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STRENGTH PROPERTIES OF FIBER REINFORCED CONCRETE BY MEANS OF A HOMOGENIZATION APPROACH

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Abstract. Addition of steel fiber to concrete results in a composite with enhanced strength properties than conventional concrete. Experimental studies have shown that fibers improve the biaxial compressive and uniaxial tensile strengths of concrete. This study investigates the strength properties at the macroscopic level of the reinforced material by means of a micromechanical reasoning which implements the static approach of limit analysis homogenization theory. Within this framework, the macroscopic strength criterion can be theoretically determined from the knowledge of the failure conditions of individual constituents, namely, concrete matrix and fibers. Adopting a Drucker-Prager failure criterion for the concrete failure properties, and assuming an isotropic spatial distribution for the fibers, an approximate static-based model is formulated for the homogenized strength properties. The expression of the latter is provided together with the corresponding geometric interpretation in the space of macroscopic stresses. The accuracy of the proposed model is assessed by means of comparison with available data. The predictions of the analytical model are found to compare well with the available experimental results.

1 INTRODUCTION

Evaluating the strength properties of materials remains fundamental concern in material and structural engineering, either for material appropriate use or for correct consideration in projects, design and verification stages. The strength capacities are classically characterized by the strength convex which defines the set of admissible stresses. Its determination is an important issue in material modeling.

As for most conventional materials, the concrete strength and behavior under multiaxial stress states have been experimentally and theoretically studied for several decades. These characteristics are mainly influenced by the physical and mechanical properties of aggregate and cement paste. In the case of fiber reinforced concrete (FRC), there are few experimental and analytical studies aimed at establishing its behavior under complex stress states. Experimental studies of Yin et al. (1989), Traina and Mansour (1991) and Swaddiwudhipong and Seow (2006) sought to evaluate the steel FRC behavior under biaxial compression. These studies showed that the reinforced concrete has a failure curve distinct than the concrete matrix alone and the increase in the biaxial compression strength due to the fibers addition can be considerable.

The limit analysis theory, in turn, provides a powerful tool not only to establish the collapse load of structural components but also to determine the strength capacity of heterogeneous materials. It is a direct method since it does not involve a step by step analysis and does not consider the full story along the loading process as in elastic-plastic analysis. For this, it uses the classical lower (static) and upper (cinematic) bound theorems. The homogenization theory applied to the limit analysis theory allows defining the macroscopic strength properties of a heterogeneous medium in a rigorous way. Through this approach the strength properties of composites reinforced by long parallel fibers has been obtained and the theoretical results established for these materials showed good agreement with experimental results (de Buhan and Taliercio, 1991).

Based on a concept of equivalence between the FRC and a fictitious medium defined by a concrete matrix reinforced by long fibers, this study aims to obtain the steel FRC strength domain analytically. Therefore, the homogenization theory, the limit analysis theory and the results obtained by de Buhan (1985) for a medium reinforced by long parallel fibers are used. The composite strength criterion is determined by considering the strength properties of its individual constituents, namely, concrete matrix and fibers, and their volume fractions. The formulation employed allows the consideration of any strength criterion for the components of the composite material. In the fibers case, only its uniaxial compressive and tensile strengths are needed. The concrete matrix failure is characterized in this work by the Drucker Prager criterion.

The next section describes how the FRC strength criterion will be determined in this study through the homogenization and limit analysis theories and the concept of a correspondence between the real and a fictitious medium. Section 2.1 presents the formulation that provides the strength criterion of a medium reinforced by long fibers arranged in one direction. Following this result is used for the analysis of a medium reinforced by fibers in three directions. Section 2.3 presents the results obtained by considering a matrix with isotropic behavior and that obeys the Drucker-Prager criterion. Finally the analytical results are compared to steel FRC experimental data available in the literature.

2 MACROSCOPIC STRENGTH DOMAIN

The construction of the macroscopic strength domain G^{hom} of a heterogeneous medium results from the resolution of a limit analysis problem placed on the representative elementary volume (REV) of this material. It requires the knowledge of the constituent (i.e., the matrix, fibers and fiber/matrix interface) strength domains, its geometric configuration and the fiber volume fraction.

The FRC is a random medium (Figure 1a) formed by a cementitious matrix and the i (integer) fiber families characterized by its direction $\underline{e}_i$ and its aspect ratio l/d (length/diameter).

In this study, some simplifications are adopted to obtain the FRC macroscopic strength domain in an analytical way. First it is assumed that the macroscopic strength of a medium reinforced by randomly distributed short fibers can be obtained by considering a correlation between this material and a fictitious medium defined by a concrete matrix reinforced by long fibers. The second simplification concerns the consideration of a fictitious medium with long fibers arranged in three perpendicular directions for the random medium study, since the analysis with a larger number of directions would involve a more complex formulation.

The described concept is depicted in Figure 1.

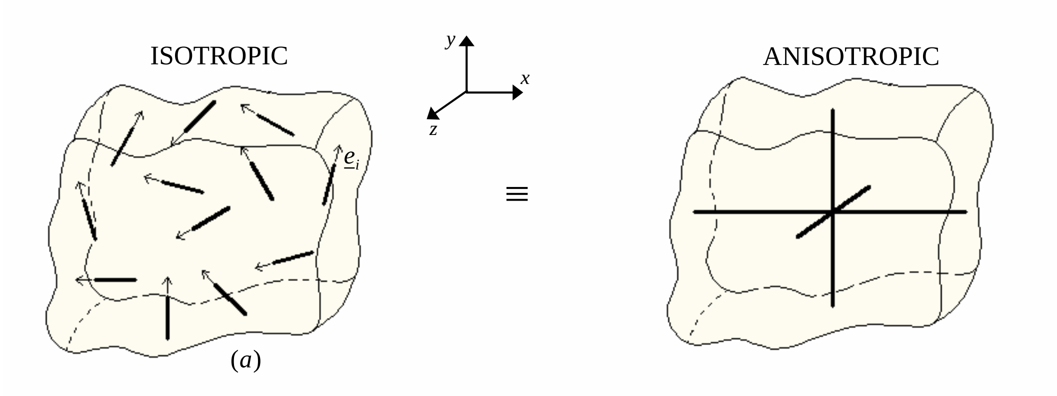

Figure 1: Medium formed by randomly distributed short fibers and an approximate fictitious medium.

This fictitious medium, with fictitious properties, should present a macroscopic strength domain able to reproduce the macroscopic strength domain of a medium reinforced by random and short fibers. To establish the correlation between the strength properties of these two media, in other words the determination of the fictitious properties to be used, comparisons between the obtained analytical results and those verified in experiments are needed.

