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ON THE USE OF THE STEKLOV ANALYSIS TO VERIFY THE STABILITY CONDITION OF MIXED FINITE ELEMENT FORMULATIONS

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Abstract. In finite element approximations of conservation laws using mixed formulations the stability depends strongly on the compatibility of the approximation space of the primary variable and dual variable. This compatibility is also known as the Ladyzenskaya-Babuska-Brezzi (LBB)-condition.

This work is dedicated to the development of a numerical procedure to verify the compatibility of the approximation spaces based on the analysis of the quality of approximations of a Steklov eigenvalue problem. It is observed that in the occurence of poorly configured approximation spaces, the numerical approximation of the eigenvalues of the Steklov problem either presents artificial (spurious) low energy eigenvalues or a reduced number of correctly approximated values. The first case is an indication of a poor constraint space and the latter is an indication of a too rich constraint space.

The advantage of using the Steklov eigenvalue problem is that the numerically obtained eigenvalues can be compared to either analytical values.

1 INTRODUCTION

Finite element approximations of mixed formulations of differential equations have gained renewed interest due to their interesting features in terms of local conservation and orders of convergence of the primal and dual variables.

The development of p and hp adaptive approximation spaces for mixed formulations present a challenge in terms of compatibility of approximation spaces of the primal variable and dual variable. If the approximation space for the dual variable is too poor, the lack of restraints on the primal variable may cause the solution to oscillate. Such oscillations are known as spurious modes. If, on the other hand, the approximation space of the dual variable is too rich, the problem of the primal variable is over restrained, leading to a fenomenon known as locking.

The compatibility of approximation spaces can be expressed by a condition known as the Ladyzenskaya-Babuska-Brezzi (LBB) condition. Although the LBB condition is mathematically elegant, it is difficult to use as a practical tool for analysing the compatibility of spaces generated in an hp-adaptive context.

Numerical approaches have been proposed to analyse the compatibility of approximation spaces. In Brezzi and Fortin (1991), it is shown that the compatibility of the spaces can be analysed by studying the eigenvalues associated with the restraint matrix. In Bathe (2001), an analysis of the inf sup condition is also based on the behaviour of the eigenvalues of the matrix of a mixed finite element approximation. Spurious modes appear as small eigenvalues. In neither of these publications objetive values are given to compare with.

In this work, the compatibility of the approximation spaces is studied by observing the evolution of the numerical approximation of a Steklov problem. The advantage of using a Steklov problem is that analytical and/or high precision eigenvalues can be computed. These values can be used for comparaison with numerically obtained values. Spurious modes appear as artificial eigenvalues, which do not correspond to Steklov eigenvalues. Locking is recognized by the inability of the approximation to represent the first eigenmodes and corresponding eigenvalues.

2 THE STEKLOV EIGENVALUE PROBLEM

A Steklov problem is known as an eigenvalue problem relating the flux over the boundary of the domain with the solution: Find $\lambda \in I\!R$ and $p \in H^1(\Omega)$ such that

$$\begin{aligned} \triangle p = & \quad 0, \text{in } \Omega \\ \frac{\partial p}{\partial \eta} = & \quad \lambda p \quad \text{in } \partial\Omega \end{aligned} \tag{1}$$

It is well known that the solution of the above problem is given by a sequence of eigenvalues and eigenvectors $(\lambda_j, u_j) \in I\!R \times H^1(\Omega)$, where λ_j, $j = 1, 2, ...$, are positive and diverge to $+\infty$ (see Babuska and Osborn (1991)). We assume that the eigenvalues are increasingly ordered,

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \lambda_j \leq \cdots , \tag{2}$$

with each eigenvalue occurring many times as given by its multiplicity.

For a square domain (Figure 1), the eigenvalue problem (1) can be solved using separation of variables $p(x_1, x_2) = v_1(x_1)v_2(x_2)$, with $v_1, v_2 \in C^2(\Omega)$. Inserting in (1) we get

$$\frac{v_1''(x_1)}{v_1(x_1)} + \frac{v_2''(x_2)}{v_2(x_2)} \quad = \quad 0 \tag{3}$$

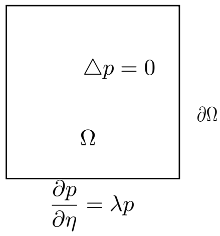

Figure 1: Square domain

with boundary conditions $-v_i'(-1) = \lambda v_i(-1)$ and $v_i'(1) = \lambda v_i(1)$.

