner

$$w_{xy} = -\frac{z}{2}\cdot\left(\frac{\partial\vartheta_y}{\partial y} + \frac{\partial\vartheta_x}{\partial x}\right)\cdot\left(1 - \frac{4\cdot z^2}{3\cdot h^2}\right)$$
B.F. Vlasov

Figure 1: Different theories of plates

On analyzing the three expressions of the rotation ω_{xy} around oz-axis, it can be seen that harmonization is possible if

$$\frac{\partial\vartheta_y}{\partial y} + \frac{\partial\vartheta_x}{\partial x} = 0 \qquad (7)$$

is postulated, being consequently the rotation ω_{xy} null in any case.

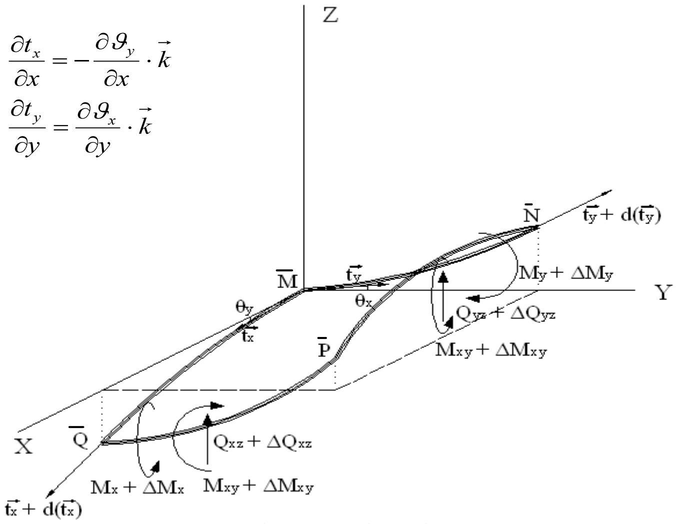


Figure 2: Plate element

Besides to obey to an intuition of the deformation of the plate, the fact that this equation is obtained if the approach of the equilibrium equations of the plate element is done in the deformed configuration, which corresponds to the reality that the equilibrium is really achieved in the bent plate, supports this new hypothesis.

In figure 2 and 3 an element of bent plate ($\bar{M}$,$\bar{N}$,$\bar{P}$,$\bar{Q}$) is shown submitted to bending and membrane efforts (which appear separately for reasons of clarity). In these figures the forces on the sides, which are not seen, have been omitted. Also, the equations:

$$\frac{\partial \boldsymbol{t}_x}{\partial x} = -\frac{\partial \theta_y}{\partial x}\boldsymbol{k}$$
$$\frac{\partial \boldsymbol{t}_y}{\partial y} = \frac{\partial \theta_x}{\partial y}\boldsymbol{k} \tag{8}$$

are easily deduced taking into account Frenet formulae (Quesada (1996)).

On raising the equilibrium equations of Statics for the element

$$\begin{aligned} &\sum \bar{F} = \bar{0}\,;\ \sum \bar{F}_{\bar{Q}\bar{M}} + \sum \bar{F}_{\bar{M}\bar{N}} + \sum \bar{F}_{\bar{P}\bar{N}} + \sum \bar{F}_{\bar{P}\bar{Q}} + \sum \bar{F}_{ext} = \bar{0} \\ &\sum \bar{M} = \bar{0}\,;\ \sum \bar{M}_{\bar{Q}\bar{M}} + \sum \bar{M}_{\bar{M}\bar{N}} + \sum \bar{M}_{\bar{P}\bar{N}} + \sum \bar{F}_{\bar{P}\bar{Q}} + \sum \bar{M}_{ext} = \bar{0} \end{aligned} \tag{9}$$

of which, as an example, the vectorial equation corresponding to the moments which act on edges $\bar{P}\bar{N}$ and $\bar{Q}\bar{M}$ has been presented.

$$\sum \vec{M}_{\overline{PN}y\,QM} = \left[-M_y \cdot \vec{t}_x - d\left(M_y \cdot \vec{t}_x\right) + M_y \cdot \vec{t}_x + M_{xy} \cdot \vec{t}_y + d\left(M_{xy} \cdot \vec{t}_y\right) - M_{xy} \cdot \vec{t}_y + Q_{yz} \cdot dy \cdot \vec{t}_x\right] dx \tag{10}$$

On differentiating and considering the value of the unitary tangent vector derivatives $\vec{t}_x$ and $\vec{t}_y$, a total of six scalar equations are obtained. The two scalar equations corresponding to $\sum F_x = 0, \sum F_y = 0$ provide the equilibrium conditions of the membrane efforts contained in the mid surface:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0\,;\ \frac{\partial N_{yx}}{\partial x} + \frac{\partial N_y}{\partial y} = 0\,. \tag{11}$$

The equation $\sum F_z = 0$, in which the membrane efforts contained in the mid surface are disregarded, and the two equations corresponding to the moments respect to x, y axes coincide with the equilibrium equations obtained in the non-deformed initial geometry, being

$$\frac{\partial Q_{xz}}{\partial x} + \frac{\partial Q_{yz}}{\partial y} + P = 0 \; ; \; \frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} = Q_{xz} \; ; \; \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} = Q_{yz} \tag{12}$$

The equation relative to the moments respect to z-axis provides us:

$$M_x \cdot \frac{\partial \vartheta_x}{\partial x} + M_y \cdot \frac{\partial \vartheta_y}{\partial y} + M_{xy} \cdot \left(\frac{\partial \vartheta_y}{\partial x} + \frac{\partial \vartheta_x}{\partial y} \right) = 0 \tag{13}$$

However, if M_x, M_y and M_{xy} are substituted with their values in the last equation, given e.g. by Bolle-Reissner theory, it becomes to

$$\frac{D \cdot (1+\mu)}{2} \cdot \left(\frac{\partial \vartheta_x}{\partial x} + \frac{\partial \vartheta_y}{\partial y} \right) \cdot \left(\frac{\partial \vartheta_y}{\partial x} - \frac{\partial \vartheta_x}{\partial y} \right) = 0 \tag{14}$$

Noticing that:

$$\left(\frac{\partial \vartheta_y}{\partial x} - \frac{\partial \vartheta_x}{\partial y} \right) = \frac{1}{D \cdot (1+\mu)} \cdot (M_x + M_y) = \frac{M}{D} \tag{15}$$

cannot be null for every point of the plate, the following expression is obtained

$$\left(\frac{\partial \vartheta_x}{\partial x} + \frac{\partial \vartheta_y}{\partial y} \right) = 0 \tag{16}$$

The magnitude $\frac{M_x + M_y}{(1+\mu)}$ is called *moment addition*.