Strength, like elastic properties, is function of the fiber aspect ratio and approaches an asymptotic limit as the aspect ratio becomes larger. The influence of the parameter l/d in the composite material strength properties can be obtained, for example, from the evaluation of experimental data of matrices reinforced by fibers of different aspect ratios. The comparison of experiments and the analytical results obtained in this study allows the definition of fictitious properties which vary in function of the fiber aspect ratios.

The Chen (1971) and Halpin and Kardos (1978) works sought to evaluate the influence of this parameter on the composite strengths. Halpin and Kardos (1978) propose a strength reduction factor in function of l/d. However their studies are based on composites reinforced by short fibers periodically arranged in one direction.

For the strength properties of fiber/matrix interface, in turn, it is considered perfect bonding between the two constituents.

2.1 Matrix reinforced by long parallel fibers

Composites formed by a matrix reinforced by long parallel fibers belong to the class of periodic media. In this case it is therefore possible to define a unit cell A, which is the structure that repeats along the composite material and contains all the necessary information for their complete description.

The determination of the macroscopic strength domain of this material is obtained from the solution of a limit analysis problem placed on their unit cell:

$$\underline{\underline{\Sigma}} \in G^{\text{hom}} \Leftrightarrow \exists \underline{\underline{\sigma}} \begin{cases} div\underline{\underline{\sigma}} = 0 \ \ (\text{e } [\![\underline{\underline{\sigma}}]\!] \cdot \underline{n}^S = 0 \ along \ S) \\ \underline{\underline{\Sigma}} = \langle \underline{\underline{\sigma}} \rangle, \ \underline{\underline{\sigma}} \cdot \underline{n} \ antiperiodic, \underline{\underline{\sigma}}(\underline{x}) \in G(\underline{x}) \ \ \forall \underline{x} \in A \end{cases} \tag{1}$$

where $\underline{\underline{\Sigma}}$ is the macroscopic stress tensor and $\underline{\underline{\sigma}}$ represents the microscopic stress field in A. $G(\underline{x})$ is the material strength domain at the current point $\underline{x}$ of A, or, the set of allowable stress tensors $\underline{\underline{\sigma}}(\underline{x})$. In the matrix case $\underline{\underline{\sigma}}(\underline{x}) \in G^m \ \ \forall \underline{x} \in V^m$ and in the fibers case $\underline{\underline{\sigma}}(\underline{x}) \in G^f \ \ \forall \underline{x} \in V^f$. $[\![\underline{\underline{\sigma}}]\!]$ is the jump of $\underline{\underline{\sigma}}$ and S are possible stress discontinuity surfaces with outer unit normal $\underline{n}^S$. The matrix and fiber strength domains could be represented in a more appropriate way employing their yield functions, respectively, F^m e F^f ($\underline{\underline{\sigma}} \in G^m \ \Leftrightarrow \ F^m(\underline{\underline{\sigma}}) \le 0$ and $\underline{\underline{\sigma}} \in G^f \ \Leftrightarrow \ F^f(\underline{\underline{\sigma}}) \le 0$). It is worth noting that the Eq. (1) represents the static definition of G^{hom}. The dual cinematic definition of G^{hom} is done through the employment of its support function $\pi^{\text{hom}}(\underline{\underline{D}})$.

Assuming the fibers parallel to the x-axis, the resolution of the problem (1) is effectuated through the static approach with the consideration of the piecewise homogeneous stress field:

$$\underline{\underline{\sigma}} = \begin{cases} \underline{\underline{\sigma}}^m & in \ the \ matrix, \ with \ \underline{\underline{\sigma}}^m \in G^m \\ \underline{\underline{\sigma}}^f = \underline{\underline{\sigma}}^m + \sigma^f \underline{e}_x \otimes \underline{e}_x & in \ the \ fiber, \ with \ \underline{\underline{\sigma}}^f \in G^f \end{cases} \tag{2}$$

The stress field (2) satisfies the problem conditions (1). The described static approach, with the consideration of piecewise homogeneous stress field, constitutes a lower bound approximation to G^{hom}, since it corresponds to the use of the limit analysis lower bound theorem of the limit analysis theory:

$$G_s^{\text{hom}} = \left\{ \begin{matrix} \underline{\underline{\Sigma}} = \underline{\underline{\sigma}}^m + f\sigma^f \underline{e}_x \otimes \underline{e}_x \\ \underline{\underline{\sigma}}^m \in G^m, \ \ \underline{\underline{\sigma}}^m + \sigma^f \underline{e}_x \otimes \underline{e}_x \in G^f \end{matrix} \right\} \quad \subset \quad G^{hom} \tag{3}$$

where s refers to static and $f = |A^f| / |A|$ is the fiber volume fraction.

In particular conditions defined by $f << 1$ and $G^f \underset{f \to 0}{\simeq} G^0 / f >> G^m$, where G^0 is a bounded fixed convex domain, de Buhan and Taliercio (1991) showed that:

$$G^{\text{hom}} \simeq G_s^{\text{hom}} = \left\{ \begin{matrix} \underline{\underline{\Sigma}} = \underline{\underline{\sigma}}^m + \sigma_f \underline{e}_x \otimes \underline{e}_x \\ \underline{\underline{\sigma}}^m \in G^m, \ \ \sigma_f \in [\sigma_0^-, \sigma_0^+] \end{matrix} \right\} \tag{4}$$

where the parameters $\sigma_0^+ = \sup\left\{ \sigma \middle| \sigma \underline{e}_x \otimes \underline{e}_x \in G^0 \right\} = f \sup\left\{ \sigma \middle| \sigma \underline{e}_x \otimes \underline{e}_x \in G^f \right\} = f\sigma_f^+$ and

$\sigma_0^- = \inf\left\{\sigma \middle| \sigma \underline{e}_x \otimes \underline{e}_x \in G^0\right\} = f \inf\left\{\sigma \middle| \sigma \underline{e}_x \otimes \underline{e}_x \in G^f\right\} = f\sigma_f^-$ represent respectively the fiber uniaxial tensile and compressive strengths per unit transverse area. σ_f^+ and σ_f^- are respectively the fiber tensile and compressive strengths.

Geometrically, the strength domain G^{hom} can be interpreted in the space $R^6 = \left\{\Sigma_{xx}, \Sigma_{yy}, \Sigma_{zz}, \Sigma_{xy}, \Sigma_{xz}, \Sigma_{yz}\right\}$ of macroscopic stress as the convex envelope of two domains obtained by translating the matrix strength domain G^{m} by algebraic distances $f\sigma_f^-$ and $f\sigma_f^+$ parallel to the axis Σ_{xx}-axis. These translations in the space of macroscopic stresses are the expression of the reinforcement due to the fibers presence.