In the case where $\frac{v_1''(x_1)}{v_1(x_1)} = \frac{v_2''(x_2)}{v_2(x_2)} = 0$ the corresponding eigenvalue is $\lambda = 1$ and the eigenvector $p(x_1, x_2) = x_1 x_2$. Otherwise, it follows that the solutions have the form

$$\begin{aligned} v_1(x_1) &= c_1\text{Sin}(x_1) + c_2\text{Cos}(x_2) \\ v_2(x_2) &= c_3\text{Sinh}(x_1) + c_4\text{Cosh}(x_2). \end{aligned} \tag{4}$$

To impose the boundary condition, the eigenvalues are obtained by finding the roots of the follow equations

$$\begin{aligned} \text{Cot}(n_1) - \text{Coth}(n_1) &= 0 \\ \text{Cot}(n_2) - \text{Tanh}(n_2) &= 0 \\ \text{Tan}(n_3) + \text{Tanh}(n_3) &= 0 \\ \text{Tan}(n_4) + \text{Coth}(n_4) &= 0 \end{aligned} \tag{5}$$

where n_i are integer multiple of π. The eigenvectors are obtained by linear combination of

$$\begin{aligned} p_1(x_1, x_2) &= \text{Sin}(n_1x_1)\text{Sinh}(n_1x_2) \\ p_2(x_1, x_2) &= \text{Cosh}(n_2x_2)\text{Sin}(n_2x_1) \\ p_3(x_1, x_2) &= \text{Cos}(n_3x_1)\text{Cosh}(n_3x_2) \\ p_4(x_1, x_2) &= \text{Cos}(n_4x_1)\text{Sinh}(n_4x_2) \end{aligned} \tag{6}$$

Some eigenvalues and corresponding eigenvectors are shown in Table 1.

i	λ_i	$p_i(x, y)$
1	0	1
2	0.688253	Cosh(0.937552y) Sin(0.937552x)
3	0.688253	Cosh(0.937552x) Sin(0.937552y)
4	1	xy
5	2.32364	Cos(2.36502 x) Cosh(2.36502y)
6	2.32364	Cos(2.36502 y) Cosh(2.36502x)
7	2.39039	Cos(2.34705 y) Sinh(2.34705x)
8	2.39039	Cos(2.34705 x) Sinh(2.34705y)
9	3.92965	Sin(3.92965 x) Sinh(3.92965y)
⋮	⋮	⋮

Table 1: The first eigenvalues ans corresponding eigenvectors for the problem 1

Figure 2 shows the plots of the eigenvectors corresponding to eigenvalues λ_2, λ_4 and λ_5. Figure 3 illustrates the eigenvalues with their multiplicity.