3 THEORY OF THIN PLATES

The specific hypotheses according to Kirchhoff are the following ones:

1.- Any fibre pertaining to the plate which is parallel to z axis, and thus perpendicular to the middle surface before the deformation, is marked perpendicular to the deformed middle surface ($\gamma_{yz} \cong 0, \gamma_{xz} \cong 0$).

2.- Any surface which is parallel to the middle surface of the plate before the deformation continues to be parallel to the deformed middle surface, that is to say that the relative distance between each other continues being the same ($\varepsilon_z \cong 0$).

3.- The loads will be perpendicular to the middle surface and the displacements of the points located in the middle surface go in the direction of the loads that produce them (inelastic middle surface).

Consequently, the parallel components to x,y axes of the displacement of a generic point, are defined by

$$u = -\frac{\partial w}{\partial x} \cdot z \; ; \; v = -\frac{\partial w}{\partial y} \cdot z \, . \tag{17}$$

The stresses will be expressed by means of

$$\sigma_x = -\frac{E\cdot z}{1-\mu^2}\cdot\left(\frac{\partial^2 w}{\partial x^2}+\mu\cdot\frac{\partial^2 w}{\partial y^2}\right);\ \sigma_y = -\frac{E\cdot z}{1-\mu^2}\cdot\left(\frac{\partial^2 w}{\partial y^2}+\mu\cdot\frac{\partial^2 w}{\partial x^2}\right);\ \sigma_z \cong 0;\ \tau_{xy} = -2\cdot G\cdot z\cdot\frac{\partial^2 w}{\partial x\partial y} \qquad (18)$$

and the moment stress resultants and transverse shear forces are expressed as

$$M_x = -D\cdot\left(\frac{\partial^2 w}{\partial x^2}+\mu\cdot\frac{\partial^2 w}{\partial y^2}\right);\ M_y = -D\cdot\left(\frac{\partial^2 w}{\partial y^2}+\mu\cdot\frac{\partial^2 w}{\partial x^2}\right);\ M_{xy} = -\frac{G\cdot h^3}{6}\cdot\frac{\partial^2 w}{\partial x\partial y} \qquad (19)$$

$$Q_{xz} = -\frac{G\cdot h^3}{6}\cdot\frac{\partial^3 w}{\partial x\partial y^2} - D\cdot\left(\frac{\partial^3 w}{\partial x^3}+\mu\cdot\frac{\partial^3 w}{\partial x\partial y^2}\right);\ Q_{yz} = -\frac{G\cdot h^3}{6}\cdot\frac{\partial^3 w}{\partial x^2\partial y} - D\cdot\left(\frac{\partial^3 w}{\partial y^3}+\mu\cdot\frac{\partial^3 w}{\partial x^2\partial y}\right)$$

The rigidity constant of the plates, D, is $= -\frac{E\cdot h^3}{12\cdot(1-\mu^2)}$

The *governing equation,* well-known as Lagrange equation, is obtained from the equilibrium of the plate element

$$\Delta\Delta w = \frac{P}{D} \qquad (20)$$

where Δ is the Laplace operator, $\Delta = \nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$.

If the variable moment addition, defined by. $M = \frac{M_x + M_y}{(1+\mu)}$, is introduced instead of eliminating Q_{xz} and Q_{yz} from the equilibrium equations, it yields

$$\Delta M = -P$$
$$\Delta w = -\frac{M}{D} \qquad (21)$$

This system is presented by Timoshenko for thin plates (1970).

Apart from the exterior transversal loads P, in the stability calculation the bending forces and moments (see Fig. 2) and the forces contained in the middle surface of the plate, N (see Fig. 3), are taken into account, which may have a great influence on the bending phenomenon. Their influence is revealed when considering the equilibrium of the plate in its deformed geometry.

The equation obtained assuming $\sum F_z = 0$, is

$$\frac{\partial^4 w}{\partial x^4}+2\cdot\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4} = \frac{1}{D}\left(P + N_x\cdot\frac{\partial^2 w}{\partial x^2}+N_y\cdot\frac{\partial^2 w}{\partial y^2}+2\cdot N_{xy}\cdot\frac{\partial^2 w}{\partial x\partial y}\right) \qquad (22)$$

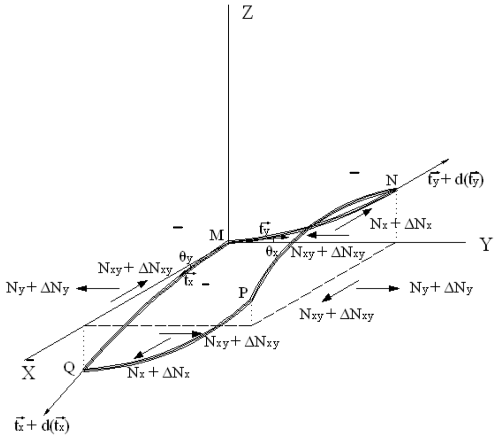


Figure 3: Plate element. Thin plates

The behaviour of thin plates after the loss of stability, for deflections *w* comparable with the thickness *h* of the plate, is analyzed with the following Karman equations

$$\frac{D}{h}\cdot\left(\frac{\partial^4 w}{\partial x^4}+2\cdot\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)=\frac{\partial^2\varphi}{\partial y^2}\cdot\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2\varphi}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2}-2\cdot\frac{\partial^2\varphi}{\partial x\partial y}\cdot\frac{\partial^2 w}{\partial x\partial y}$$

$$\frac{\partial^4\varphi}{\partial x^4}+2\cdot\frac{\partial^4\varphi}{\partial x^2\partial y^2}+\frac{\partial^4\varphi}{\partial y^4}=E\left(\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2}\right) \qquad (23)$$

while φ is the stress function which complies with

$$\frac{N_x}{h}=\frac{\partial^2\varphi}{\partial y^2};\ \frac{N_y}{h}=\frac{\partial^2\varphi}{\partial x^2};\ \frac{N_{xy}}{h}=-\frac{\partial^2\varphi}{\partial x\partial y} \qquad (24)$$

