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RELIABILITY SENSITIVITY OF LINEAR DYNAMICAL SYSTEMS SUBJECT TO GAUSSIAN EXCITATION

Marcos A. Valdebenito and Héctor A. Jensen

Department of Civil Engineering, Santa Maria University, Casilla 110-V, Valparaiso, Chile, {marcos.valdebenito,hector.jensen}@usm.cl

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Abstract. Probability theory offers an appropriate framework for quantifying the level of safety of a structural system. Thus, it is possible to evaluate the chances that a structure fulfills prescribed performance criteria during its lifetime due to uncertainties in loadings, deterioration processes, etc. The aforementioned chances are usually termed as the *reliability* associated with a structure. Although reliability constitutes a fundamental metric of safety, knowledge on the sensitivity of the reliability with respect to variables such as material properties, cross section of structural members, etc. is also of paramount importance. The information on reliability sensitivity is most valuable as it allows, for example, taking decisions on the final design of a structure such that its level of safety is relatively invariant in case unexpected changes occur.

Within the aforementioned framework, this contribution presents a novel approach for estimating the reliability sensitivity of a particular type of structures, namely linear structural systems subject to dynamic loading modeled as a Gaussian stochastic process. The sensitivity is computed with respect to deterministic variables that affect the structural performance (such as mass, stiffness, cross section of structural members, etc.). The proposed approach is based on an efficient simulation technique and local approximations of the functions used to model the structural performance. The reliability measure is expressed in terms of the first excursion probability, which is a criterion widely used in stochastic structural dynamics.

1 INTRODUCTION

One of the key issues for the analysis and design of civil engineering systems is the appropriate identification of the actions (such as water wave loading, wind loading, ground acceleration, etc.) that affect the structural performance during the life time. In most cases, such actions cannot be characterized deterministically due to inherent uncertainties (Freudenthal, 1956). Probability theory offers an appropriate framework for accounting for these uncertainties by means of, e.g. stochastic processes (Soong and Grigoriu, 1993). In this way, it is possible to generate a metric of the level of safety of a structural system. This metric refers to the reliability, i.e. the probability that a structure fulfills certain performance requirements during its lifetime. The complement of the reliability is the probability of failure (P_F), i.e. the probability that a structure violates prescribed performance criteria.

Besides determining the level of safety associated with a particular structural system, it is also of interest analyzing the sensitivity of the reliability with respect to variations in the properties of the structure (Au, 2005; Bjerager and Krenk, 1989; Ditlevsen and Madsen, 1996). For example, the determination of the variation in the reliability due to a change in the size of a structural member can provide useful information to increase the safety level or to identify the most influential design parameters.

In cases of practical interest, the quantification of the reliability of a structure is an extremely challenging task, as it requires evaluating integrals in high dimensional spaces involving implicit domains (Schuëller et al., 2004). In consequence, the evaluation of the sensitivity of the reliability with respect to system parameters will be usually even more involved than the reliability assessment. In view of this issue, this contribution proposes a novel approach for estimating reliability sensitivity for a particular class of structural systems, namely linear structures subject to dynamic loading modeled as a Gaussian stochastic process. The sensitivity is computed with respect to deterministic variables that affect the structural performance (such as mass, stiffness, cross section of structural members, etc.). The proposed approach is based on an efficient simulation technique and local approximations of the functions used to model the structural performance. The reliability measure is expressed in terms of the first excursion probability, which is a criterion widely used in stochastic structural dynamics.

The subject of efficient reliability sensitivity analysis has been considered several times in the literature. For example, sensitivity analysis of reliability estimates generated using approximate methods such as the First- and Second-Order Reliability Methods (FORM and SORM, respectively; see, e.g. Ditlevsen and Madsen (1996)) has been thoroughly studied in the literature for problems involving component or system reliability, see e.g. Bjerager and Krenk (1989); Karamchandani and Cornell (1992); Enevoldsen and Sørensen (1993, 1994). These approaches take advantage of the so-called design point associated with a particular reliability problem and eventual correlations between different individual failure modes in order to estimate the sought sensitivity efficiently. Reliability sensitivity analysis has also been studied within the framework of simulation methods. For example, in Melchers and Ahammed (2004), a method combining Monte Carlo Simulation (MCS) with a linear approximation scheme allows estimating the sought sensitivity. In Wu (1994), an estimator for the sensitivity based on Importance Sampling (IS) is proposed.

A common aspect of all approaches described above is that they are applicable whenever sensitivity is estimated with respect to a parameter characterizing the random variables of the reliability problem, such as the mean or standard deviation. However, these approaches are not directly applicable in case one seeks to estimate the sensitivity with respect to a deterministic variable. Thus, alternative approaches have been developed to address this case. For example, in Royset and Polak (2004), an algorithm for estimating the gradient of the failure probability using either MCS or IS is presented. This algorithm requires solving equations characterizing the structural performance either analytically or numerically; however, this can be an involved task. An alternative approach for reliability sensitivity analysis was introduced in Au (2005). The key issues of this approach are the association of an *instrumental variability* with the deterministic design variables involved in the sensitivity analysis and the application of the Bayes' theorem. Nonetheless, the range of applicability of this approach is limited as no more than 3 or 4 design variables can be considered simultaneously when performing sensitivity analysis (Koutsourelakis, 2008). A different strategy for estimating reliability sensitivity within the context of non linear structural dynamics was introduced in Jensen et al. (2009); Valdebenito and Schuëller (2009). The main aspect of this strategy is the introduction of a series of approximation concepts both at the level of reliability estimation and the functions modeling the structural performance. This approach has been tested successfully in problems involving up to 10 design variables.

In this contribution, the strategy developed in Jensen et al. (2009); Valdebenito and Schuëller (2009) is applied in order to estimate the reliability sensitivity of linear systems subject to dynamic Gaussian excitation. Novel aspects explored herein are the integration of the aforementioned strategy with an efficient IS scheme developed in Au and Beck (2001b) and an efficient means for generating approximations of functions related with the structural performance based on the sensitivity analysis of the spectral properties of the linear system under study, following the method proposed in Nelson (1976). These two novel aspects lead to a considerable reduction in the numerical costs associated with reliability sensitivity estimation when compared with the case of non linear structural systems.

The structure of this paper is as follows. Section 2 presents the details on the problem studied in this contribution. The approach for evaluating the structural response considering uncertainties in the excitation is discussed in Section 3. Then, Section 4 addresses an efficient simulation technique introduced in Au and Beck (2001b) for reliability evaluation of linear systems subject to Gaussian excitation. The proposed approach for reliability sensitivity estimation is discussed in Sections 5 and 6. A numerical example demonstrating the properties of the proposed approach is presented in Section 7. Finally, the paper closes with some final remarks and outlook.