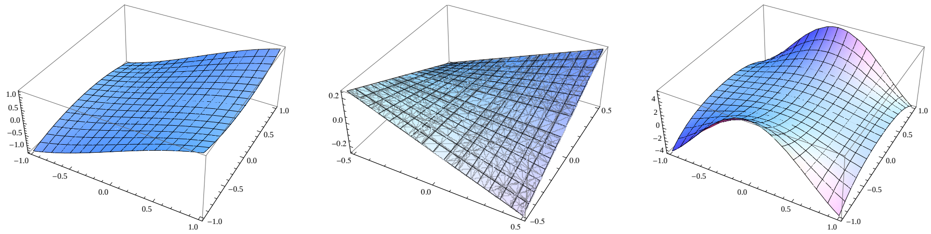

Figure 2: First eigenvectors for regular boundary conditions

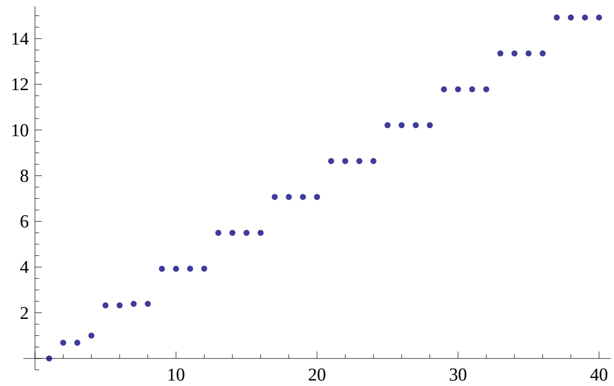

Figure 3: Eigenvalues for regular boundary conditions

3 A MIXED FORMULATION APPLIED TO A STEKLOV PROBLEM

The mixed formulation of the Steklov problem applied to the Laplace equation can be written as. Find $\lambda \in I\!R$ and $(\boldsymbol{u}, p) \in H(div) \times L^2(\Omega)$ such that

$$\begin{aligned} \boldsymbol{u} = &\quad \nabla p \quad \text{in } \Omega \\ div(\boldsymbol{u}) = &\quad 0 \quad \text{in } \Omega \\ \boldsymbol{u} \cdot \boldsymbol{\eta} = &\quad \lambda p \quad \text{in } \partial\Omega \end{aligned} \tag{7}$$

Setting $(\boldsymbol{V}_h, W_h)$ finite element spaces from $H(div) \times L^2(\Omega)$ and using a substructuring approach, the algebraic problem from (7) can be written as

$$\begin{pmatrix} A_{I\,I} & B_{I\,I} & A_{I\,\Gamma} \\ B_{I\,I}^T & 0 & B_{I\,\Gamma} \\ A_{I\,\Gamma}^T & B_{I\,\Gamma}^T & A_{\Gamma\,\Gamma} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_I \\ p_I \\ \boldsymbol{u}_\Gamma \end{pmatrix} = \sigma \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C_{\Gamma\,\Gamma} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_I \\ p_I \\ \boldsymbol{u}_\Gamma \end{pmatrix} \tag{8}$$

This system is associate to a generalized eigenvalue problem $\tilde{A}\boldsymbol{w} = \sigma\tilde{B}\boldsymbol{w}$, where $\boldsymbol{w} = (\boldsymbol{u}_I, p_I, \boldsymbol{u}_\Gamma)^T$.

Following the classical results from Babuska and Osborn (1991) we know that the problem (8) has a sequence of positive eigenvalues

$$\lambda_{1,h} \leq \lambda_{2,h} \leq \cdots \leq \lambda_{2,n} \tag{9}$$

and corresponding eigenvectors

$$(\boldsymbol{u}_{1,h}, p_{1,h}), (\boldsymbol{u}_{2,h}, p_{2,h}), \cdots, (\boldsymbol{u}_{n,h}, p_{n,h})$$

with $n = dim(\boldsymbol{V}_h \times W_h)$.

4 DEVELOPING H(DIV) APPROXIMATION SPACES

In this section we describe a systematic way to construct finite element spaces $\boldsymbol{V}_h$. For details, we refer to Siqueira et al. (2010). The approximation spaces W_h are formed by piecewise discontinuous polynomials.

Quadrilateral Meshes

Let $\hat{K} = \{(\xi,\eta) : -1 \leq \xi,\eta \leq 1\}$ be the master element with vertices $a_0 = (-1,-1)$, $a_1 = (1,-1)$, $a_2 = (1,1)$ and $a_3 = (-1,1)$. The edges $l_k, k = 0,\cdots 3$ correspond to the sides linking the vertices a_k to $a_{k+1(mod4)}$.

In Devloo et al. (2009), a hierarchy of finite element subspaces in $H^1(\Omega)$ is constructed, where the basics functions in $\hat{K}$ are classified by:

- 4 vertex functions

$$\varphi^{a_0}(\xi,\eta) = \frac{(1-\xi)}{2}\frac{(1-\eta)}{2}, \quad \varphi^{a_1}(\xi,\eta) = \frac{(1+\xi)}{2}\frac{(1-\eta)}{2} \tag{10}$$

$$\varphi^{a_2}(\xi,\eta) = \frac{(1+\xi)}{2}\frac{(1+\eta)}{2}, \quad \varphi^{a_3}(\xi,\eta) = \frac{(1-\xi)}{2}\frac{(1+\eta)}{2} \tag{11}$$

  Note that the value of φ^{a_k} is one at a_k and zero at the other vertices.