4 TECHNICAL THEORY FOR THICK PLATES BY BOLLE-REISSNER AND ITS HARMONIZATION

The study of moderately thick rectangular plates implies an awareness of the influence of the shear phenomenon. The theory which Bolle (1947) - Reissner (1945) suggested is based on the assumption of a parabolic distribution of the shearing stresses throughout the thickness. Thus, taking into account that the dimensions of the middle surface of the plate do not vary, the authors suggest a modification of the hypothesis relative to the *fibres of the plate perpendicular to the middle surface before the deformation*, in the sense that after the deformation the fibre which was perpendicular to the middle surface continues being straight, but it does not have to continue being perpendicular. As a consequence the shearing strains γ_{yz} and γ_{xz} will not be zero. The displacements of a generic point located at a distance z from the middle surface (which is assumed to coincide with the coordinate x or y) are given by

$$u=\vartheta_y\cdot z\ ;\ v=-\vartheta_x\cdot z\ ;\ w=w(x,y) \qquad (25)$$

Additionally a parabolic distribution of the shearing stresses, given by

$$\tau_{xz}=\beta\cdot G\cdot\left(1-\frac{4\cdot z^2}{h^2}\right)\cdot\gamma_{xz}\ ;\ \tau_{yz}=\beta\cdot G\cdot\left(1-\frac{4\cdot z^2}{h^2}\right)\cdot\gamma_{yz} \qquad (26)$$

is postulated along the thickness, where β takes the value 5/4.

The strains can be deduced from the displacements

$$\varepsilon_x = \frac{\partial \vartheta_y}{\partial x} \cdot z \,;\; \varepsilon_y = -\frac{\partial \vartheta_x}{\partial y} \cdot z \,;\; \gamma_{xy} = \left(\frac{\partial \vartheta_y}{\partial y} - \frac{\partial \vartheta_x}{\partial x} \right) \cdot z \,;\; \gamma_{xz} = \vartheta_y + \frac{\partial w}{\partial x} \,;\; \gamma_{yz} = -\vartheta_x + \frac{\partial w}{\partial y} \tag{27}$$

The rotations around the axes are

$$\omega_{xz} = \frac{1}{2} \left(-\vartheta_y + \frac{\partial w}{\partial x} \right) ;\; \omega_{yz} = \frac{1}{2} \left(\vartheta_x + \frac{\partial w}{\partial y} \right) ;\; \omega_{xy} = -\frac{1}{2} \cdot \left(\frac{\partial \vartheta_y}{\partial y} + \frac{\partial \vartheta_x}{\partial x} \right) \cdot z \tag{28}$$

According to the hypotheses, the shearing stresses are provided by

$$\tau_{xz} = \beta \cdot G \cdot \left(1 - \frac{4 \cdot z^2}{h^2} \right) \cdot \left(\vartheta_y + \frac{\partial w}{\partial x} \right) ;\; \tau_{yz} = \beta \cdot G \cdot \left(1 - \frac{4 \cdot z^2}{h^2} \right) \cdot \left(-\vartheta_x + \frac{\partial w}{\partial y} \right) \tag{29}$$

and the normal stresses are given by

$$\sigma_x = \frac{E \cdot z}{1-\mu^2} \cdot \left(\frac{\partial \vartheta_y}{\partial x} - \mu \cdot \frac{\partial \vartheta_x}{\partial y} \right) ;\; \sigma_y = \frac{E \cdot z}{1-\mu^2} \cdot \left(-\frac{\partial \vartheta_x}{\partial y} + \mu \cdot \frac{\partial \vartheta_y}{\partial x} \right) ;\; \sigma_z \cong 0 \,; \tag{30}$$

The moment stress resultants and transverse shear forces are expressed by

$$\begin{gathered} M_x = D \cdot \left(\frac{\partial \vartheta_y}{\partial x} - \mu \cdot \frac{\partial \vartheta_x}{\partial y} \right) ;\; M_y = D \cdot \left(-\frac{\partial \vartheta_x}{\partial y} + \mu \cdot \frac{\partial \vartheta_y}{\partial x} \right) ;\; M_{xy} = -\frac{1-\mu}{2} \cdot D \cdot \left(-\frac{\partial \vartheta_y}{\partial y} + \frac{\partial \vartheta_x}{\partial x} \right) \\ Q_{xz} = \frac{5 \cdot E \cdot h}{12 \cdot (1+\mu)} \cdot \left(\frac{\partial w}{\partial x} + \vartheta_y \right) ;\; Q_{yz} = \frac{5 \cdot E \cdot h}{12 \cdot (1+\mu)} \cdot \left(\frac{\partial w}{\partial y} - \vartheta_x \right) \end{gathered} \tag{31}$$

The governing equations are obtained considering the equilibrium of the plate element and they are as follows

$$\begin{gathered} \frac{\partial \vartheta_y}{\partial x} - \frac{\partial \vartheta_x}{\partial y} + \Delta w = -\frac{12 \cdot (1+\mu)}{5 \cdot E \cdot h} \cdot P \\ \Delta \vartheta_x - \frac{(1+\mu)}{2} \cdot \frac{\partial}{\partial x} \left(\frac{\partial \vartheta_y}{\partial y} + \frac{\partial \vartheta_x}{\partial x} \right) = \frac{5 \cdot (1-\mu)}{h^2} \cdot \left(\vartheta_x - \frac{\partial w}{\partial y} \right) \\ \Delta \vartheta_y - \frac{(1+\mu)}{2} \cdot \frac{\partial}{\partial y} \left(\frac{\partial \vartheta_x}{\partial x} + \frac{\partial \vartheta_y}{\partial y} \right) = \frac{5 \cdot (1-\mu)}{h^2} \cdot \left(\vartheta_y + \frac{\partial w}{\partial x} \right) \end{gathered} \tag{32}$$

The equations (32) are the ones deduced according to the Bolle-Reissner theory (1945). They will be transformed below according to our harmonization hypothesis. In order to do so, the equation (16) is substituted in system (32) and, as a result

$$\begin{gathered} \frac{\partial \vartheta_y}{\partial x} - \frac{\partial \vartheta_x}{\partial y} + \Delta w = -\frac{12 \cdot (1+\mu)}{5 \cdot E \cdot h} \cdot P \\ \Delta \vartheta_x - \frac{5 \cdot (1-\mu)}{h^2} \cdot \left(\vartheta_x - \frac{\partial w}{\partial y} \right) = 0 \\ \Delta \vartheta_y - \frac{5 \cdot (1-\mu)}{h^2} \cdot \left(\vartheta_y + \frac{\partial w}{\partial x} \right) = 0 \end{gathered} \tag{33}$$

which will be transformed applying the Laplace operator in the first equation of (33)

$$\Delta\left(\frac{\partial \vartheta_y}{\partial x}-\frac{\partial \vartheta_x}{\partial y}\right)+\Delta\Delta w=-\frac{12\cdot(1+\mu)}{5\cdot E\cdot h}\cdot\Delta P \tag{34}$$

or

$$\frac{\partial}{\partial x}\left(\Delta\vartheta_y\right)-\frac{\partial}{\partial y}\left(\Delta\vartheta_x\right)+\Delta\Delta w=-\frac{12\cdot(1+\mu)}{5\cdot E\cdot h}\cdot\Delta P \tag{35}$$