2 GENERAL FORMULATION OF THE PROBLEM

Consider a structural system subject to a stochastic excitation of duration T. Let this excitation be represented by a random variable vector $\boldsymbol{z} \in \Omega_z \subset \mathbb{R}^{n_z}$ of dimension n_z characterized by a probability density function $p(\boldsymbol{z})$. Moreover, consider a set of responses of interest $r_i,\ i = 1, \ldots, n_r$ of the system modeling the structural performance due to the stochastic excitation such as, e.g. displacements. As the excitation acting over the structure is uncertain, the responses of interest will be uncertain as well. In addition, suppose there is a set of design variables grouped in a vector $\boldsymbol{y}$ of dimension n_y modeling properties of the system (such as cross section of structural members, material properties, etc.) that can be changed in order to alter the responses of interest. Clearly, the aforementioned responses of interest will be a function of both the design variable vector $\boldsymbol{y}$ and the random variable vector $\boldsymbol{z}$ characterizing the excitation. As this last vector also depends on time t, the responses will be also a function of time, i.e. $r_i = r_i(t, \boldsymbol{y}, \boldsymbol{z}),\ i = 1, \ldots, n_r,\ t \in [0, T]$.

For design purposes, the responses of interest $r_i,\ i = 1, \ldots, n_r$ are compared against acceptable threshold levels $r_i^*,\ i = 1, \ldots, n_r$. Within a deterministic framework, the objective would be ensuring that these responses do not exceed their prescribed thresholds. However, when uncertainties are considered, the aforementioned condition can be fulfilled only in a fraction of the possible cases, i.e. there is a probability that the responses do exceed their prescribed thresholds. Thus, reliability offers the means for quantifying the level of safety associated with a structural system. A criterion widely used to characterize the level of safety of a structure is the first excursion probability (see, e.g. Soong and Grigoriu (1993)). This probability measures the chances that the uncertain responses exceed in magnitude the prescribed thresholds within a specified time interval. In other words, this probability measures the chances of occurrence of the following *failure event* F.

$$F = \left\{ \boldsymbol{z} \in \Omega_z : \max_{i=1,\ldots,n_r} \left(\max_{t\in[0,T]} \left(\frac{|r_i(t,\boldsymbol{y},\boldsymbol{z})|}{r_i^*} \right) \right) \geq 1 \right\} \tag{1}$$

It is important to remark that *failure* does not necessarily imply collapse. In fact, the criterion proposed in eq. (1) may refer to, e.g. partial damage states or unacceptable system performance. Alternatively, the failure event defined in eq. (1) can be expressed in terms of the so-called *normalized demand* $D_N(\boldsymbol{y},\boldsymbol{z})$ (Au and Beck, 2001a):

$$F = \{ \boldsymbol{z} \subset \mathbb{R} : D_N(\boldsymbol{y},\boldsymbol{z}) \geq 1 \} \tag{2}$$

where:

$$D_N(\boldsymbol{y},\boldsymbol{z}) = \max_{i=1,\ldots,n_r} \left(\max_{t\in[0,T]} \left(\frac{|r_i(t,\boldsymbol{y},\boldsymbol{z})|}{r_i^*} \right) \right) \tag{3}$$

Once the failure event has been defined, it is possible to express its probability of occurrence by means of the following classical probability integral.

$$P_F(\boldsymbol{y}) = \int_{D_N(\boldsymbol{y},\boldsymbol{z})\geq 1} p(\boldsymbol{z}) d\boldsymbol{z} = \int_{\boldsymbol{z}\in\Omega_z} I_F(\boldsymbol{y},\boldsymbol{z}) p(\boldsymbol{z}) d\boldsymbol{z} \tag{4}$$

In the last equation, $P_F(\boldsymbol{y})$ denotes the probability of failure (i.e. probability of occurrence of the event F) and $I_F(\boldsymbol{y},\boldsymbol{z})$ is an indicator function that is equal to 1 in case the normalized demand is equal or larger than 1 and 0 otherwise.

As indicated in eq. (4), the probability of failure depends on the design variable vector $\boldsymbol{y}$. This has a clear physical interpretation: modifications on this vector (such as an increase in mass or stiffness) will certainly alter the response of the structure and also the probability of exceeding the prescribed thresholds. Thus, for design purposes, it is important to know not only the probability of failure $P_F(\boldsymbol{y})$ but also the change that this probability experiments due to changes in the design vector, as this allows determining the most influential design variables. This is denoted as the reliability sensitivity. A classical measure for sensitivity is calculating the gradient of the quantity of interest. However, within the context of linear dynamics, the estimation of the gradient of a probability may not be feasible as the gradient may not exist due to the non smooth normalized demand (see, e.g. Kang et al. (2006)). Therefore, in this contribution, an approximate representation of the failure probabilities that is differentiable is constructed and then, the gradient of this approximation is estimated. Details on this strategy are described in Sections 5 and 6.

3 STRUCTURAL RESPONSE EVALUATION

3.1 Mechanical Model

The first step for assessing reliability and its sensitivity is characterizing the structural response. The differential equation describing the motion of a linear structure of n degrees of freedom excited by a dynamical excitation is given by (see, e.g. Chopra (1995)):

$$\boldsymbol{M}\ddot{\boldsymbol{x}}(t) + \boldsymbol{C}\dot{\boldsymbol{x}}(t) + \boldsymbol{K}\boldsymbol{x}(t) = \boldsymbol{G}\boldsymbol{f}(t) \tag{5}$$

where $\boldsymbol{x}$(t) is the displacement response vector of dimension n; $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ are the mass, damping and stiffness matrices of dimension $n \times n$; $\boldsymbol{f}(t)$ is a vector of dimension n_f modeling the excitations acting over the structure; finally, $\boldsymbol{G}$ is a $n \times n_f$ dimensional matrix that couples the excitation components of the vector $\boldsymbol{f}(t)$ with the degrees of freedom of the structure. Note that the excitation vector $\boldsymbol{f}(t)$ depends on the vector of random variables $\boldsymbol{z}$ described in Section 2; the detailed relation between these vectors is discussed in Section 3.2. The responses of interest $r_i,\ i = 1, \ldots, n_r$ associated with a particular set of values of $\boldsymbol{y}$ can be evaluated using a convolution integral including the so-called impulse response functions $h_{i,j}(t, \boldsymbol{y}),\ i = 1, \ldots, n_r,\ j = 1, \ldots, n_f$, as indicated below.

$$r_i(t, \boldsymbol{y}, \boldsymbol{z}) = \sum_{j=1}^{n_f} \int_0^t h_{i,j}(t - \tau, \boldsymbol{y}) f_j(\tau) d\tau \tag{6}$$