A cross-sectional view of the strength domain G^{hom} in an arbitrary plane of macroscopic stresses $(\Sigma_{xx}, \Sigma_{ij})$ is sketched in Figure 2.

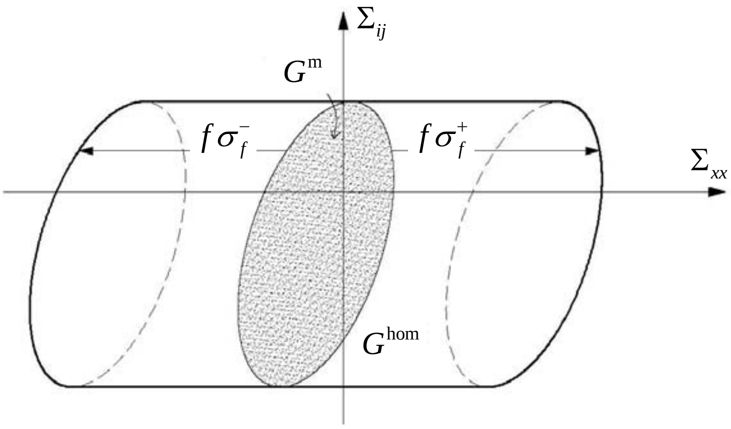

Figure 2: Geometrical representation of the macroscopic strength domain G^{hom} of a matrix reinforced by long fibers parallel to the x-axis

The strength criterion G^{hom} can also be represented employing its yield function F^{hom}, in other words the stress states $\underline{\underline{\Sigma}}$ that satisfy the macroscopic criterion (4) form the convex domain G^{hom} defined by the yield function $F^{hom}(\underline{\underline{\Sigma}}) \leq 0$:

$$F^{hom}(\underline{\underline{\Sigma}}) \leq 0 \Leftrightarrow \begin{cases} \underline{\underline{\Sigma}} = \underline{\underline{\sigma}}^m + \sigma_f \underline{e}_x \otimes \underline{e}_x \\ F^m(\underline{\underline{\sigma}}^m) \leq 0, \quad \sigma_f \in I \end{cases} \tag{5}$$

with $I = \left[f\sigma_f^-, f\sigma_f^+\right]$.

The yield function of the homogenized material defined by Eq. 5 can be rewritten as:

$$F^{hom}(\underline{\underline{\Sigma}}) = \min_{\sigma \in I} F^m(\underline{\underline{\Sigma}} - \sigma \underline{e}_x \otimes \underline{e}_x) \tag{6}$$

2.2 Matrix reinforced by long fibers in three directions

For the situation of a matrix reinforced by long fibers arranged in three directions, the considerations employed are similar to those described before when only on fiber direction was evaluated. Thus the strength criterion is defined by the equation:

$$F^{hom}(\underline{\underline{\Sigma}}) \leq 0 \Leftrightarrow \begin{cases} \underline{\underline{\Sigma}} = \underline{\underline{\sigma}}^m + \sum_{i=1}^{3} \sigma_i \underline{e}_i \otimes \underline{e}_i \\ F^m(\underline{\underline{\sigma}}^m) \leq 0, \quad \sigma_i \in I_i \end{cases} \tag{7}$$

with $I_i = \left[f_i\sigma_i^-, f_i\sigma_i^+ \right]$.

The geometric representation of the strength domain in the space of macroscopic stresses for the case of fibers oriented in three directions, for example, x, y and z, is then defined as the convex envelope of eight domains obtained by translating the matrix strength domain by algebraic distances $f_x\sigma_x^-$ and $f_x\sigma_x^+$ along the Σ_{xx}-axis, $f_y\sigma_y^-$ and $f_y\sigma_y^+$ along the Σ_{yy}-axis and $f_z\sigma_z^-$ and $f_z\sigma_z^+$ along the Σ_{zz}-axis. The geometric representation of the strength domain G^{hom} of a composite material reinforced by fibers in three or more directions is not trivial. Along the convex envelope can be identified zones where one of the parameters related to the stress in the fibers (σ_x, σ_y and σ_z) has a limit value and also zones where two or three of these parameters have a limit value.

Figure 3, adapted from Taliercio et al. (1991), shows in a generic way the macroscopic strength domain of a composite material formed by a matrix reinforced by fibers in the directions x, y and z, in the space of the macroscopic stress components Σ_{xx}, Σ_{yy} and Σ_{zz}. For the sake of clearness, the particular case of a bounded and polyhedric matrix strength domain has been depicted. The zones identified as A are those in which only one parameter (σ_x, σ_y and σ_z) has a limit value whilst zones B and C are those in which two and three parameters, respectively, have a limit value.

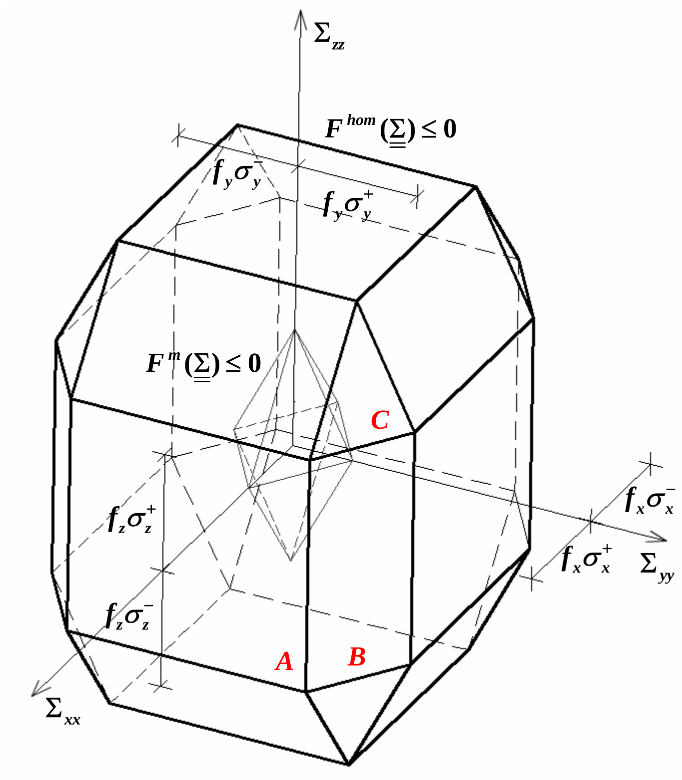

Figure 3: Macroscopic strength domain $F^{hom}(\underline{\underline{\Sigma}}) \le 0$ of a material reinforced by fibers in the directions x, y and z in the space of macroscopic stress components Σ_{xx}, Σ_{yy} and Σ_{zz}.