- $4(p-1)$ edge functions

$$\begin{aligned}
\varphi^{l_0,n}(\xi,\eta) &= \varphi^{a_0}(\xi,\eta)[\varphi^{a_1}(\xi,\eta)+\varphi^{a_2}(\xi,\eta)]f_n(\xi),\\
\varphi^{l_1,n}(\xi,\eta) &= \varphi^{a_1}(\xi,\eta)[\varphi^{a_2}(\xi,\eta)+\varphi^{a_3}(\xi,\eta)]f_n(\eta),\\
\varphi^{l_2,n}(\xi,\eta) &= \varphi^{a_2}(\xi,\eta)[\varphi^{a_3}(\xi,\eta)+\varphi^{a_0}(\xi,\eta)]f_n(-\xi),\\
\varphi^{l_3,n}(\xi,\eta) &= \varphi^{a_3}(\xi,\eta)[\varphi^{a_0}(\xi,\eta)+\varphi^{a_1}(\xi,\eta)]f_n(-\eta),
\end{aligned}$$

  where f_n are the Chebychev polynomials of degree n, $n = 0,1,\cdots,p-2$. The edge functions $\varphi^{l_k,n}$ vanish on all edges l_m, $m \neq k$;

- $(p-1)^2$ surface functions

$$\varphi^{C,n_0,n_1}(\xi,\eta) = \varphi^{a_0}(\xi,\eta)\varphi^{a_2}(\xi,\eta)f_{n_0}(\xi)f_{n_1}(\eta), \tag{12}$$

  with $0 \leq n_0, n_1 \leq p-2$. These functions are zero on all edges.

Let us consider a set of eighteen vectors $\boldsymbol{v}_m$, as indicated in Figure 4, satisfying the properties

1. $\boldsymbol{v}_{2+3k} = \overrightarrow{\eta}_k$ is the outward unit normal, and $\boldsymbol{v}_{k+12}$ is tangent to l_k.
2. for $m = 3k$, $\boldsymbol{v}_m \cdot \boldsymbol{v}_{m+1} = \boldsymbol{v}_m \cdot \boldsymbol{v}_{m+2} = \boldsymbol{v}_{m+1} \cdot \boldsymbol{v}_{m+2} = 1$.
3. on the surface element, v_{16} and $\boldsymbol{v}_{17}$ are orthogonal vectors $\boldsymbol{v}_{16} \perp \boldsymbol{v}_{17}$.

We propose the construction of a family of vector functions by multiplication this vector field by the hierarchical scalar basis according to the following procedure:

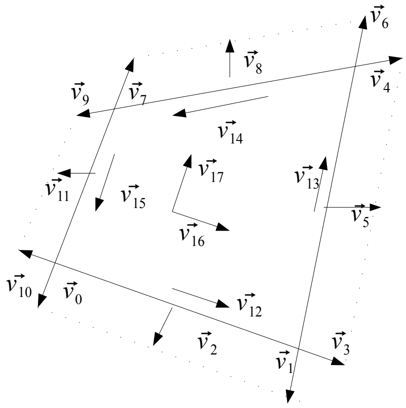

Figure 4: Vector field for H(div)-quadrilateral elements

$4(p+1)$ edge vector functions

$$
\begin{aligned}
k=0: &\quad \overrightarrow{\varphi}^{l_0,a_0}=\varphi^{a_0}\,\overrightarrow{v_0}, \quad \overrightarrow{\varphi}^{l_0,a_1}=\varphi^{a_1}\,\overrightarrow{v}_1, \quad \overrightarrow{\varphi}^{l_0,n}=\varphi^{l_0,n}\,\overrightarrow{v}_2 && (13)\\
k=1: &\quad \overrightarrow{\varphi}^{l_1,a_1}=\varphi^{a_1}\,\overrightarrow{v_3}, \quad \overrightarrow{\varphi}^{l_1,a_2}=\varphi^{a_2}\,\overrightarrow{v}_4, \quad \overrightarrow{\varphi}^{l_1,n}=\varphi^{l_1,n}\,\overrightarrow{v}_5 && (14)\\
k=2: &\quad \overrightarrow{\varphi}^{l_2,a_2}=\varphi^{a_2}\,\overrightarrow{v_6}, \quad \overrightarrow{\varphi}^{l_2,a_3}=\varphi^{a_3}\,\overrightarrow{v}_7, \quad \overrightarrow{\varphi}^{l_2,n}=\varphi^{l_2,n}\,\overrightarrow{v}_8 && (15)\\
k=3: &\quad \overrightarrow{\varphi}^{l_3,a_3}=\varphi^{a_3}\,\overrightarrow{v_9}, \quad \overrightarrow{\varphi}^{l_3,a_0}=\varphi^{a_0}\,\overrightarrow{v}_{10}, \quad \overrightarrow{\varphi}^{l_3,n}=\varphi^{l_3,n}\,\overrightarrow{v}_{11} && (16)
\end{aligned}
$$