In the last equation, $f_j(t)$ is the j-th component of the excitation vector and $h_{i,j}(t, \boldsymbol{y})$ is the impulse response function for the response function r_i at time t due to a unit impulse applied at the j-th input at time 0, where zero initial conditions have been assumed without loss of generality. It is noted that the dependence of the response function r_i on the vector of random variables $\boldsymbol{z}$ is due to the characterization of the stochastic excitation $\boldsymbol{f}(t)$. In case the response function of interest is given, for example, as a linear combination of the components of the displacement vector, that is, $r_i = \boldsymbol{\gamma}_i^T \boldsymbol{x}(t)$ and in case the system possesses classical damping, the corresponding impulse response function is given by:

$$h_{i,j}(t, \boldsymbol{y}, \boldsymbol{z}) = \sum_{r=1}^{n} \alpha_r^{i,j} \frac{1}{\omega_{d,r}} e^{-\zeta_l \omega_r t} \sin(\omega_{d,r} t), \quad \alpha_r^{i,j} = \frac{\boldsymbol{\gamma}_i^T \boldsymbol{\phi}_r \boldsymbol{\phi}_r^T \boldsymbol{g}_j}{\boldsymbol{\phi}_r^T \boldsymbol{M} \boldsymbol{\phi}_r} \tag{7}$$

where $\alpha_r^{i,j},\ r = 1, \ldots, n,\ i = 1, \ldots, n_r,\ j = 1, \ldots, n_f$ are mode factors, $\boldsymbol{\phi}_r,\ r = 1, \ldots, n$ are the eigenvectors associated with the eigenproblem of the undamped equation of motion, $\omega_r,\ r = 1, \ldots, n$ are the natural frequencies of the system, $\zeta_r,\ r = 1, \ldots, n$ are the corresponding damping ratios, $\omega_{d,r} = \omega_r \sqrt{(1 - \zeta_r^2)},\ r = 1, \ldots, n$ are the damped frequencies and $\boldsymbol{g}_j$ is the j-th column of the $\boldsymbol{G}$ matrix. The last equation can be interpreted as the modal superposition formula for the impulse response functions. One advantage of this representation is that in general only a relatively small number of modes m will be needed in for performing dynamic analysis, i.e. $m \ll n$. Eventually, the contribution of the remaining modes can also be considered in the formulation by using the static solution of the structural modes of higher order (Bathe, 1996).

3.2 Stochastic Excitation

Stochastic processes offer an adequate means for describing quantities fluctuating randomly in time such as earthquake ground motion, water wave loading, etc. In particular, a Gaussian process $X_t,\ t \in [0, T]$ is such that for every finite set $\{t_1, t_2, \ldots, t_{n_T}\} \in [0, T]$, the random variables $X_{t_1},\ X_{t_2}, \ldots, X_{t_{n_T}}$ are jointly normally distributed (see, e.g. Soong and Grigoriu (1993)). A quite general means for representing Gaussian processes is by applying the Karhunen-Loève (K-L) expansion of the corresponding covariance function (see, e.g. Loève (1963)). This representation is applicable to stationary as well as to non stationary stochastic processes. In order to apply this representation for the excitation vector $\boldsymbol{f}(t)$ acting over a particular structural system (see eq. (5)), consider a discrete representation of time such that $\Delta t = T/(n_T - 1)$ where T is the duration of the excitation and n_T the number of time points so that the time instants of analysis are $t_k = (k-1)\Delta t,\ k = 1, \ldots, n_T$. Thus, the discrete K-L representation of the j-th component of $\boldsymbol{f}(t)$ is the following.

$$f_j(t_k) \approx f_j^0(t_k) + \sum_{s=1}^{n_{KL}} f_j^s(t_k) z_s^j,\ k = 1, \ldots, n_T,\ j = 1, \ldots, n_f \tag{8}$$

In the equation above, $z_s^j,\ s = 1, \ldots, n_{KL}$ are independent, identically distributed standard Gaussian random variables, $f_j^0(t_k)$ and $f_j^s(t_k)$ denote the mean function and the s-th K-L component of $f_j(t_k)$, respectively, and n_{KL} is the order of truncation of the series expansion. The K-L vectors are determined by solving an eigenproblem of the corresponding covariance matrix of the discrete process. In the remaining part of this contribution, it is assumed – without loss of generality – that the excitation is a zero-mean process, i.e. $f_j^0(t_l) = 0,\ l = 1, \ldots, n_T$.
Considering the characterization of the components of the excitation vector described above and considering the convolution integral in eq. (6), the dynamic response of interest evaluated at time t_k can be written as:

$$r_i(t_k, \boldsymbol{y}, \boldsymbol{z}) = \Delta t \sum_{j=1}^{n_f} \sum_{s=1}^{n_{KL}} \left(\sum_{l=1}^{k} \epsilon_l h_{i,j}(t_k - t_l, \boldsymbol{y}) f_j^s(t_l) \right) z_s^j \tag{9}$$

where ϵ_l is a coefficient depending on the numerical integration scheme used in the evaluation of the convolution integral.

4 RELIABILITY ANALYSIS

4.1 Characterization of Elementary Failure Events

According to the definition in eq. (1), failure takes place whenever the absolute value of any of the responses exceeds the prescribed thresholds at any time within the duration of the stochastic excitation. Considering the discretization scheme proposed above, the occurrence of the failure event reduces to comparing the value of the n_r responses during the n_T discrete time steps against the prescribed thresholds. Thus, the failure event F can be expressed as the union of *elementary failure events* $F_{i,k},\ i = 1, \ldots, n_r,\ j = 1, \ldots, n_T$, i.e.:

$$F = \bigcup_{i=1}^{n_r} \bigcup_{k=1}^{n_T} F_{i,k} \tag{10}$$

where:

$$F_{i,k} = \left\{ \boldsymbol{z} \in \Omega_z : \frac{|r_i(t_k, \boldsymbol{y}, \boldsymbol{z})|}{r_i^*} \geq 1 \right\} \tag{11}$$