Now, for the second and third one it is found

$$\frac{\partial}{\partial x}\left(\Delta\vartheta_y\right)-\frac{\partial}{\partial y}\left(\Delta\vartheta_x\right)=\frac{5\cdot(1-\mu)}{h^2}\cdot\left(\frac{\partial \vartheta_y}{\partial x}+\frac{\partial^2 w}{\partial x^2}-\frac{\partial \vartheta_x}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)=-\frac{5\cdot(1-\mu)}{h^2}\cdot\frac{12\cdot(1+\mu)}{5\cdot E\cdot h}\cdot P=-\frac{P}{D} \tag{36}$$

and making the substitution in (34) one obtains

$$\Delta\Delta w=\frac{P}{D}-\frac{12\cdot(1+\mu)}{5\cdot E\cdot h}\cdot\Delta P \tag{37}$$

This leads to a harmonized governing equation system decoupled in displacements-rotations which is

$$\Delta\Delta w=\frac{P}{D}-\frac{12\cdot(1+\mu)}{5\cdot E\cdot h}\cdot\Delta P\ ;\quad \Delta\vartheta_x-\frac{5\cdot(1-\mu)}{h^2}\cdot\vartheta_x=-\frac{5\cdot(1-\mu)}{h^2}\cdot\frac{\partial w}{\partial y}\ ;\quad \Delta\vartheta_y-\frac{5\cdot(1-\mu)}{h^2}\cdot\vartheta_y=\frac{5\cdot(1-\mu)}{h^2}\cdot\frac{\partial w}{\partial x} \tag{38}$$

which constitutes a partial differential equation system of general character, with the unknown functions (w , ϑ_x , ϑ_y) decoupled, for the linear study of plates. From (31) it may be formed

$$\frac{\partial \vartheta_y}{\partial x}-\frac{\partial \vartheta_x}{\partial y}=\frac{M_x+M_y}{D\cdot(1+\mu)}=\frac{M}{D} \tag{39}$$

After that, this last expression can be replaced in (33) and (36)

$$\begin{gathered}\frac{\partial \vartheta_y}{\partial x}-\frac{\partial \vartheta_x}{\partial y}+\Delta w=\frac{M}{D}+\Delta w=-\frac{12\cdot(1+\mu)}{5\cdot E\cdot h}\cdot P\\ \frac{\partial}{\partial x}\left(\Delta\vartheta_y\right)-\frac{\partial}{\partial y}\left(\Delta\vartheta_x\right)=\Delta\left(\frac{\partial \vartheta_y}{\partial x}-\frac{\partial \vartheta_x}{\partial y}\right)=\frac{\Delta M}{D}=-\frac{P}{D}\end{gathered} \tag{40}$$

and it yields

$$\begin{gathered}\Delta M=-P\\ \Delta w=-\frac{M}{D}-\frac{6}{5\cdot G\cdot h}\cdot P\end{gathered} \tag{41}$$

In order to know the behaviour of plates for deflections *w,* which are comparable with the plate thickness *h*, the equilibrium of the plate is raised in its deformed geometry and taking some more exact expressions into account for the strains, in which the values of the displacements given in (25) are substituted

$$\begin{gathered}\varepsilon_x=\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2=z\cdot\frac{\partial \vartheta_y}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\\ \varepsilon_y=\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2=-z\cdot\frac{\partial \vartheta_x}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\end{gathered} \tag{42}$$

$$\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}=\left(\frac{\partial \vartheta_y}{\partial y}-\frac{\partial \vartheta_x}{\partial x}\right)\cdot z+\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}$$

The expression $\frac{\partial^2 \varepsilon^{pm}{}_x}{\partial y^2}+\frac{\partial^2 \varepsilon^{pm}{}_y}{\partial x^2}-\frac{\partial^2 \gamma^{pm}{}_{xy}}{\partial y \partial y}$ might be formed with the strains for the points in the middle surface (z=0), which are distinguished by the superscript *pm*, and obtain

$$\frac{\partial^2 \varepsilon^{pm}{}_x}{\partial y^2}+\frac{\partial^2 \varepsilon^{pm}{}_y}{\partial x^2}-\frac{\partial^2 \gamma^{pm}{}_{xy}}{\partial y \partial y}=\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} \tag{43}$$

The first member can be expressed in function of the axial and shear forces contained in the middle surface: N_x, N_y, N_{xy} as

$$\frac{\partial^2 \varepsilon^{pm}{}_x}{\partial y^2}+\frac{\partial^2 \varepsilon^{pm}{}_y}{\partial x^2}-\frac{\partial^2 \gamma^{pm}{}_{xy}}{\partial y \partial y}=\frac{1}{E\cdot h}\left(\frac{\partial^2 N_x}{\partial y^2}-\mu\cdot\left(\frac{\partial^2 N_x}{\partial x^2}+\frac{\partial^2 N_y}{\partial x^2}\right)+\frac{\partial^2 N_y}{\partial x^2}\right) \tag{44}$$

On defining the stress function φ with

$$\frac{N_x}{h}=\frac{\partial^2 \varphi}{\partial y^2};\ \frac{N_y}{h}=\frac{\partial^2 \varphi}{\partial x^2};\ \frac{N_{xy}}{h}=-\frac{\partial^2 \varphi}{\partial x \partial y} \tag{45}$$

it yields

$$\frac{\partial^4 \varphi}{\partial x^4}+2\cdot\frac{\partial^4 \varphi}{\partial x^2 \partial y^2}+\frac{\partial^4 \varphi}{\partial y^4}=E\left(\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2}\right) \tag{46}$$

together with the equilibrium equation $\sum F_z=0$, which provides us the following equation

$$\frac{D}{h}\cdot\left(\frac{\partial^4 w}{\partial x^4}+2\cdot\frac{\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)=\frac{\partial^2 \varphi}{\partial y^2}\cdot\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 \varphi}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2}-2\cdot\frac{\partial^2 \varphi}{\partial x \partial y}\cdot\frac{\partial^2 w}{\partial x \partial y} \tag{47}$$

constitute the governing system of Karman given in (23) for thin plates. This means that the behaviour of plates for deflections *w*, which are comparable with the thickness *h* of the plate, is given by the same Karman equation system for both thin and moderately thick plates.