The determination of the mathematical expressions that define the macroscopic strength domain, i.e., the convex envelope of the eight domains obtained by translating the matrix strength domain, is possible by defining the parameters σ_x, σ_y e σ_z that minimize the function:

$$g(\sigma_x, \sigma_y, \sigma_z) = F^m(\underline{\underline{\Sigma}} - \sigma_x \underline{e}_x \otimes \underline{e}_x - \sigma_y \underline{e}_y \otimes \underline{e}_y - \sigma_z \underline{e}_z \otimes \underline{e}_z) \tag{8}$$

$\underline{\underline{\Sigma}}$ being prescribed. In other words:

$$F^{hom}(\underline{\underline{\Sigma}}) = \min_{\substack{\sigma_x \in I_x \\ \sigma_y \in I_y \\ \sigma_z \in I_z}} F^m(\underline{\underline{\Sigma}} - \sigma_x \underline{e}_x \otimes \underline{e}_x - \sigma_y \underline{e}_y \otimes \underline{e}_y - \sigma_z \underline{e}_z \otimes \underline{e}_z) \tag{9}$$

The following situations can be identified:

- $\begin{cases} \sigma_x \in \partial I_x \\ \sigma_y \in \partial I_y \\ \sigma_z \in \partial I_z \end{cases}$, where ∂I_i represents the limit values of the interval I_i defined before. In

this case, zones of the macroscopic strength domain G^{hom} defined as C are obtained.

- $\begin{cases} \sigma_x \in \partial I_x \\ \sigma_y \in \partial I_y \\ \sigma_z \in \mathring{I}_z \end{cases}$ and permutations of x, y and z, where $\mathring{I}_i$ represents values within the interval I_i. In this case, zones of the macroscopic strength domain G^{hom} defined as B are obtained.

- $\begin{cases} \sigma_x \in \partial I_x \\ \sigma_y \in \overset{\circ}{I}_y \\ \sigma_z \in \overset{\circ}{I}_z \end{cases}$ and permutations of x, y and z. In this case, zones of the macroscopic strength domain G^{hom} defined as A are obtained.

- $\begin{cases} \sigma_x \in \overset{\circ}{I}_x \\ \sigma_y \in \overset{\circ}{I}_y \\ \sigma_z \in \overset{\circ}{I}_z \end{cases}$. In this case, the zones of the macroscopic strength domain G^{hom} are located in the complementary space $\{\Sigma_{xy}, \Sigma_{xz}, \Sigma_{yz}\}$ of R^3.

In the second situation described above, when two of the parameters have reached their limit values, the third is obtained by annulling the correspondent derivative of $g(\sigma_x, \sigma_y, \sigma_z)$. For example, if $\sigma_x \in \partial I_x$ and $\sigma_y \in \partial I_y$, σ_z is obtained through:

$$\begin{aligned} &\frac{\partial g}{\partial \sigma_z}(\sigma_x, \sigma_y, \sigma_z) = \frac{\partial F^m}{\partial \sigma_z}(\underline{\underline{\Sigma}} - \sigma_x \underline{e}_x \otimes \underline{e}_x - \sigma_y \underline{e}_y \otimes \underline{e}_y - \sigma_z \underline{e}_z \otimes \underline{e}_z) = \\ &-\frac{\partial F^m}{\partial \sigma_{zz}}(\underline{\underline{\Sigma}} - \sigma_x \underline{e}_x \otimes \underline{e}_x - \sigma_y \underline{e}_y \otimes \underline{e}_y - \sigma_z \underline{e}_z \otimes \underline{e}_z) = 0 \end{aligned} \tag{10}$$

Due to the symmetry in relation to the three considered directions, the resolution of the remaining cases are obtained through permutations of x, y and z.

In the third situation described above, when only one of the parameters have reached its limit values, the other two parameters are obtained through the system formed by the correspondent derivatives of $g(\sigma_x, \sigma_y, \sigma_z)$ annulled. For example, if $\sigma_z \in \partial I_z$, σ_x and σ_y are obtained through the system:

$$\frac{\partial g}{\partial \sigma_x}(\sigma_x, \sigma_y, \sigma_z) = 0 \quad e \quad \frac{\partial g}{\partial \sigma_y}(\sigma_x, \sigma_y, \sigma_z) = 0 \tag{11}$$

The resolution of the remaining cases are obtained through permutations of x, y and z.

2.3 Matrix characterized by the Drucker Prager Criterion

The concepts previously described are generalized, i. e., they can be applied to matrices satisfying any strength criterion G^{m} (defined by their yield function $F^m(\underline{\underline{\sigma}}) \leq 0$). This section describes the formulation for a matrix with isotropic behavior and that obeys the Drucker-Prager criterion.

The Drucker-Prager criterion was initially developed to study the soils behavior. However, it has also been applied to the study of rocks, polymers, foams, concrete and other materials which depend on the hydrostatic pressure. In the three-dimensional principal stress space, the Drucker-Prager surface has a conical form with its vertex on the hydrostatic axis

$(\sigma_{xx} = \sigma_{yy} = \sigma_{zz})$. In a two-dimensional stress space (for example, considering a cross section of the cone in the plane of principal stresses $(\sigma_{xx}, \sigma_{yy})$), it represents a domain with an ellipsoidal shape.

The Drucker-Prager criterion may be expressed in the following form:

$$F^m(\underline{\underline{\sigma}}) = \sqrt{\frac{3}{2}}\left\|\underline{\underline{s}}\right\| + \alpha_m\left(tr\underline{\underline{\sigma}} - \sigma_m\right) - \sigma_m \le 0 \tag{12}$$

where $\left\|\underline{\underline{s}}\right\| = (\underline{\underline{s}} : \underline{\underline{s}})^{1/2}$ is the norm of the second-order tensor $\underline{\underline{s}}$, which is the deviatoric part of $\underline{\underline{\sigma}}$, i.e., $\underline{\underline{s}} = dev(\underline{\underline{\sigma}})$. σ_m represents the elastic limit of the material under uniaxial tensile stress. The scalar α_m is a non-dimensional parameter ranging between 0 (von Mises criterion) and 1, which accounts for the criterion dependence on the hydrostatic stress.