Based on eq. (10), it is clear that the probability of failure is actually the probability of the union of a number of elementary failure elements. According to eq. (11), the elementary failure regions are defined as the region in the random variables space which cause a barrier crossing at instant t_k due to the i-th response function. Then, it is seen that the evaluation of $P_F(\boldsymbol{y})$ corresponds to a reliability problem of a series system of $n_r \times n_T$ failure elements. Using the linear relations between input and response in terms of the Gaussian random variables $\boldsymbol{z}$ (see eq. (9)),

the so-called design points can be established in a straightforward manner defining uniquely the elementary failure regions $F_{i,k}$, $i = 1, \ldots, n_r$, $j = 1, \ldots, n_T$, see e.g. Der Kiureghian (2000); Ditlevsen and Madsen (1996). It is reminded that the design point associated with the (i, k)-th elementary failure domain – which is denoted as $\boldsymbol{z}^*_{i,k}$ – can be defined using two equivalent criteria (Freudenthal, 1956). According to the geometrical criterion, the design point is the realization in the standard normal space that belongs to $F_{i,k}$ with the minimum Euclidean norm with respect to the origin. According to the probabilistic interpretation, the design point is the point belonging to $F_{i,k}$ with highest probability density $p(\boldsymbol{z})$.
Once the design point $\boldsymbol{z}^*_{i,k}$ has been established, the calculation of the probability of occurrence of the elementary failure events is straightforward. In particular, this probability is equal to $2\Phi(-\beta_{i,k}(\boldsymbol{y}))$, where $\Phi(\cdot)$ is the cumulative distribution function of the standard Gaussian distribution, and $\beta_{i,k}(\boldsymbol{y})$ is the so-called reliability index. Note that this reliability index is the norm of the corresponding design point, i.e. $\beta_{i,k}(\boldsymbol{y}) = ||\boldsymbol{z}^*_{i,k}||$. For the particular case under study, the expression for calculating the reliability index associated with the i-th response and k-th discrete time is the following (see, e.g. Der Kiureghian (2000)).

$$\beta_{i,k}(\boldsymbol{y}) = ||\boldsymbol{z}^*_{i,k}|| = \frac{r^*_i}{\Delta t \sqrt{\sum_{j=1}^{n_f} \sum_{s=1}^{n_{KL}} \left(\sum_{l=1}^{k} \epsilon_l h_{i,j}(t_k - t_l, \boldsymbol{y}) f^s_j(t_l) \right)^2}} \tag{12}$$

Additionally, the (s, j)-th component of the vector $\boldsymbol{z}^*_{i,k}$ is equal to:

$$\left(\boldsymbol{z}^*_{i,k}\right)^s_j = \frac{r^*_i \left(\sum_{l=1}^{k} \epsilon_l h_{i,j}(t_k - t_l, \boldsymbol{y}) f^s_j(t_l) \right)}{\Delta t \sum_{j=1}^{n_f} \sum_{s=1}^{n_{KL}} \left(\sum_{l=1}^{k} \epsilon_l h_{i,j}(t_k - t_l, \boldsymbol{y}) f^s_j(t_l) \right)^2}, \; s = 1, \ldots, n_{KL}, \; j = 1, \ldots, n_f \tag{13}$$

4.2 Simulation Strategy

As the elementary failure regions are fully described by the design points, Importance Sampling (IS) arises as an option for evaluating the probability of occurrence (see, e.g. Schuëller and Stix (1987)). Thus, the probability integral in eq. (4) is estimated as:

$$P_F(\boldsymbol{y}) = \int_{\boldsymbol{z} \in \Omega_z} I_F(\boldsymbol{y}, \boldsymbol{z}) \frac{p(\boldsymbol{z})}{p_{IS}(\boldsymbol{z})} p_{IS}(\boldsymbol{z}) d\boldsymbol{z} \approx \frac{1}{N} \sum_{v=1}^{N} I_F(\boldsymbol{y}, \boldsymbol{z}^{(v)}) \frac{p(\boldsymbol{z}^{(v)})}{p_{IS}(\boldsymbol{z}^{(v)})} \tag{14}$$

where $p_{IS}(\boldsymbol{z})$ is the Importance Sampling density (ISD) function and $\boldsymbol{z}^{(v)}$, $v = 1, \ldots, N$ are samples of the vector of uncertain parameters simulated according to $p_{IS}(\boldsymbol{z})$. The key issue for the application of the IS scheme described in eq. (14) is the design of an appropriate ISD function ensuring a low variability of the estimator of the probability. In particular,the ISD function proposed in Au and Beck (2001b) is applied here. This importance sampling density is defined as a weighted sum of probability density functions conditioned on the elementary failure events described previously, that is:

$$p_{IS}(\boldsymbol{z}) = \sum_{i=1}^{n_r} \sum_{k=1}^{n_T} \omega_{i,k}(\boldsymbol{y}) p_{IS}(\boldsymbol{z}/F_{i,k}) \tag{15}$$

where $\omega_{i,k} \geq 0$ is the weight associated with the elementary failure domain $F_{i,k}$ and it is defined such that:

$$\omega_{i,k}(\boldsymbol{y}) = \frac{\Phi(-\beta_{i,k}(\boldsymbol{y}))}{\sum_{l=1}^{n_r} \sum_{s=1}^{n_T} \Phi(-\beta_{l,s}(\boldsymbol{y}))} \tag{16}$$

Additionally, the distribution of $\boldsymbol{z}$ conditioned on the elementary failure domain $F_{i,k}$ is defined as (according to Bayes' theorem):

$$p_{IS}(\boldsymbol{z}/F_{i,k}) = \frac{p(\boldsymbol{z}) I_{F_{i,k}}(\boldsymbol{y}, \boldsymbol{z})}{2\Phi(-\beta_{i,k}(\boldsymbol{y}))} \tag{17}$$

where $I_{F_{i,k}}(\boldsymbol{y}, \boldsymbol{z})$ is an indicator function equal to one in case the (i, k)-th failure event takes place and zero, otherwise. Replacing the expressions in eqs. (15), (16) and (17) into eq. (14) yields the following estimator for the failure probability:

$$P_F(\boldsymbol{y}) \approx \hat{P}_F(\boldsymbol{y}) = \frac{2}{N} \left(\sum_{i=1}^{n_r} \sum_{k=1}^{n_T} \Phi(-\beta_{i,k}(\boldsymbol{y})) \right) \sum_{v=1}^{N} \frac{1}{\sum_{i=1}^{n_r} \sum_{k=1}^{n_T} I_{F_{i,k}}(\boldsymbol{y}, \boldsymbol{z}^{(v)})} \tag{18}$$

where $\hat{P}_F(\boldsymbol{y})$ denotes an estimator of the failure probability. It should be noted that in the estimator in eq. (18), the indicator function $I_F(\boldsymbol{y})$ is set equal to one, as the generation of samples of $\boldsymbol{z}$ conditioned on an elementary failure event ensures that the failure events occurs. In summary, the estimation of the failure probability using the estimator in eq. (18) requires the characterization of the elementary failure domains through the design points and reliability indexes. Samples of the vector $\boldsymbol{z}$ conditioned on the elementary failure events are required as well; these samples can be generated according to the technique proposed in Au and Beck (2001b). Finally, the structural response associated with each of the samples of $\boldsymbol{z}$ must be evaluated and compared with the threshold levels in order to compute the indicator function $I_{F_{i,k}}(\boldsymbol{y}, \boldsymbol{z})$.