Observe that the vector functions associated to the edge l_0 satisfy

$$\overrightarrow{\varphi}^{l_0,a_0}\cdot\overrightarrow{\eta_0}=\varphi^{a_0}\in Q_1(K), \quad \overrightarrow{\varphi}^{l_0,a_1}\cdot\overrightarrow{\eta_0}=\varphi^{a_1}\in Q_1(K), \quad \overrightarrow{\varphi}^{l_0,n}\cdot\overrightarrow{\eta}_0=\varphi^{l_0,n}\in Q_n(K). \tag{17}$$

Similar results hold for the vectors functions associated to l_k, $k=1,2$ and 3

$$\overrightarrow{\varphi}^{l_k,a_j}\cdot\overrightarrow{\eta_k}=\varphi^{a_j}\in Q_1(K), \quad \text{for } j=k,k+1(mod4), \quad \overrightarrow{\varphi}^{l_k,n}\cdot\overrightarrow{\eta}_k=\varphi^{l_k,n}\in Q_n(K). \tag{18}$$

$2(p^2-1)$ internal vector functions

To complete the space, we add three types of functions

$$\overrightarrow{\varphi}_1^{C,n_0,n_1}=\varphi^{C,n_0,n_1}\,\overrightarrow{v}_{16}, \quad \overrightarrow{\varphi}_2^{C,n_0,n_1}=\varphi^{C,n_0,n_1}\,\overrightarrow{v}_{17}, \quad \text{and} \quad \overrightarrow{\varphi}_3^{l_k,n}=\varphi^{l_k,n}\,\overrightarrow{v}_{k+12}. \tag{19}$$

The normal components of these internal vector functions vanishes at all edges.

The numbers of edge and internal vector functions sums $2(p+1)^2$, coinciding with the dimension of $\boldsymbol{V}_K=Q_p(K)\times Q_p(K)$.

Triangular Meshes

Consider the master triangular element $\hat{K}=\{(\xi,\eta):0\le\xi\le 1, 0\le\eta\le 1-\xi\}$, with vertices $a_0=(0,0)$, $a_1=(1,0)$ and $a_2=(0,1)$, and edges l_k, $k=0,1,2$ linking the vertex a_k to $a_{k+1(mod3)}$. For the hierarchy of finite element subspaces in $H^1(\Omega)$ constructed in Devloo et al. (2009), the basic functions are classified by:

- 3 vertex functions

$$\varphi^{a_0}(\xi,\eta)=1-\xi-\eta, \quad \varphi^{a_1}(\xi,\eta)=\xi, \quad \varphi^{a_2}(\xi,\eta)=\eta \tag{20}$$

that have unit value on the corresponding vertex and zero on the other ones;

- $3(p-1)$ edge functions

$$\begin{aligned} \varphi^{l_0,n}(\xi,\eta) &= \varphi^{a_0}(\xi,\eta)\varphi^{a_1}(\xi,\eta)f_n(\eta+2\xi-1), & (21)\\ \varphi^{l_1,n}(\xi,\eta) &= \varphi^{a_1}(\xi,\eta)\varphi^{a_2}(\xi,\eta)f_n(\eta-\xi), & (22)\\ \varphi^{l_2,n}(\xi,\eta) &= \varphi^{a_2}(\xi,\eta)\varphi^{a_0}(\xi,\eta)f_n(1-\xi-2\eta) & (23)\end{aligned}$$

- $\frac{(p-2)(p-1)}{2}$ surface functions

$$\varphi^{C,n_0,n_1}(\xi,\eta) = \varphi^{a_0}(\xi,\eta)\varphi^{a_1}(\xi,\eta)\varphi^{a_2}(\xi,\eta)f_{n_0}(2\xi-1)f_{n_1}(2\eta-1) \quad (24)$$

  with $0 \leq n_0 + n_1 \leq p-3$.