It is observed that:

$$\frac{\partial F^m}{\partial \underline{\underline{\sigma}}} = \sqrt{\frac{3}{2}}\frac{\underline{\underline{s}}}{\left\|\underline{\underline{s}}\right\|} + \alpha_m \underline{\underline{1}} \tag{13}$$

Considering Eq. (12) and Eq (13), which respectively express the Drucker-Prager criterion and the derivative of its function in relation to $\underline{\underline{\sigma}}$, the resolution of Eq. (10) leads to:

$$\sigma_z = \frac{3}{2}\left(S_{zz} + \frac{\left(\sigma_x + \sigma_y\right)}{3} + \sqrt{\frac{2}{3}}\frac{\alpha_m}{\sqrt{1-\alpha_m^2}}\sqrt{\underline{\underline{S}} : \underline{\underline{S}} - \frac{3}{2}S_{zz}^2 + \frac{\left(\sigma_x - \sigma_y\right)^2}{2} - \left(\sigma_x - \sigma_y\right)(S_{xx} - S_{yy})}\right) \tag{14}$$

where $\underline{\underline{S}} = dev(\underline{\underline{\Sigma}})$ is the deviatoric part of the macroscopic stress tensor.

Two zones B of the macroscopic strength domain are then obtained considering:

$$\sigma_z(\underline{\underline{\Sigma}}) = \begin{cases} f_z\sigma_z^- & if \quad \sigma_z \le f_z\sigma_z^- \\ \sigma_z \quad \begin{matrix} with\ \sigma_x = f_x\sigma_x^+\ and\ \sigma_y = f_y\sigma_y^- \\ or\ \sigma_x = f_x\sigma_x^-\ and\ \sigma_y = f_y\sigma_y^+ \end{matrix} & if \quad \sigma_z \in I_z \\ f_z\sigma_z^+ & if \quad \sigma_z \ge f_z\sigma_z^+ \end{cases} \tag{15}$$

in the function $F^{hom}(\underline{\underline{\Sigma}}) = F^m(\underline{\underline{\Sigma}} - \sigma_x \underline{e}_x \otimes \underline{e}_x - \sigma_y \underline{e}_y \otimes \underline{e}_y - \sigma_z \underline{e}_z \otimes \underline{e}_z)$. The remaining zones B are obtained by permutations of x, y and z in Eq. (10), Eq. (14) and Eq. (15).

The expressions obtained for the zones B of the macroscopic strength domain are described below in a simplified form:

$$\text{Zone } B_1:\ \Sigma_{xx} = \frac{\left(f_1 + f_2\right)\left(f_7 - f_8\right)}{f_9}, \text{ with } \sigma_x = f_x\sigma_x^+ \ and\ \sigma_z = f_z\sigma_z^-$$

$$\text{Zone } B_2:\ \Sigma_{yy} = \frac{\left(f_3 + f_4\right)\left(-f_7 + f_8\right)}{f_9}, \text{ with } \sigma_y = f_y\sigma_y^+ \ and\ \sigma_z = f_z\sigma_z^-$$

$$\text{Zone } B_3: \ \Sigma_{yy} = \frac{(f_5 - f_6)(f_7 - f_8)}{f_9}, \text{ with } \sigma_x = f_x\sigma_x^- \ and \ \sigma_y = f_y\sigma_y^+ \tag{16}$$

$$\text{Zone } B_4: \ \Sigma_{xx} = \frac{(-f_1 + f_2)(f_7 + f_8)}{f_9}, \text{ with } \sigma_x = f_x\sigma_x^- \ and \ \sigma_z = f_z\sigma_z^+$$

$$\text{Zone } B_5: \ \Sigma_{yy} = \frac{(f_3 - f_4)(f_7 + f_8)}{f_9}, \text{ with } \sigma_y = f_y\sigma_y^- \ and \ \sigma_z = f_z\sigma_z^+$$

$$\text{Zone } B_6: \ \Sigma_{yy} = \frac{(f_5 + f_6)(-f_7 - f_8)}{f_9}, \text{ with } \sigma_x = f_x\sigma_x^+ \ and \ \sigma_y = f_y\sigma_y^-$$

where $f_1 = \sqrt{3}\left(1-\alpha_m^2\right)\left(-\sigma_x + \sigma_z - \Sigma_{zz}\right)$, $f_2 = \sqrt{1-\alpha_m^2}\left(3\alpha_m\left(-\sigma_x - \sigma_z + \Sigma_{zz}\right) - 2\sigma_m\left(1+\alpha_m\right)\right)$, $f_3 = \sqrt{3}\left(1-\alpha_m^2\right)\left(\sigma_y - \sigma_z + \Sigma_{zz}\right)$, $f_4 = \sqrt{1-\alpha_m^2}\left(3\alpha_m\left(\sigma_y + \sigma_z - \Sigma_{zz}\right) + 2\sigma_m\left(1+\alpha_m\right)\right)$, $f_5 = \sqrt{3}\left(1-\alpha_m^2\right)\left(\sigma_x - \sigma_y - \Sigma_{xx}\right)$, $f_6 = \sqrt{1-\alpha_m^2}\left(3\alpha_m\left(\sigma_x + \sigma_y - \Sigma_{xx}\right) + 2\sigma_m\left(1+\alpha_m\right)\right)$, $f_7 = 3\alpha_m\sqrt{1-\alpha_m^2}$, $f_8 = \sqrt{3}\left(1-\alpha_m^2\right)$ e $f_9 = 3\left(1-\alpha_m^2\right)\left(1-4\alpha_m^2\right)$.

While the expressions that define the zones B_1, B_2, B_4 and B_5 are constants for each choice of the matrix and reinforcement properties and for each Σ_{zz}, the expressions of the zones B_3 and B_6 are functions $f\left(\Sigma_{xx}, \Sigma_{yy}\right) = 0$.

Considering Eq. (12) and Eq. (13), the solution of the equation system (11) gives the expressions for one of the zones A. System (11) gives:

$$\begin{aligned} \sigma_x &= 2S_{xx} + S_{yy} + \sigma_z + 2\sqrt{3}\frac{\alpha_m}{\sqrt{1-4\alpha_m^2}}\sqrt{S_{xy}^2 + S_{xz}^2 + S_{yz}^2} \quad e \\ \sigma_y &= 2S_{yy} + S_{xx} + \sigma_z + 2\sqrt{3}\frac{\alpha_m}{\sqrt{1-4\alpha_m^2}}\sqrt{S_{xy}^2 + S_{xz}^2 + S_{yz}^2} \end{aligned} \tag{17}$$

Zone A_3 of the macroscopic strength domain is then obtained considering:

$$\begin{aligned} \sigma_x(\underline{\underline{\Sigma}}) &= \begin{cases} f_x\sigma_x^- & if \quad \sigma_x \le f_x\sigma_x^- \\ \sigma_x \quad with \ \sigma_z = f_z\sigma_z^+ & if \quad \sigma_x \in I_x \\ f_x\sigma_x^+ & if \quad \sigma_x \ge f_x\sigma_x^+ \end{cases} \\ \sigma_y(\underline{\underline{\Sigma}}) &= \begin{cases} f_y\sigma_y^- & if \quad \sigma_y \le f_y\sigma_y^- \\ \sigma_y \quad with \ \sigma_z = f_z\sigma_z^+ & if \quad \sigma_y \in I_y \\ f_y\sigma_y^+ & if \quad \sigma_y \ge f_y\sigma_y^+ \end{cases} \end{aligned} \tag{18}$$

in the function $F^{hom}(\underline{\underline{\Sigma}}) = F^m(\underline{\underline{\Sigma}} - \sigma_x \underline{e}_x \otimes \underline{e}_x - \sigma_y \underline{e}_y \otimes \underline{e}_y - \sigma_z \underline{e}_z \otimes \underline{e}_z)$. The remaining zones A are obtained by permutations of x, y and z in Eq. (11), Eq. (17) and Eq. (18).