5 FRAMEWORK FOR ESTIMATING RELIABILITY SENSITIVITY

As already mentioned above, the sought sensitivity of the probability of failure is expressed as the gradient of an approximate representation of the probability. This approximation is continuous and smooth, ensuring differentiability. In particular, consider the following approximate representation of the normalized demand function:

$$D_N(\boldsymbol{y} + \Delta\boldsymbol{y}, \boldsymbol{z}) \approx \tilde{D}_N(\boldsymbol{y} + \Delta\boldsymbol{y}, \boldsymbol{z}) = D_N(\boldsymbol{y}, \boldsymbol{z}) + \sum_{q=1}^{n_y} a_q \Delta y_q \tag{19}$$

where $\Delta\boldsymbol{y}$ is a certain perturbation of the design variable vector and $a_q,\ q = 1, \ldots, n_y$ are real coefficients (the issue on how to compute these coefficients is discussed in Section 6). Now, consider the following limit, expressing the derivative of the approximate failure probability $\tilde{P}_F(\boldsymbol{y})$ with respect to $y_q,\ q = 1, \ldots, n_y$.

$$\frac{\partial \tilde{P}_F(\boldsymbol{y})}{\partial y_q} = \lim_{\Delta y_q \to 0} \frac{\tilde{P}_F(\boldsymbol{y} + \boldsymbol{v}(q)\Delta y_q) - \tilde{P}_F(\boldsymbol{y})}{\Delta y_q} \tag{20}$$

$$= \lim_{\Delta y_q \to 0} \frac{P\left[\tilde{D}_N(\boldsymbol{y} + \boldsymbol{v}(q)\Delta y_q, \boldsymbol{z}) \geq 1\right] - P\left[\tilde{D}_N(\boldsymbol{y}, \boldsymbol{z}) \geq 1\right]}{\Delta y_q} \tag{21}$$

where $P[\cdot]$ denotes the probability of occurrence of the event between brackets and where $\boldsymbol{v}(q)$ is a vector of length n_y with zero entries, except by the q-th entry, which is equal to one.

Introducing the approximation of the normalized demand function of eq. (19) in eq. (21) yields the following result.

$$\frac{\partial \tilde{P}_F(\boldsymbol{y})}{\partial y_q} = \lim_{\Delta y_q \to 0} \frac{P\left[D_N(\boldsymbol{y},\boldsymbol{z}) \geq 1 - a_q \Delta_{y_q}\right] - P\left[D_N(\boldsymbol{y},\boldsymbol{\theta}) \geq 1\right]}{\Delta y_q} \tag{22}$$

To evaluate the limit in eq. (22), it is necessary to estimate the probability that the normalized demand (for a given design $\boldsymbol{y}$) exceeds two different threshold levels. That is, it is necessary to estimate $P[D_N(\boldsymbol{y},\boldsymbol{z}) \geq b]$ for the values $b = 1$ and $b = 1 - a_q \Delta_{y_q}$. Numerical experience has shown that this relation can be approximated by means of an exponential function (Gasser and Schuëller, 1997), i.e.:

$$P\left[D_N(\boldsymbol{y},\boldsymbol{z}) \geq b\right] \approx e^{\psi_0 + \psi_1 (b-1)}, \quad b \in [1-\epsilon, 1+\epsilon] \tag{23}$$

where ϵ is a small constant, e.g. $\epsilon = 0.02$ and where $\psi_0,\ \psi_1$ are real coefficients (the details on the calculation of these coefficients are discussed in Section 6). After introducing eq. (23) in (22), an expression for computing the probability sensitivity can be obtained, as shown below.

$$\frac{\partial \tilde{P}_F(\boldsymbol{y})}{\partial y_q} = \lim_{\Delta y_q \to 0} \frac{e^{\psi_0 - \psi_1 a_q \Delta y_q} - e^{\psi_0}}{\Delta y_q} = -\psi_1 a_q e^{\psi_0} = -\psi_1 a_q \hat{P}_F(\boldsymbol{y}) \tag{24}$$

It is most interesting to note that the expression for estimating the derivative of the failure probability in eq. (24) depends on the estimate of the failure probability $\hat{P}_F(\boldsymbol{y})$, the coefficient ψ_1 and on $a_q,\ q = 1, \ldots, n_y$. While the estimate of the failure probability can be computed with the procedure outlined in Section 4, the remaining coefficients are obtained using the procedure outlined in the following Section.

6 IMPLEMENTATION OF RELIABILITY SENSITIVITY ANALYSIS

6.1 Exponential Approximation of Probability

As discussed in the previous Section, the implementation of the proposed strategy for reliability sensitivity requires the generation of an exponential approximation of the probability that the normalized demand exceeds a threshold b. The procedure for reliability estimation described in Section 4 allows estimating the probability that the normalized demand exceeds 1. However, this procedure can be easily extended for estimating the required relation. As this required relation must be estimated for a range $b \in [1-\epsilon, 1+\epsilon]$, the definition of the elementary failure domains is slightly altered. In particular, the reliability indexes and design points defined in eqs. (12) and (13) are modified as follows: the term that denotes the threshold level r_i^* is replaced by $r_i^*(1-\epsilon)$. This change ensures that the samples of $\boldsymbol{z}$ are such that the values of the associated normalized demands are equal or larger than $(1-\epsilon)$. In the second place, an indicator function $I_{F,b}(\boldsymbol{y},\boldsymbol{z})$ is introduced in the estimator of eq. (18). This indicator function is defined such that:

$$I_{F,b}(\boldsymbol{y},\boldsymbol{z}) = \begin{cases} 1 & \text{if } D_N(\boldsymbol{y},\boldsymbol{z}) \geq b \\ 0 & \text{if } D_N(\boldsymbol{y},\boldsymbol{z}) < b \end{cases} \tag{25}$$

The indicator function in the equation above is dependent on the threshold level b. This function is then included in the probability estimator of eq. (18) as shown below.

$$\hat{P}_F\left[D_N(\boldsymbol{y},\boldsymbol{z}) \geq b\right] = \frac{2}{N}\left(\sum_{i=1}^{n_r}\sum_{k=1}^{n_T}\Phi(-\beta_{i,k}(\boldsymbol{y}))\right)\sum_{v=1}^{N}\frac{I_{F,b}(\boldsymbol{y},\boldsymbol{z}^{(v)})}{\sum_{i=1}^{n_r}\sum_{k=1}^{n_T} I_{F_{i,k}}(\boldsymbol{y},\boldsymbol{z}^{(v)})} \tag{26}$$