5 TECHNICAL THEORY FOR THICK BY B. F. VLASOV AND ITS HARMONIZATION

As regards the previous theory, this author suggests modifying the hypothesis relative to the fibre which is perpendicular to the middle surface, in the sense that after the deformation the fibre which was perpendicular to the middle surface does not continue being perpendicular and does not lengthen $(\varepsilon_z=0)$ but it does bend. Additionally, $\sigma_z=0$ is postulated for the thickness and a parabolic distribution throughout the thickness for the shearing strains defined by

$$\gamma_{xz}=\left(1-\frac{4\cdot z^2}{h^2}\right)\hat{\gamma}_{xz}\ ;\ \gamma_{yz}=\left(1-\frac{4\cdot z^2}{h^2}\right)\hat{\gamma}_{yz}, \tag{48}$$

with $\hat{\gamma}_{xz}$ and $\hat{\gamma}_{yz}$ being the shearing strains in the points of the middle surface.

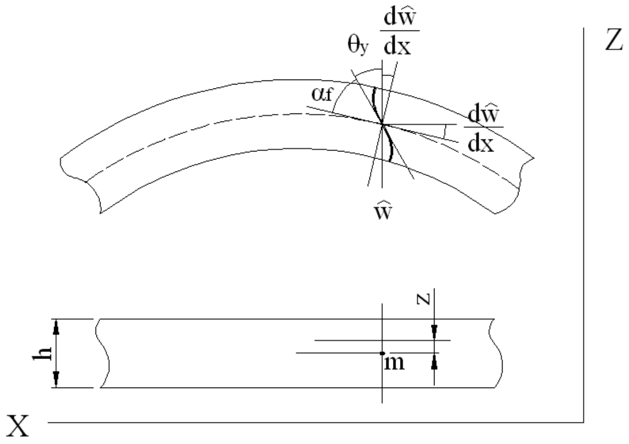


Figure 4: Bending of plates

The angle rotated by the rectilinear segment normal to the middle surface around ox-axis is called ϑ_x and the one around oy-axis is called ϑ_y, figure 4.

The shearing strains in the xz, yz surfaces, in the point *m* located on the middle surface, are

$$\hat{\gamma}_{xz} = \vartheta_y + \frac{\partial \hat{w}}{\partial x} \text{ , } \hat{\gamma}_{yz} = -\vartheta_x + \frac{\partial \hat{w}}{\partial y} \text{ .} \tag{49}$$

They can also be written

$$\hat{\gamma}_{xz} = \frac{\hat{\tau}_{xz}}{G} \text{ and } \hat{\gamma}_{yz} = \frac{\hat{\tau}_{yz}}{G} \tag{50}$$

being $\hat{\tau}_{xz}$ and $\hat{\tau}_{yz}$ the shearing stresses in the points located on the middle surface. Therefore the shearing stresses and strains in the points of the middle surface are

$$\frac{\hat{\tau}_{xz}}{G} = \vartheta_y + \frac{\partial w}{\partial x} \text{ ; } \frac{\hat{\tau}_{yz}}{G} = -\vartheta_x + \frac{\partial w}{\partial y} \text{ ; } \hat{\gamma}_{xz} = \vartheta_y + \frac{\partial w}{\partial x} \text{ ; } \hat{\gamma}_{yz} = -\vartheta_x + \frac{\partial w}{\partial y} \tag{51}$$

The shearing strains in a generic point are

$$\gamma_{xz} = \vartheta_y + \frac{\partial w}{\partial x} - \frac{4 \cdot z^2}{h^2} \cdot \frac{\hat{\tau}_{xz}}{G} \text{ , } \gamma_{yz} = = -\vartheta_x + \frac{\partial w}{\partial y} - \frac{4 \cdot z^2}{h^2} \cdot \frac{\hat{\tau}_{yz}}{G} \tag{52}$$

and the displacements

$$u = \vartheta_y \cdot z - \frac{4 \cdot z^3}{3 \cdot h^2} \cdot \frac{\hat{\tau}_{xz}}{G} \text{ ; } v = -\vartheta_x \cdot z - \frac{4 \cdot z^3}{3 \cdot h^2} \cdot \frac{\hat{\tau}_{yz}}{G} \text{ ; } w = w(x, y). \tag{53}$$

The rotations around the axes are

$$\omega_{xz} = -\frac{1}{2} \cdot \left(\vartheta_y - \frac{4 \cdot z^2}{h^2} \cdot \frac{\hat{\tau}_{xz}}{G} - \frac{\partial w}{\partial x} \right)$$
$$\omega_{yz} = -\frac{1}{2} \cdot \left(-\vartheta_x - \frac{4 \cdot z^2}{h^2} \cdot \frac{\hat{\tau}_{yz}}{G} - \frac{\partial w}{\partial y} \right) \tag{54}$$

$$\omega_{xy} = -\frac{z}{2}\cdot\left(\frac{\partial\vartheta_y}{\partial y}+\frac{\partial\vartheta_x}{\partial x}\right)\cdot\left(1-\frac{4\cdot z^2}{3\cdot h^2}\right)$$

The rest of the strains are

$$\varepsilon_x = \frac{\partial\vartheta_y}{\partial x}\cdot z - \frac{4\cdot z^3}{3\cdot G\cdot h^2}\cdot\frac{\partial\hat{\tau}_{xz}}{\partial x}\;;\;\varepsilon_y = -\frac{\partial\vartheta_x}{\partial y}\cdot z - \frac{4\cdot z^3}{3\cdot G\cdot h^2}\cdot\frac{\partial\hat{\tau}_{yz}}{\partial y}$$
$$\gamma_{xy} = -\frac{\partial\vartheta_x}{\partial x}\cdot z - \frac{4\cdot z^3}{3\cdot G\cdot h^2}\cdot\frac{\partial\hat{\tau}_{yz}}{\partial x} + \frac{\partial\vartheta_y}{\partial y}\cdot z - \frac{4\cdot z^3}{3\cdot G\cdot h^2}\cdot\frac{\partial\hat{\tau}_{xz}}{\partial y} = \left(z-\frac{4\cdot z^3}{3\cdot h^2}\right)\cdot\left(\frac{\partial\vartheta_y}{\partial y}+\frac{\partial\vartheta_x}{\partial x}\right)-\frac{8\cdot z^3}{3\cdot h^2}\cdot\frac{\partial^2 w}{\partial x\partial y} \tag{55}$$