Consider a field of fourteen vectors associated to a triangular element, as illustrated in Figure 5. These vectors satisfy the properties

1. $\boldsymbol{v}_{2+3k} = \overrightarrow{\eta}_k$ is the outward unit normal, and $\boldsymbol{v}_{k+9}$ is tangent to the edge l_k.
2. for $m = 3k$, $\boldsymbol{v}_m \cdot \boldsymbol{v}_{m+1} = \boldsymbol{v}_m \cdot \boldsymbol{v}_{m+2} = \boldsymbol{v}_{m+1} \cdot \boldsymbol{v}_{m+2} = 1$.
3. $\boldsymbol{v}_{12} \perp \boldsymbol{v}_{13}$.

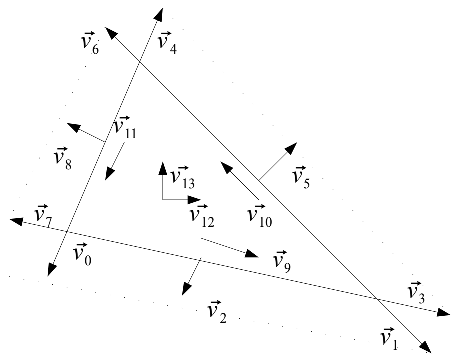

Figure 5: Vector field for Hdiv-triangular elements

As in the quadrilateral case, we introduce the vector functions associated to the edges

$$k=0: \qquad \overrightarrow{\varphi}^{l_0,a_0} = \varphi^{a_0}\,\overrightarrow{v_0}, \quad \overrightarrow{\varphi}^{l_0,a_1} = \varphi^{a_1}\,\overrightarrow{v}_1, \quad \overrightarrow{\varphi}^{l_0,n} = \varphi^{l_0,n}\,\overrightarrow{v}_2 \quad (25)$$

$$k=1: \qquad \overrightarrow{\varphi}^{l_1,a_1} = \varphi^{a_1}\,\overrightarrow{v_3}, \quad \overrightarrow{\varphi}^{l_1,a_2} = \varphi^{a_2}\,\overrightarrow{v}_4, \quad \overrightarrow{\varphi}^{l_1,n} = \varphi^{l_1,n}\,\overrightarrow{v}_5 \quad (26)$$

$$k=2: \qquad \overrightarrow{\varphi}^{l_2,a_2} = \varphi^{a_2}\,\overrightarrow{v_6}, \quad \overrightarrow{\varphi}^{l_2,a_3} = \varphi^{a_3}\,\overrightarrow{v}_7, \quad \overrightarrow{\varphi}^{l_2,n} = \varphi^{l_2,n}\,\overrightarrow{v}_8 \quad (27)$$

and internal vector functions

$$\overrightarrow{\varphi}_1^{C,n_0,n_1} = \varphi^{C,n_0,n_1}\,\overrightarrow{v}_{12} \qquad \overrightarrow{\varphi}_2^{C,n_0,n_1} = \varphi^{C,n_0,n_1}\,\overrightarrow{v}_{13} \qquad \overrightarrow{\varphi}_3^{l_k,n} = \varphi^{l_k,n}\,\overrightarrow{v}_{9+k}. \quad (28)$$

Again, the normal components of the vector functions associated to the edge l_k are given by

$$\overrightarrow{\varphi}^{l_k,a_j} \cdot \overrightarrow{\eta_k} = \varphi^{a_j} \in P_1(K), \quad \text{for } j = k, k+1(mod3), \quad \overrightarrow{\varphi}^{l_k,n} \cdot \overrightarrow{\eta}_k = \varphi^{l_k,n} \in P_n(K), \tag{29}$$

and the normal components of the internal vector functions vanish at all edges. Furthermore, for triangular elements the total number of edge and internal vector functions is $(p+1)(p+2)$, also coinciding with the dimension of $V_K = P_p(K) \times P_p(K)$.