The estimator above can be used directly to estimate the probability that the normalized demand exceeds a certain threshold level . Thus, the estimator is applied as follows. Firstly, samples of the vector $\boldsymbol{z}$ conditioned on the modified elementary failure events are generated. Then, response and normalized demand associated with each of these samples is computed. In the next step, the probability that the normalized demand exceeds a threshold level is computed over a suitable grid $b \in [1-\epsilon, 1+\epsilon]$. This allows generating pairs $(b, \hat{P}_F[D_N(\boldsymbol{y}, \boldsymbol{z}) \geq b])$. Finally, the coefficients $\psi_0,\ \psi_1$ of the exponential approximation proposed in eq. (23) are calculated on a least square sense using the aforementioned pairs.
An important feature of the approach proposed above for calculating the coefficients $\psi_0,\ \psi_1$ is that it requires a single run of the simulation algorithm. Moreover, it should be noted that only minor modifications are introduced to the original approach for estimation the probability of failure presented in Au and Beck (2001b) and described in Section 4 of this contribution.

6.2 Linear Approximation of Normalized Demand Function

The expression in eq. (19) is as a *linear* expansion of the normalized demand function with respect to a perturbation in the design variable vector. In most cases, this expansion will not be exact, as changes in the normalized demand function due to perturbations in the design variables are non linear, implicit functions. Nonetheless, as long as the coefficients $a_q,\ q = 1, \ldots, n_y$ are calibrated appropriately, it could be expected that $\tilde{D}_N(\boldsymbol{y} + \Delta\boldsymbol{y}, \boldsymbol{z})$ can approximate $D_N(\boldsymbol{y} + \Delta\boldsymbol{y}, \boldsymbol{z})$ reasonably well. For a better understanding of the last point, consider the definition of normalized demand function in eq. (3). As this function includes the $\max(\cdot)$ operator, several elementary failure domains may be relevant in the calculation of $P_F(\boldsymbol{y})$. The sensitivity of each these elementary domains with respect to the design variables will be – in general – different. Thus, the linear expansion shown in eq. (19) will not capture those individual sensitivities. However, the coefficients a_q represent an *average sensitivity*, i.e. they are an approximate representation of the sensitivity of the individual elementary failure domains.
In view of the issues discussed above, the proposed approach for calibrating the coefficients a_q consists in two steps. In the first one, samples of the vector modeling the uncertain excitation $\boldsymbol{z}$ are generated such that they lie precisely in the boundary of the failure event. The second step consists in calculating the sensitivity of the normalized demand function with respect to the design variables at the samples generated at the previous step. Thus, the sought coefficients can be calculated as the average of the sensitivities calculated for each sample.
From the two steps described above, the first one can be easily achieved by scaling the samples of $\boldsymbol{z}$ generated at the reliability analysis step. Suppose that for a particular sample $\boldsymbol{z}^\star$, the associated normalized demand function is equal to $D_N(\boldsymbol{y}, \boldsymbol{z}^\star) = b^\star$. Then, the required sample is equal to the original sample multiplied by the reciprocal of $b^\star$, i.e. $D_N(\boldsymbol{y}, (1/b^\star)\boldsymbol{z}^\star) = 1$. This property holds due to the linearity of the equation of motion with respect to the excitation (see eq. (5)).
The second step of the proposed approach for calibrating the coefficients a_q consists in estimating the sensitivity of the normalized demand function with respect to the design variables at the sample point $\boldsymbol{z}^\star$. Suppose that the sample $\boldsymbol{z}^\star$ lies in the boundary of the $(i^\star, k^\star)$-th elementary failure domain. That means that the failure event is reached because the i-th response achieved the threshold value r_i^* at the k-th discrete time step. Thus, the sensitivity of the associated normalized demand function can be approximated by considering the derivatives of corresponding

response at that particular time, i.e.:

$$\frac{\Delta D_N(\boldsymbol{y}, (1/b^\star)\boldsymbol{z}^\star)}{\Delta y_q} \approx \frac{1}{r_i^*}\text{sgn}\left(r_i(t_k, \boldsymbol{y}, (1/b^\star)\boldsymbol{z}^\star)\right)\frac{\partial r_i(t_k, \boldsymbol{y}, (1/b^\star)\boldsymbol{z}^\star)}{\partial y_q},\ q = 1, \ldots, n_y \quad (27)$$

In the last equation, $\text{sgn}(\cdot)$ is the sign function. The derivative of the response function r_i at the k-th time can be computed by calculating the derivative of the impulse response function with respect to the design variables. Thus, considering the expression for calculating the response of interest in eq. (9), the expression for estimating the sensitivity of the response is as follows.

$$\frac{\partial r_i(t_k, \boldsymbol{y}, \boldsymbol{z})}{\partial y_q} = \Delta t \sum_{j=1}^{n_f}\sum_{s=1}^{n_{KL}}\left(\sum_{l=1}^{k}\epsilon_l \frac{\partial h_{i,j}(t_k - t_l, \boldsymbol{y})}{\partial y_q} f_j^s(t_l)\right) z_s^j \quad (28)$$

In the equation above, $\partial h_{i,j}(t_l, \boldsymbol{y})/\partial y_q$ represents the derivative of the impulse response function with respect to the q-th design variable. Moreover, it is assumed that the coefficients of the K-L expansion of the stochastic excitation do not depend on the design variable $y_q,\ q = 1, \ldots, n_y$.

The calculation of the derivative of the impulse response function implies calculating the derivative of the spectral properties (frequencies and eigenvectors associated with the eigenproblem of the undamped equation of motion). In this contribution, an efficient procedure for determining eigenvector derivatives developed in Nelson (1976) is used. In that procedure the calculation of the derivatives of a given eigenvector requires only eigendata associated with that eigenvector. The simplified procedure is much more efficient than standard methods in which eigenvector derivatives are expressed in terms of all eigenvectors.

Once the sensitivities of the responses are calculated according to eq. (28), the sought coefficients a_q are calculated taking the average of the sensitivities calculated for all the samples generated.

$$a_q = \frac{1}{N}\sum_{v=1}^{N}\frac{\Delta D_N(\boldsymbol{y}, (1/b^{\star,(v)})\boldsymbol{z}^{\star,(v)})}{\Delta y_q} \quad (29)$$

An interesting feature of the approach proposed above is that it requires a sensitivity analysis of the spectral properties. The numerical costs associated with this analysis can be considerably lower than solving the eigenproblem for perturbed values of the design variables.