The expressions obtained for the zones A of the macroscopic strength domain are described below in a simplified form:

$$\text{Zone } A_1: \ \Sigma_{xx} = \frac{3\alpha_m \sigma_x + \sigma_m(\alpha_m + 1)}{3\alpha_m}, \text{ with } \sigma_x = f_x \sigma_x^+$$

$$\text{Zone } A_2: \ \Sigma_{yy} = \frac{3\alpha_m \sigma_y + \sigma_m(\alpha_m + 1)}{3\alpha_m}, \text{ with } \sigma_y = f_y \sigma_y^+ \tag{19}$$

$$\text{Zone } A_3: \ \Sigma_{zz} = \frac{3\alpha_m \sigma_z + \sigma_m(\alpha_m + 1)}{3\alpha_m}, \text{ with } \sigma_z = f_z \sigma_z^+$$

The above expressions show that the zones A_1, A_2, and A_3 are constant with respect to the macroscopic stress components Σ_{xx}, Σ_{yy} and Σ_{zz}, respectively. The planes defined by expressions (19) correspond to three planes placed on four vertices of four domains obtained by translating the matrix strength domain.

Figure 4 displays the intersection of the plane $\Sigma_{zz} / \Sigma_0 = 0$ with the macroscopic strength domain of a composite reinforced by fibers in the directions x, y and z and whose matrix obeys the Drucker-Prager criterion. Σ_0 is a reference stress. The intersections of the same plane with the matrix strength domain and with the eight domains obtained by its translations $f_x \sigma_x^{\pm} \underline{e}_x + f_y \sigma_y^{\pm} \underline{e}_y + f_z \sigma_z^{\pm} \underline{e}_z$ are also displayed. It is possible to verify that the macroscopic strength domain G^{hom} is the convex envelope of these eight domains.

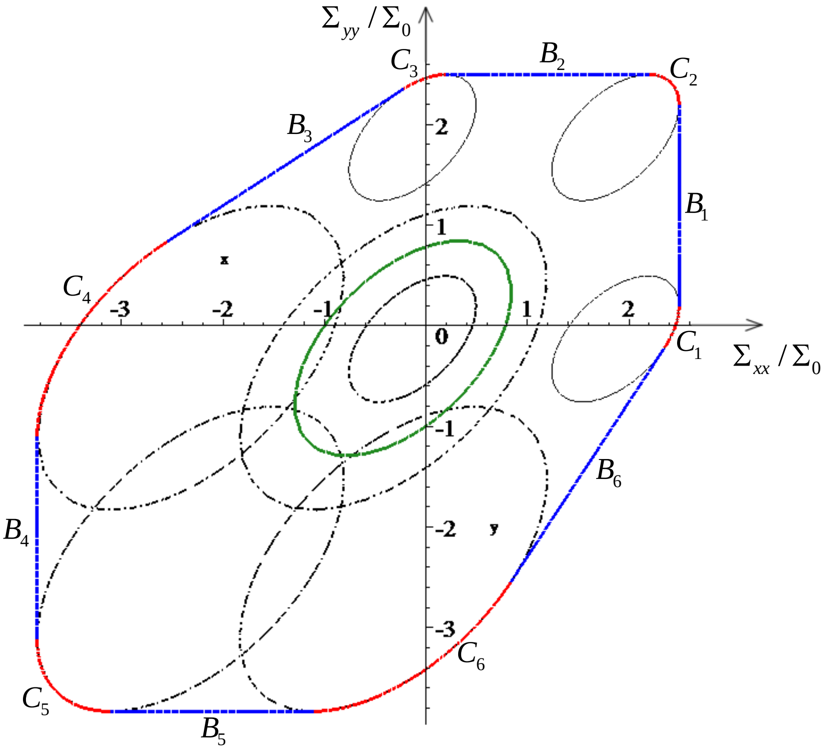

Figure 4: Macroscopic strength domain in the plane Σ_{xx}, Σ_{yy} of a composite reinforced in the directions x, y e z and whose matrix strength is represented by the Drucker-Prager criterion.

Figure 5 displays the intersections of G^{hom} with the planes $\Sigma_{zz}/\Sigma_0 = 0$, $\Sigma_{zz}/\Sigma_0 = 0.7$, $\Sigma_{zz}/\Sigma_0 = 1.41$, $\Sigma_{zz}/\Sigma_0 = 2$ and $\Sigma_{zz}/\Sigma_0 = 3.41$. The intersections of the matrix strength domain with these planes (ellipses) are also displayed.

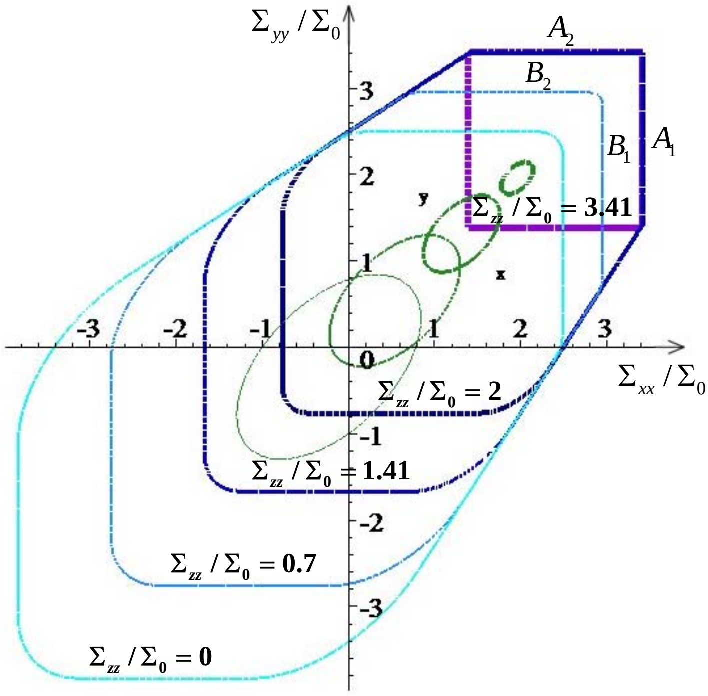

Figure 5: Intersections of G^{hom} and of the matrix strength domain with the planes $\Sigma_{zz}/\Sigma_0 = 0$, $\Sigma_{zz}/\Sigma_0 = 0.7$, $\Sigma_{zz}/\Sigma_0 = 1.41$, $\Sigma_{zz}/\Sigma_0 = 2$ and $\Sigma_{zz}/\Sigma_0 = 3.41$.