Having defined the two set of hierarchical vector functions in V_K, both for quadrilateral and triangular elements, it remains to verify that they indeed form bases for V_K. Furthermore, if they to be combined to span H(div) spaces $V(\tau_h)$, we need to show that the normal components on the elements interfaces are continuous. As proved in Siqueira et al. (2010) the following results hold.

Theorem 1 The edge and internal vector functions defined in equations (13–19), for quadrilateral elements, and in formulae (25-28) for triangular elements, form a hierarchical basis for V_K.

Theorem 2 Using the hierarchical vector bases defined by equations (13–19), for quadrilateral elements, and in formulae (25-28) for triangular elements, H(div)-conforming spaces $V(\tau_h)$ can be created by imposing that the sum of the multiplying coefficients associated with the edge vector functions of neighboring elements is zero .

5 NUMERICAL RESULTS

The purpose of this section is to present some numerical results for the problem (7) using finite element spaces $(\boldsymbol{V}_h, W_h)$ based on uniform meshes τ_h. The primary spaces $\boldsymbol{V}_h = V(\tau_h)$ are $H(div)$ conforming, as described in the previews section.

In order to guarantee stability and uniqueness of discrete solution, the couple of finite element spaces must satisfy the LBB condition: for all $w_h \in W_h$ exist $\beta_h > 0$ and $\boldsymbol{u}_h \in \boldsymbol{V}_h$ such that

$$\sup_{0\neq \boldsymbol{u}_h \in \boldsymbol{V}_h} \frac{(div\, \boldsymbol{u}_h, w_h)}{\|\boldsymbol{u}_h\|_{\boldsymbol{V}}} \geq \beta_h \|w_h\|_W \tag{30}$$

This inf-sup condition has been proved for some pair of spaces Raviart and Thomas (1977), Boffi et al. (2006). In general analytical procedures are hard to be established, and numerical methods are required. For instance, in order to give a numerical procedure to choose the correct pair of spaces $(\boldsymbol{V}_h, W_h)$, Bathe (2001) shows that $\beta_h = \lambda_{min}$, with λ_{min} denoting the smallest eigenvalue of a constraint matrix. On this case if β_h is kept away from zero, independently of h, the stability condition is satisfied.

In this paper, we propose to analyse the stability condition for the pair of spaces $(\boldsymbol{V}_h, W_h)$ by observing the evolution of numerical approximation of the Steklov problem. If the spaces are compatible, it is expected that the eigenvalue problem will be well approximated.

In our simulation, the approximation spaces are of type $Q_p P_{p-1}$, which means that in each cell K the bases for $\boldsymbol{V}_h$ are in $Q_p \times Q_p$, where Q_p denotes the polynomial of tensorial degree less or equal p, and the bases for W_h are in P_{p-1}, with total degree less or equal than $p-1$. The results are shown in next tables.

Figure 6 illustrates the case $Q_1 P_0$ the first three different eigenvalues λ_3, Λ_6 and λ_8. It can be verified the rate of convergence is of second order, as indicated in the first three lines of Table 6. In our tests, similar results also hold for the first twenty eigenvalues, suggesting that the pair $Q_1 P_0$ is compatible.

p	16 elements	64 elements	256 elements	1024 elements
1	0.0108845	0.00267593	0.000666242	0.00016639
2	0.0117147	0.00488317	0.00154731	0.000429736
3	0.000341	0.0000231604	1.49399*10^{-6}	–

Table 2: Error for $\lambda_3 = 0.688253$

p	16 elements	64 elements	256 elements	1024 elements
1	0.0763622	0.0195096	0.00489751	0.00122555
2	0.0208147	0.0285629	0.0133846	0.0043857
3	0.0305218	0.00277274	0.000192544	–

Table 3: Error for $\lambda_6 = 2.32364$

p	16 elements	64 elements	256 elements
1	0.0379973	0.00901995	0.00222836
2	0.26372	0.0879632	0.0253418
3	0.0337548	0.00279347	0.00019258