The stresses are defined by Hooke's Law

$$\sigma_x = -\frac{E\cdot z}{1-\mu^2}\cdot\left(-\frac{\partial\vartheta_y}{\partial x}+\mu\cdot\frac{\partial\vartheta_x}{\partial y}\right)-\frac{4\cdot z^3}{3\cdot G\cdot h^2}\cdot\frac{E}{1-\mu^2}\cdot\left(\frac{\partial\hat{\tau}_{xz}}{\partial x}+\mu\cdot\frac{\partial\hat{\tau}_{yz}}{\partial y}\right)$$
$$\sigma_y = -\frac{E\cdot z}{1-\mu^2}\cdot\left(\frac{\partial\vartheta_x}{\partial y}-\mu\cdot\frac{\partial\vartheta_y}{\partial x}\right)-\frac{4\cdot z^3}{3\cdot G\cdot h^2}\cdot\frac{E}{1-\mu^2}\cdot\left(\frac{\partial\hat{\tau}_{yz}}{\partial y}+\mu\cdot\frac{\partial\hat{\tau}_{xz}}{\partial x}\right)$$
$$\sigma_z \cong 0$$
$$\tau_{xy} = G\cdot\left(z-\frac{4\cdot z^3}{3\cdot h^2}\right)\cdot\left(\frac{\partial\vartheta_y}{\partial y}+\frac{\partial\vartheta_x}{\partial x}\right)-\frac{8\cdot G\cdot z^3}{3\cdot h^2}\cdot\frac{\partial^2 w}{\partial x\partial y}\;;\;\tau_{xz} = \left(1-\frac{4\cdot z^2}{h^2}\right)\cdot\hat{\tau}_{xz}\;;\;\tau_{yz} = \left(1-\frac{4\cdot z^2}{h^2}\right)\cdot\hat{\tau}_{yz} \tag{56}$$

The moment stress resultants and transverse shear forces are expressed by

$$M_x = -D\cdot\left(-\frac{\partial\vartheta_y}{\partial x}+\mu\cdot\frac{\partial\vartheta_x}{\partial y}\right)-\frac{D}{5\cdot G}\cdot\left(\frac{\partial\hat{\tau}_{xz}}{\partial x}+\mu\cdot\frac{\partial\hat{\tau}_{yz}}{\partial y}\right);\;M_y = -D\cdot\left(\frac{\partial\vartheta_x}{\partial y}-\mu\cdot\frac{\partial\vartheta_y}{\partial x}\right)-\frac{D}{5\cdot G}\cdot\left(\frac{\partial\hat{\tau}_{yz}}{\partial y}+\mu\cdot\frac{\partial\hat{\tau}_{xz}}{\partial x}\right)$$
$$M_{xy} = -\frac{1-\mu}{2}\cdot D\cdot\left(-\frac{\partial\vartheta_y}{\partial y}+\frac{\partial\vartheta_x}{\partial x}\right)-\frac{(1-\mu)D}{10\cdot G}\cdot\left(\frac{\partial\hat{\tau}_{xz}}{\partial y}+\mu\cdot\frac{\partial\hat{\tau}_{yz}}{\partial x}\right)$$
$$Q_{xz} = \frac{2\cdot h}{3}\cdot\hat{\tau}_{xz} = \frac{2\cdot h\cdot G}{3}\cdot\left(\vartheta_y+\frac{\partial w}{\partial x}\right);\quad Q_{yz} = \frac{2\cdot h}{3}\cdot\hat{\tau}_{yz} = \frac{2\cdot h\cdot G}{3}\cdot\left(-\vartheta_x+\frac{\partial w}{\partial y}\right) \tag{57}$$

The *governing equations* are obtained when considering the equilibrium of the plate element and they are

$$\frac{\partial\vartheta_y}{\partial x}-\frac{\partial\vartheta_x}{\partial y}+\Delta w = -\frac{3}{2\cdot G\cdot h}\cdot P$$
$$\Delta\vartheta_x-\frac{(1+\mu)}{2}\frac{\partial}{\partial x}\left(\frac{\partial\vartheta_y}{\partial y}+\frac{\partial\vartheta_x}{\partial x}\right)+\frac{1}{4}\cdot\frac{\partial}{\partial y}(\Delta w) = \frac{5\cdot(1-\mu)}{h^2}\cdot\left(\vartheta_x-\frac{\partial w}{\partial y}\right) \tag{58}$$
$$\Delta\vartheta_y-\frac{(1+\mu)}{2}\frac{\partial}{\partial y}\left(\frac{\partial\vartheta_x}{\partial x}+\frac{\partial\vartheta_y}{\partial y}\right)-\frac{1}{4}\cdot\frac{\partial}{\partial x}(\Delta w) = \frac{5\cdot(1-\mu)}{h^2}\cdot\left(\vartheta_y+\frac{\partial w}{\partial x}\right)$$

Comparing system (58) with system (32) the difference in the terms $\frac{1}{4}\cdot\frac{\partial}{\partial y}(\Delta w)$ can be noticed, being these terms consequence of the first hypothesis. On substituting (16) in system (58) a harmonized system similar to the one in (33) is found

$$\frac{\partial\vartheta_y}{\partial x}-\frac{\partial\vartheta_x}{\partial y}+\Delta w = -\frac{3}{2\cdot G\cdot h}\cdot P$$
$$\Delta\vartheta_x+\frac{1}{4}\cdot\frac{\partial}{\partial y}(\Delta w) = \frac{5\cdot(1-\mu)}{h^2}\cdot\left(\vartheta_x-\frac{\partial w}{\partial y}\right) \tag{59}$$

$$\Delta\vartheta_y - \frac{1}{4}\cdot\frac{\partial}{\partial x}(\Delta w) = \frac{5(1-\mu)}{h^2}\left(\vartheta_y + \frac{\partial w}{\partial x}\right)$$

applying the Laplace operator to the first equation, it is transformed

$$\Delta\left(\frac{\partial\vartheta_y}{\partial x} - \frac{\partial\vartheta_x}{\partial y}\right) + \Delta\Delta w = -\frac{3}{2\cdot G\cdot h}\cdot\Delta P \text{, or } \frac{\partial}{\partial x}(\Delta\vartheta_y) - \frac{\partial}{\partial y}(\Delta\vartheta_x) + \Delta\Delta w = -\frac{3}{2\cdot G\cdot h}\cdot\Delta P \quad (60)$$