The vertices of the domains obtained by the translations vectors $f_x\sigma_x^{\pm}\underline{e}_x + f_y\sigma_y^{\pm}\underline{e}_y + f_z\sigma_z^{-}\underline{e}_z$ of the matrix strength domain are located in the plane $\Sigma_{zz} = (3\alpha_m\sigma_z + \sigma_m(\alpha_m+1))/3\alpha_m$, with $\sigma_z = f_z\sigma_z^{-}$. In this plane the zones C_1, C_2 and C_3 represent three points (the vertices) and the intersections between the zones A_1 and B_1 and A_2 and B_2 happen. Below this plane only zones defined as B and C occur, as it was shown in Figure 4. Above this plane, the equations of zones A_1 and A_2 substitute, respectively, the equations of the zones B_1 and B_2, as it can be seen in Figure 5.

The vertices of the domains obtained by the translations vectors $f_x\sigma_x^{\pm}\underline{e}_x + f_y\sigma_y^{\pm}\underline{e}_y + f_z\sigma_z^{+}\underline{e}_z$ of the matrix strength domain are located in the plane $\Sigma_{zz} = (3\alpha_m\sigma_z + \sigma_m(\alpha_m+1))/3\alpha_m$, with $\sigma_z = f_z\sigma_z^{+}$. The zone A_3 of the macroscopic strength domain is located in this plane. Stress states where Σ_{zz} has a value greater than this plan represents the situation of the composite material rupture.

2.4 Comparison with experimental results

The results obtained from the limit analysis and the homogenization theories are compared in this section with the experimental results of Yin et al. (1989), Swaddiwudhipong and Seow (2006) and Peres (2008) which were performed on concrete reinforced by steel fibers.

Yin et al. (1989) employed in their experiments fibers with aspect ratio $l/d = 60$ and volume fractions $f = 1\%$ and $f = 2\%$. In this analysis the real properties of the random medium are used to characterize the fictitious medium. The uniaxial and the biaxial compressive strength of the concrete matrix were, respectively, $f_c =$ 37.6 MPa and

$f_{cb} = 1.34 f_c$. Thus the following values for the parameters of the Drucker-Prager criterion are obtained: $\alpha_m = 0.2023$ and $\sigma_m = 24.94$ MPa. The uniaxial tensile strength of the fibers was 414 MPa. Figure 6 shows the experimental data of Yin et al. (1989) and the results obtained in this study (solid lines).

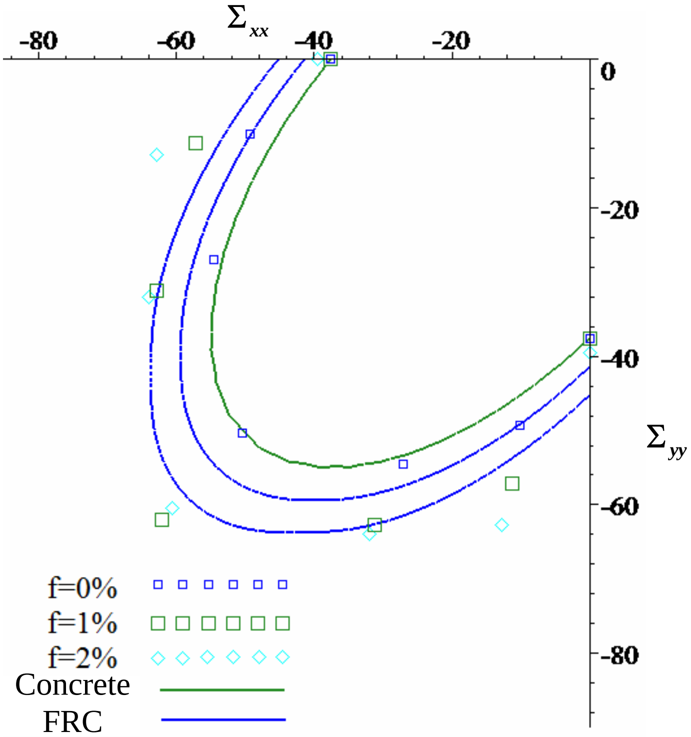

Figure 6: Experimental data of Yin et al. (1989) and results obtained in this study.

Swaddiwudhipong e Seow (2006) employed in their experiments fibers with aspect ratio $l/d =$ 55 and volume fraction $f =$ 0,5% 1% and 1,5%. In this analysis the real properties of the random medium are used to characterize the fictitious medium. The uniaxial and the biaxial compressive strength of the concrete matrix were, respectively, $f_c =$ 22.96 MPa and $f_{cb} = 1.23 f_c$. Thus the following values for the parameters of the Drucker-Prager criterion are obtained: $\alpha_m = 0.1575$ and $\sigma_m = 16.754$ MPa. The uniaxial tensile strength of the fibers was 1100 MPa. Figure 7 shows the experimental data of Swaddiwudhipong e Seow (2006) and the results obtained in this study (solid lines).

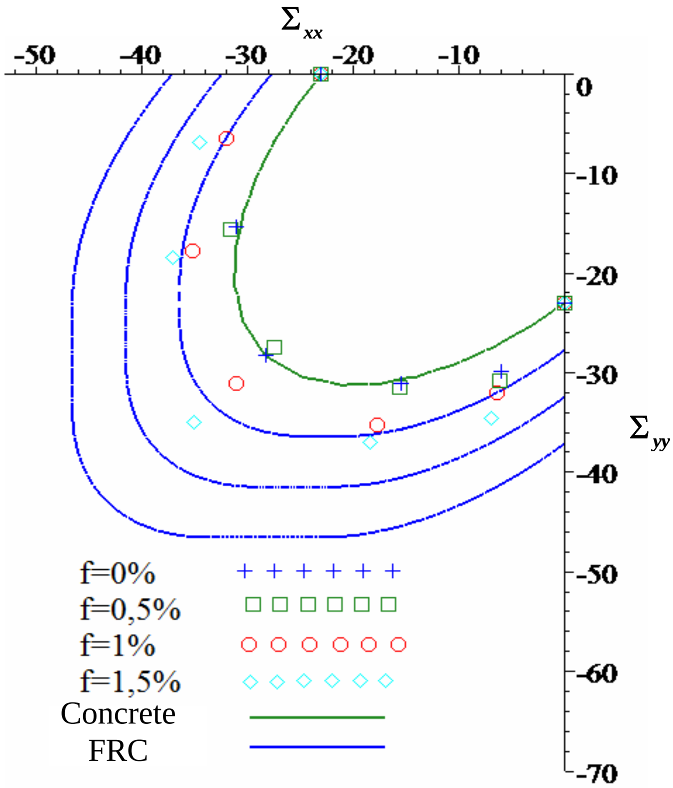

Figure 7: Experimental data of Swaddiwudhipong e Seow (2006) and results obtained in this study.