Table 4: Error to $\lambda_8 = 2.39039$

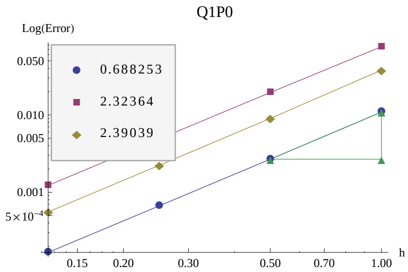

Figure 6: Q_1P_0 approximations for the first three different eigenvalues

λ_i	16 elements	64 elements	256 elements	1024 elements
0.688253	2.02417	2.00592	2.00147	2.00037
2.32364	1.96867	1.99407	1.99861	1.99966
2.39039	2.07471	2.01714	2.00419	2.00104
3.92433	1.00267	1.88059	1.97367	1.99361
3.92965	4.8101	1.42524	1.90003	1.97682

Table 5: Rates of convergence using $Q1P0$

For comparison, Figure 7 shows the convergence analysis for the pair of spaces with Q_pP_{p-1} with $p = 1, 2, 3$. For higher degree p it is observed a decay of the rates of converge for increasing eigenvalues suggesting that those spaces are not compatible.

It can also be observed that if $\boldsymbol{V}_h$ is set much richer than W_h, for instance spaces of type Q_2P_0 the eigenvalues are not approximated well as indicated in Figure 8.

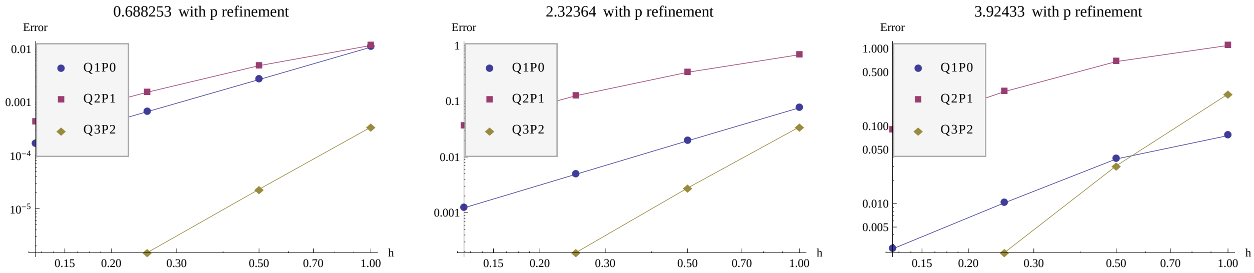

Figure 7: Comparative Q_pP_{p-1} approximation

λ_i	16 elements	64 elements	256 elements
0.688253	1.26243	1.65806	1.84824
2.32364	-0.456535	1.09356	1.6097
2.39039	1.22521	1.58404	1.79538
3.92433	0.97561	1.31838	1.56633
3.92965	0.97561	1.31838	1.56633
5.49762	0.805723	1.06492	1.53416

Table 6: Rates of convergence using $Q2P1$

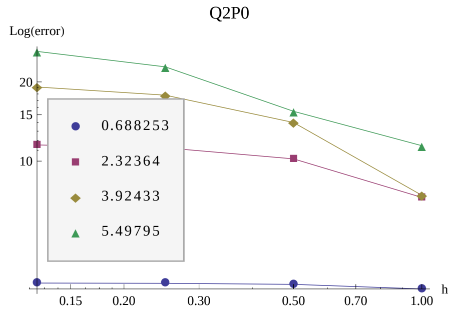

Figure 8: Error in $Q_2\,P_0$ approximations of eigenvalues

6 CONCLUSIONS

The present paper describes a numerical analysis of compatibility for a class of spaces Q_pP_{p-1} to be used in Mixed Formulation. The approach is based on the quality of the approximation of such spaces when applied to the resolution of a Steklov eigenvalue problem. The results suggest that for lower degree $p = 1$, Q_1P_0 seems to be compatible, presenting an optimal second order rate of convergence. However for higher degrees $p = 2, 3$, the rates of convergence are not optimal and decay for increasing eigenvalues, suggesting that those spaces are not compatible. The advantage of the approach is that it leads to an objective criteria to numerically determine the compatibility.

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