Then, from the second and third previous equations

$$\frac{\partial}{\partial x}(\Delta\vartheta_y) - \frac{\partial}{\partial y}(\Delta\vartheta_x) = \frac{1}{4}\cdot\frac{\partial^2}{\partial x^2}(\Delta w) + \frac{1}{4}\cdot\frac{\partial^2}{\partial y^2}(\Delta w) + \frac{5(1-\mu)}{h^2}\left(\frac{\partial\vartheta_y}{\partial x} + \frac{\partial^2 w}{\partial x^2} - \frac{\partial\vartheta_x}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right) \quad (61)$$

which according to (59) leaves us with

$$\frac{\partial}{\partial x}(\Delta\vartheta_y) - \frac{\partial}{\partial y}(\Delta\vartheta_x) = \frac{1}{4}\cdot\Delta\Delta w - \frac{5(1-\mu)}{h^2}\cdot\frac{3}{2\cdot G\cdot h}\cdot P \quad (62)$$

So substituting in (60) it is obtained

$$\frac{1}{4}\cdot\Delta\Delta w - \frac{5(1-\mu)}{h^2}\cdot\frac{3}{2\cdot G\cdot h}\cdot P + \Delta\Delta w = -\frac{3}{2\cdot G\cdot h}\cdot\Delta P \quad (63)$$

which leaves us with the following, after operating and reordering,

$$\Delta\Delta w = \frac{P}{D} - \frac{12(1+\mu)}{5\cdot E\cdot h}\cdot\Delta P \quad (64)$$

The harmonized governing equation system decoupled in displacements-rotations is therefore

$$\begin{aligned} \Delta\Delta w &= \frac{P}{D} - \frac{12(1+\mu)}{5\cdot E\cdot h}\cdot\Delta P \\ \Delta\vartheta_x - \frac{5(1-\mu)}{h^2}\cdot\vartheta_x &= -\frac{1}{4}\cdot\frac{\partial}{\partial y}(\Delta w) - \frac{5(1-\mu)}{h^2}\cdot\frac{\partial w}{\partial y} \\ \Delta\vartheta_y - \frac{5(1-\mu)}{h^2}\cdot\vartheta_y &= \frac{1}{4}\cdot\frac{\partial}{\partial x}(\Delta w) + \frac{5(1-\mu)}{h^2}\cdot\frac{\partial w}{\partial x} \end{aligned} \quad (65)$$

The first one of the equations, which provides us the displacements *w*, is identical to the first equation in system (33), this is the reason why the displacements provided are the same. The other two, which provide us the rotations ϑ_x, ϑ_y, only differ in the terms $\frac{1}{4}\cdot\frac{\partial}{\partial y}(\Delta w)$ of the equation system (33), and therefore they only contribute to a more accurate approach to the value of the referred rotations.

Due to the fact that a unique solution (*uniqueness guarantee*) and quick convergence are being looked for, and the calculation is done by numerical methods, system (65) must be transformed**.**

According to (58) it may be formed

$$\frac{\partial\vartheta_y}{\partial x} - \frac{\partial\vartheta_x}{\partial y} = \frac{M}{D} + \frac{1}{5\cdot G}\left(\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y}\right) \quad (66)$$

However the derivatives of the shearing stresses in the middle surface are obtained from (51) from which it is deduced

$$\frac{\partial\hat{\tau}_{xz}}{\partial x}+\frac{\partial\hat{\tau}_{yz}}{\partial y}=G\left(\frac{\partial\vartheta_y}{\partial x}+\frac{\partial^2 w}{\partial x^2}-\frac{\partial\vartheta_x}{\partial x}+\frac{\partial^2 w}{\partial y^2}\right)=G\left(\frac{\partial\vartheta_y}{\partial x}-\frac{\partial\vartheta_x}{\partial x}+\Delta w\right) \tag{67}$$

Thus, (66) is transformed into:

$$4\left(\frac{\partial\vartheta_y}{\partial x}-\frac{\partial\vartheta_x}{\partial x}\right)=5\frac{M}{D}+\Delta w\,. \tag{68}$$

According to (58)

$$\frac{\partial\hat{\tau}_{xz}}{\partial x}+\frac{\partial\hat{\tau}_{yz}}{\partial y}=-\frac{3}{2\cdot h}\cdot P \tag{69}$$

so (66) gives us

$$\frac{\partial\vartheta_y}{\partial x}-\frac{\partial\vartheta_x}{\partial y}=\frac{M}{D}-\frac{3\cdot P}{10\cdot G\cdot h} \tag{70}$$

Then the equations decoupled in displacements – moments addition can be obtained, by eliminating the parenthesis in which the variations of the rotations appear, between this last equation and the first of (58), obtaining

$$\Delta w=-\frac{M}{D}-\frac{6}{5\cdot G\cdot h}\cdot P \tag{71}$$

From (61) it is deduced

$$\frac{5\cdot(1-\mu)}{h^2}\cdot\left(\frac{\partial\vartheta_y}{\partial x}+\frac{\partial^2 w}{\partial x^2}-\frac{\partial\vartheta_x}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)=\Delta[\ \frac{\partial}{\partial x}(\vartheta_y)-\frac{\partial}{\partial y}(\vartheta_x)\ ]-\frac{1}{4}\cdot\Delta[\frac{\partial^2}{\partial x^2}(w)+\frac{\partial^2}{\partial y^2}(w)] \tag{72}$$

According to (59), the first member is:

$$-\frac{5\cdot(1-\mu)}{h^2}\cdot\frac{3\cdot(1+\mu)}{E\cdot h}\cdot P=-\frac{5P}{4\cdot D}\,. \tag{73}$$

The first addend of the second member, according to (66), is:

$$\frac{\Delta M}{D}-\frac{3\Delta P}{10\cdot G\cdot h} \tag{74}$$

and the second addend of the second member, according to (65), is:

$$-\frac{P}{4D}+\frac{12\cdot(1+\mu)}{20\cdot E\cdot h}\cdot\Delta P\,. \tag{75}$$

Substituting, one obtains

$$\frac{\Delta M}{D}=-\frac{5P}{4\cdot D}+\frac{P}{4\cdot D}+\frac{3\Delta P}{10\cdot G\cdot h}-\frac{12\cdot(1+\mu)}{20\cdot E\cdot h}\cdot\Delta P\,. \tag{76}$$