Peres (2008) employed in her experiments fibers with aspect ratio $l/d=$ 44 and volume fraction $f=$ 0,5% 1% and 1,5%. In this analysis the real properties of the random medium are used to characterize the fictitious medium. The uniaxial and the biaxial compressive strength of the concrete matrix were, respectively, $f_c=$ 28.24 MPa and $f_{cb}=1.16f_c$. Thus the following values for the parameters of the Drucker-Prager criterion are obtained: $\alpha_m=0.121$ and $\sigma_m=23.13$ MPa. The uniaxial tensile strength of the fibers was 1200 MPa. Figure 8 shows the experimental data of Peres (2008) and the results obtained in this study (solid lines).

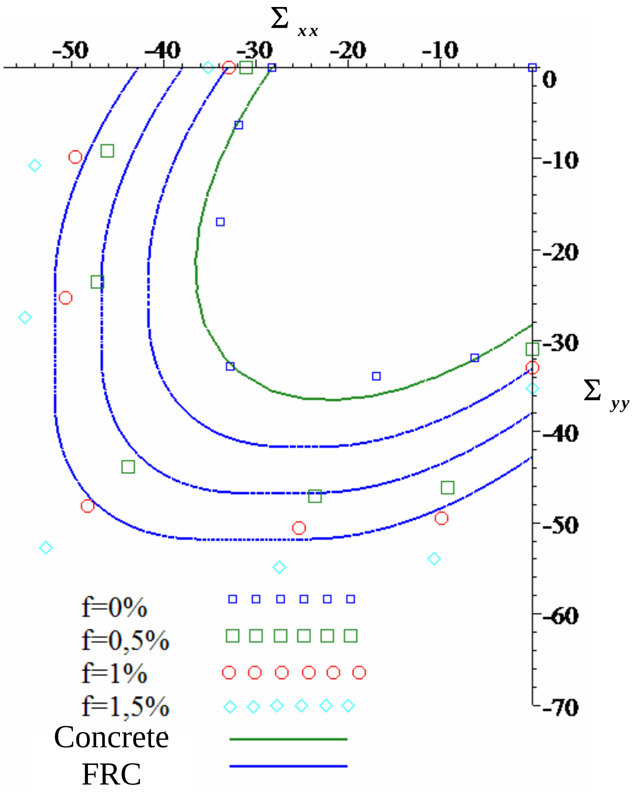

Figure 8: Experimental data of Peres (2008) and results obtained in this study.

The results presented above confirm that the use of the Drucker Prager criterion is suitable to the study of the concrete behavior under biaxial compression. This criterion has a simpler formulation than the Ottosen and the Willam-Warnke criteria, for example, not allowing the characterization of the concrete behavior under tensile and compressive stress states with the same parameters. Nevertheless, it allows performing the proposed study, employing the homogenization and limit analysis theories, in an analytical way.

The "expansion effect" that occurs on the concrete strength criterion when fibers are added to the matrix can be obtained through the employed approach. However, the comparison of the analytical results with the Yin et al. (1989) and Peres (2008) experimental data shows that the obtained results underestimate the biaxial compressive strength values and overestimate the uniaxial compressive strength values. It is observed that other studies (Williamson (1974), Narayanan and Darwish (1987), Bentur and Mindess (1990), Lim and Oh (1999)) have indicated that the increase of the uniaxial compressive strength in function of the fiber addition can be greater than the increase found in the Yin et al. (1989) and Swaddiwudhipong and Seow (2006) experiments.

Since the present study considered a fictitious medium with long fibers and the real material properties for the steel FRC analysis, it was expected that the analytical results of strength criterion of the reinforced material were higher than the experimental data of the concrete reinforced by short fibers. Further studies will be conducted to evaluate the influence of l/d on the composite material strength and on the fictitious properties of the fictitious medium used in this study. The comparison of the obtained results with the Yin et al. (1989) and Peres (2008) experimental data indicates that for usual values of the aspect ratio l/d, the use of real properties for the characterization of the fictitious medium may be appropriate. Following this reasoning, the fact that the analytical results are higher than the

Swaddiwudhipong and Seow (2006) experimental data may be related to the consideration of perfect bonding between the composite constituents.

3 CONCLUSIONS

The strength properties of steel FRC were evaluated in this work through the homogenization and limit analysis theories. Assuming the existence of a correlation between the random medium (FRC) and a fictitious medium defined by a concrete matrix reinforced by long fibers, the strength domain of this material was obtained analytically. The composite strength criterion was determined considering the constituents (fibers and matrix) volume fractions and strength properties. In the fibers case, only its uniaxial compressive and tensile strengths are needed. The matrix concrete strength, in turn, was characterized by the Drucker Prager criterion. This criterion has a simpler formulation than the Ottosen and the Willam-Warnke criteria, for example, not allowing the characterization of the concrete behavior under tensile and compressive stress states with the same parameters. Nevertheless, it allows performing the proposed study, employing the homogenization and limit analysis theories, in an analytical way.

The "expansion effect" that occurs on the concrete strength criterion when fibers are added to the matrix can be obtained through the employed approach. The comparison of the analytical results with the experimental data available in the literature suggests that the proposed model satisfactorily approximate the experiments. In all the analyzed cases, the micromechanical predictions seem to overestimate the strength properties of FRC under uniaxial compressive stresses. This aspect is likely to be explained by the assumption of perfect bonding at the fiber/matrix interface, which amounts to consider infinite resistance interface material. A more comprehensive approach would account for the strength properties of interface in the formulation of the macroscopic strength domain G^{hom}.

Studies aiming the evaluation of the fiber aspect ratio influence in the composite strength properties are still needed. Moreover, the verification of the influence of the fiber/matrix interface strength on the macroscopic strength and the consideration of the Otossen criterion for the characterization of the concrete matrix behavior are tasks which remain to be done.

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