7 NUMERICAL EXAMPLE

7.1 Description

A 2-DOF linear shear beam model is considered in the example. The model is depicted in fig. (1). The system is excited by a horizontal ground acceleration $g_A(t)$ of 15 [s] duration, which is modeled as a discrete white noise. The responses of the system to be controlled involve the displacements of each floor due to the ground acceleration. These responses are compared with prescribed thresholds in order to define whether or not failure occurs. The objective is estimating the sensitivity of the probability of failure with respect to the stiffnesses of each of floor.

As shown in fig. (1), each floor of the model is supported by two columns. It is assumed that the columns are perfectly clamped and axially very stiff; thus, each column can be described by its lateral stiffness $k_i/2 = 9 \times 10^6$ [N/m], $i = 1, 2$. The mass of each floor of the model is equal to $m_1 = m_2 = 30 \times 10^3$ [kg]. Additionally, two viscous dampers are located between

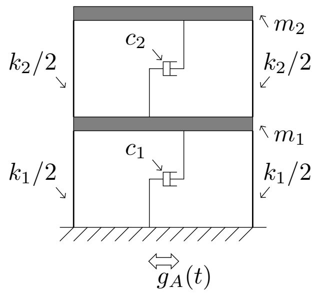

Figure 1: Example - 2-DOF shear beam model

each floor; the damping ratios associated with these dampers are equal to 4% for each vibration mode. Thus, the equation of motion of the mechanical model can be written as:

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \ddot{\boldsymbol{x}}(t) + \begin{pmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{pmatrix} \dot{\boldsymbol{x}}(t) + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \boldsymbol{x}(t) = \boldsymbol{f}(t) \tag{30}$$

The external force vector is defined as $\boldsymbol{f}(t) = -\boldsymbol{M} g_A(t)$, where $\boldsymbol{M}$ is the mass matrix and $g_A(t)$ is a horizontal ground acceleration, which is modeled as a discrete white noise of 15 [s] duration. A time discretization step equal to $\Delta t = 0.01$ [s] is used to model the ground acceleration, i.e. a total of 1501 time instants are involved in the problem. The discrete representation of the white noise signal is such that the coefficients of the K-L expansion of the ground acceleration g_A are equal to:

$$f^s(t_k) = \begin{cases} \sqrt{\frac{2\pi S}{\Delta t}} & \text{if } s = k \\ 0 & \text{if } s \neq k \end{cases}, \ k = 1, \ldots, 1501 \tag{31}$$

In the equation above, $S = 10^{-4}$ [m^2/s^3] is the spectral density of the white noise.
The failure event is formulated as a first passage problem during the time of analysis; the structural responses to be controlled are the 2 interstorey drift displacements and the roof displacement. The threshold values are chosen equal to 6×10^{-3} [m] for each of these three responses. In addition, the design variable vector $\boldsymbol{y}$ groups the stiffnesses of the columns of each floor, i.e. $\boldsymbol{y} = \langle k_1, k_2 \rangle^T$. Thus, the reliability problem is formally defined as:

$$P_F(\boldsymbol{y}) = \int_{D_N(\boldsymbol{y},\boldsymbol{z}) \geq 1} p(\boldsymbol{z}) d\boldsymbol{z} \tag{32}$$

where:

$$D_N(\boldsymbol{y}, \boldsymbol{z}) = \max_{i=1,2,3} \left(\max_{k=1,\ldots,1501} \left(\frac{|r_i(\Delta t(k-1), \boldsymbol{y}, \boldsymbol{z})|}{r_i^*} \right) \right) \tag{33}$$

7.2 Results

In order to estimate the reliability sensitivity of the failure probability in eq. (32) with respect to the stiffnesses, two different approaches are considered, namely:

1. Firstly, the sensitivity is estimated by performing finite differences, i.e.:

$$\frac{\Delta \hat{P}_F(\boldsymbol{y})}{\Delta y_q} = \frac{\hat{P}_F(\boldsymbol{y} + \boldsymbol{v}(q)\Delta y_q) - \hat{P}_F(\boldsymbol{y} - \boldsymbol{v}(q)\Delta y_q)}{2\Delta y_q}, \quad l = 1, 2 \tag{34}$$

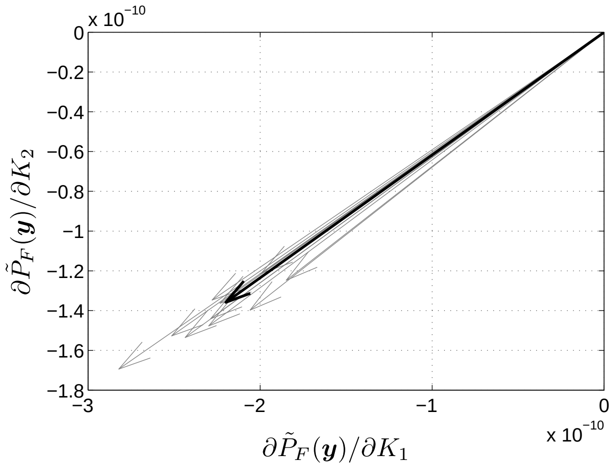

Figure 2: Sensitivity Analysis - 10 independent estimations using proposed approach compared to reference solution

In the estimator of the equation above, the perturbation introduced is such that $\Delta y_q/y_q = 0.1\%$. As it can be noted from the formula, two reliability analyses are required for estimating the sensitivity with respect to one design variable. For each of these reliability analyses, a total of 200 samples of the vector modeling the excitation are generated. As the sensitivity measure in eq. (34) may be affected by the variability of the probability estimator $\hat{P}_F(\boldsymbol{y})$, a total of 200 independent estimations of the sensitivity are performed and then, an average result is considered. Thus, the estimate of the sensitivity generated using this procedure is considered as a reference.

2. Secondly, the sensitivities are estimated using the approach proposed in Sections 5 and 6. A total of 200 independent runs of this approach are performed, in order to assess the variability of the estimates. Each individual simulation involves generating 200 samples of the vector modeling the stochastic excitation.

The results obtained are shown in figs. (2) and (3), respectively. Figure (2) shows the estimates of probability sensitivity in the form of arrows indicating the magnitude of the sensitivity; the results obtained with the proposed approach are presented with gray, thin line (only ten simulations were plotted for the sake of clarity) and the reference result with black, thick line. It can be observed that the estimates generated with the proposed approach present very little dispersion. Moreover, on the average, they tend to converge to the reference result. Figure (3) also shows the probability sensitivity estimates for the two approaches described above. Note that this figure includes the results for the proposed approach in terms of the mean plus/minus one standard deviation. As it can be observed, the variability of the proposed approach is quite low, producing accurate estimates of the probability sensitivity.