After operating and reordering it yields $\Delta M=-P$. Finally, along with equation (71) it is obtained

$$\begin{aligned}\Delta M&=-P\\ \Delta w&=-\frac{M}{D}-\frac{6}{5\cdot G\cdot h}\cdot P\end{aligned} \tag{77}$$

Each one of the equations (77) constitutes a typical Dirichlet problem, in which the uniqueness of the solution is guaranteed, and the solution known as the *Green Function* (1996). It allows for resolution by numerical methods and is strongly convergent, even for meshes which are not very dense, using finite differences or finite elements approximating the integral formulation by means of the Galerkin method (very common in the elementary elliptical problems, Haberman, R (1998)). The system (77), decoupled in displacements – moments addition, is identical to the one described in (41) and formally analogue to the one deduced by Reismann and Pawlik (1980), formulae 6.60 and 6.61, page 235 and section 3 of the present paper], and it is a generalisation for moderately thick plates of what is shown in **5** and what was presented by Timoshenko (1997). (Note: the coefficient $\frac{6}{5}$ is replaced with $\frac{1}{\kappa^2}$ in the Reismann and Pawlik theory, which adopts the value $\frac{1}{\kappa^2}=1.16$ for $\mu=0.3$).

On the other hand, if it is called

$$w = w_1 + w_2 \tag{78}$$

so that

$$\Delta w_1 = -\frac{M}{D} \text{ and } \Delta w_2 = -\frac{6}{5 \cdot G \cdot h} \cdot P, \tag{79}$$

one gets

$$\Delta\Delta w_1 = \frac{P}{D} \text{ and } \Delta w_2 = -\frac{6}{5 \cdot G \cdot h} \cdot P. \tag{80}$$

The last one is identical to the one presented by Panc (1947) in his Component Theory for thick plates.

Regarding the precision obtained with these equations, in a second part of this article it will be shown that the analytical solutions obtained for simply supported or clamped plates are identical to the ones obtained by the authors.

6 CONCLUSIONS

In this article an harmonization hypothesis of the technical theories of thin-thick plates with a constant thickness is presented. The conclusions obtained are:

1.- The second Bolle equation (16) $\Delta\alpha = \frac{10}{h^2}\alpha$ is inappropriate because of this parameter being null for the whole plate.

2.- The coupled and harmonized differential equation system (33)

$$\begin{aligned} \Delta\Delta w &= \frac{P}{D} - \frac{12(1+\mu)}{5 \cdot E \cdot h} \cdot \Delta P \\ \Delta\vartheta_x - \frac{5(1-\mu)}{h^2} \cdot \vartheta_x &= -\frac{1}{4} \cdot \frac{\partial}{\partial y}(\Delta w) - \frac{5(1-\mu)}{h^2} \cdot \frac{\partial w}{\partial y} \\ \Delta\vartheta_y - \frac{5(1-\mu)}{h^2} \cdot \vartheta_y &= \frac{1}{4} \cdot \frac{\partial}{\partial x}(\Delta w) + \frac{5(1-\mu)}{h^2} \cdot \frac{\partial w}{\partial x} \end{aligned} \tag{81}$$

or the decoupled and harmonized differential equation system (77)

$$\Delta M = -P$$

$$\Delta w = -\frac{M}{D} - \frac{6}{5 \cdot G \cdot h} \cdot P \qquad (82)$$

provide correct solutions in the case of thin as well as moderately thick plates.
3.- The decoupled system (77) is formally analogue to the one deduced by Reismann (1980) and to the one presented by Panc (1947) in his Component Theory for thick plates, and that it is a generalisation for moderately thick plates of what is said in section **5** and what was presented by Timoshenko (1970).

REFERENCES

Bolle L. Contribution on problema linéare deflexión d´une plaque élastique. Part 1. *Bulletin Tecnique de la Suisse Romande*, 21, 281-285, 1947.

Donnell Ll. H. *Beams, Plates and Shells*. McGraw-Hill, New York, USA, 1976.

Eisenberger, M. Dynamic Stiffness vibration Analysis for Higher Order Plate Models. *Lectures Notes in Applied and Computational Mechanic*, 16, 29-38, 2004

Haberman. R. *Elementary Applied Partial Differential Equations*. Prentice Hall, 1998.

Hencky H. Über die Berücksichtigung der Schubverzerrungen in ebenen Platten *Ing.-Arch*. 16, 1947.

Kienzler R.J. On consistent Second-Order Plate Theories Lectures Notes in *Applied and Computational Mechanic*, 16, 85-96, 2004.

Krom A. Verallgemeinerte theorie der plattenstatik *Ing.Arch*., 21, 1953.

Love A. E. H. A treatise on the Mathematical Theory of Elasticity. *Dover Publications*. New York, USA, 1944.

Meenen J. An assesment of Classical and Refined Plate Theories Derived from the principle of virtual displacements. *Lectures Notes in Applied and Computational Mechanic*, 16, 149-156, 2004.

Mindlin R. D. *Journal Applied Mechanics*. 18, 31-38, 1951.

Muhammad A.K., Baluch, M.H., Azad, A.K. Generalized Theory for Bending of Thick Plates, *Advances in the Theory of Plates and Shells*, Elsevier Science Publishers B.V., Amsterdam, The Netherlands, 1990.

Panc V.*Theories of Elastic Plates*, Noordhoff International Publishing, Leyden, The Netherlands, 1975.

Quesada Molina J.J. *Ecuaciones Diferenciales, Análisis Numérico y Métodos Matemáticos*, Santa Rita, Granada, Spain, 1996.

Reismann H.and Pawlik *Elasticity: Theory and applications*, John Wiley, New York, USA, 1980.

Reissner E. The effect of transverse shear deformation on the bending of elastic plates. *Journal Applied Mechanic*, 12, 69-77, 1945.

Timoshenko S., Woinowski S. *Teoría de Placas y Láminas*, Urmo, Bilbao, Spain, 1970.

Vlásov V. Z. Mecánica de construcción de sistemas especiales de paredes delgadas. ВЛасоb В.З.*СТРОИТеЛЪНаЯ МеХаНИКа ТОНКОСТеННЪIХ ПРОСТРаНСТВеННЪIХ СИСТеМ. СТРОЙИЗдаТ*, 1949, 1958.

Voyiaddjis G. Z. and Karamanlidis D. *Advances in the Theory of Pates and Shells*, Elsevier Science Publishers B.V., Amsterdam, The Netherlands, 1990.

Zienkiewicz O. C., Taylor L.R. *El Método de los Elementos Finitos*, McGraw-Hill and CIMNE, Barcelona, Spain, 1994.