8 CONCLUSIONS AND OUTLOOK

This contribution has presented an approach for estimating reliability sensitivity of linear structures subject to Gaussian stochastic excitation. The approach is based on a general framework that was proposed for non linear systems. Taking advantage of the linearity of the problem studied here, this general framework was modified appropriately.
The key issue of the proposed approach is replacing the original reliability problem by one that

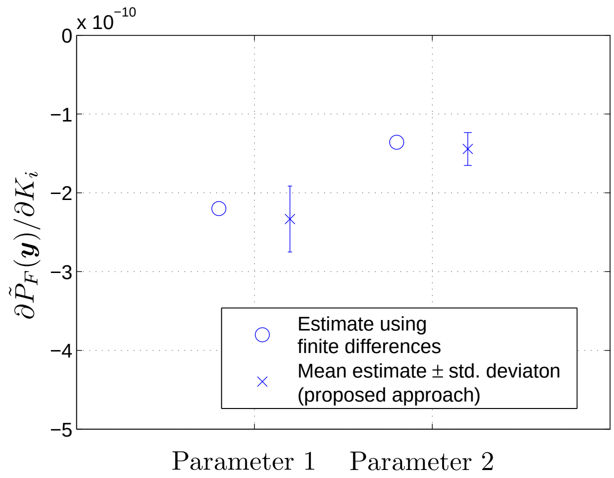

Figure 3: Sensitivity Analysis - mean $\pm$ std. deviation using results of proposed approach compared to reference solution

is differentiable. Thus, the sought sensitivities are calculated by combining an efficient simulation technique with two approximation concepts. The first of these approximations involves the construction of an exponential function modeling the probability that the normalized demand exceeds a prescribed threshold. This approximation can be constructed using the information generated by the simulation technique when estimating reliability. The second approximation involves the sensitivity analysis of the responses of interest of the structural system with respect to design variables. This sensitivity analysis can be performed most efficiently by applying a specific procedure for computing the sensitivity of spectral properties.

A salient feature of the proposed approach for reliability sensitivity is that it can be very efficient from a numerical point of view. The same samples generated for reliability sensitivity can be employed for constructing the aforementioned exponential approximation. In addition, the second approximation mentioned above requires a sensitivity analysis of the spectral properties, which is a problem that is numerically inexpensive when compared to solving a full eigenproblem for perturbed values of the design variable vector.

Future research efforts on the approach proposed here aim at testing the proposed approach in examples involving a larger number of design variables, subject not only to white noise but also to other types of Gaussian processes. Particular emphasis will be given at investigating the variability of the sensitivity estimators produced with the method presented in this contribution

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REFERENCES

Au S. Reliability-based design sensitivity by efficient simulation. *Computers & Structures*, 83(14):1048–1061, 2005.

Au S. and Beck J. Estimation of small failure probabilities in high dimensions by subset simulation. *Probabilistic Engineering Mechanics*, 16(4):263–277, 2001a.

Au S. and Beck J. First excursion probabilities for linear systems by very efficient importance sampling. *Probabilistic Engineering Mechanics*, 16(3):193–207, 2001b.

Bathe K. *Finite Element Procedures*. Prentice Hall, New Jersey, 1996.

Bjerager P. and Krenk S. Parametric sensitivity in first order reliability theory. *Journal of Engineering Mechanics*, 115(7):1577–1582, 1989.

Chopra A. *Dynamics of structures: theory and applications to earthquake engineering*. Prentice Hall, 1995.

Der Kiureghian A. The geometry of random vibrations and solutions by FORM and SORM. *Probabilistic Engineering Mechanics*, 15(1):81–90, 2000.

Ditlevsen O. and Madsen H. *Structural Reliability Methods*. John Wiley and Sons, 1996.

Enevoldsen I. and Sørensen J. Reliability-based optimization of series systems of parallel systems. *Journal of Structural Engineering*, 119(4):1069–1084, 1993.

Enevoldsen I. and Sørensen J. Reliability-based optimization in structural engineering. *Structural Safety*, 15(3):169–196, 1994.

Freudenthal A. Safety and the probability of structural failure. *ASCE Transactions*, 121:1337–1397, 1956.

Gasser M. and Schuëller G. Reliability-based optimization of structural systems. *Mathematical Methods of Operations Research*, 46(3):287–307, 1997.

Jensen H., Valdebenito M., Schuëller G., and Kusanovic D. Reliability-based optimization of stochastic systems using line search. *Computer Methods in Applied Mechanics and Engineering*, 198(49-52):3915–3924, 2009.

Kang B.S., Park G.J., and Arora J. A review of optimization of structures subjected to transient loads. *Structural and Multidisciplinary Optimization*, 31(2):81–95, 2006.

Karamchandani A. and Cornell C.A. Sensitivity estimation within first and second order reliability methods. *Structural Safety*, 11(2):95–107, 1992.

Koutsourelakis P. Design of complex systems in the presence of large uncertainties: A statistical approach. *Computer Methods in Applied Mechanics and Engineering*, 197(49-50):4092–4103, 2008.

Loève M. *Probability theory*. D. Van Nostrand Company, Inc., Princeton, New Jersey, third edition, 1963.

Melchers R. and Ahammed M. A fast approximate method for parameter sensitivity estimation in Monte Carlo structural reliability. *Computers & Structures*, 82(1):55–61, 2004.

Nelson R. Simplified calculation of eigenvector derivatives. *AIAA Journal*, 14(9):1201–1205, 1976.

Royset J. and Polak E. Reliability-based optimal design using sample average approximations. *Probabilistic Engineering Mechanics*, 19(4):331–343, 2004.

Schuëller G., Pradlwarter H., and Koutsourelakis P. A critical appraisal of reliability estimation procedures for high dimensions. *Probabilistic Engineering Mechanics*, 19(4):463–474, 2004.

Schuëller G. and Stix R. A critical appraisal of methods to determine failure probabilities. *Structural Safety*, 4(4):293–309, 1987.

Soong T. and Grigoriu M. *Random Vibration of Mechanical and Structural Systems*. Prentice Hall, Englewood Cliffs, New Jersey, 1993.

Valdebenito M. and Schuëller G. Efficient reliability sensitivity estimation for optimization under uncertainty. In K. Kalinski, editor, *80th Annual Meeting of the International Association of Applied Mathematics and Mechanics (GAMM)*, volume 9 of *Proc. Appl. Math. Mech. (PAMM)*, pages 577–578. WILEY-VCH, Gdansk, Poland, EU, 2009.

Wu Y. Computational methods for efficient structural reliability and reliability sensitivity analysis. *AIAA Journal*, 32(8):1717–1723, 1994.