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A SONIC FIX FOR IDEAL MAGNETOGASDYNAMICS EQUATIONS

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Abstract. A plasma thruster is being developed in Córdoba, Argentina and local researchers are developing theoretical models and scientific software to solve the magnetogasdynamics (MGD) equations. Computational MGD represents one of the most promising interdisciplinary computational technologies for aerospace design. For ideal MGD the numerical simulations are a very important tool, by reducing expensive, and sometimes unviable, experimental parametric studies. However, the numerical simulations always are limited by the ability to analyze and to solve accurately the hyperbolic non-linear differential equations system. In this work is presented a modification of the original Harten and Yee TVD scheme by incorporating a new sonic fix for the acoustic causality points using the finite volume technique. The proposed sonic fix is implemented to solve the transient, two-dimensional ideal MGD equations. Two test cases were simulated: MGD Riemann problem and the Hartmann flow. The numerical results obtained using the new sonic fix have shown to reduce the oscillations and to be robust.

1 1. INTRODUCTION

The electric propulsion can be defined as the acceleration of gases for propulsion by electromagnetic means; according to the propulsive effects underlying physics, the electric thrusters can be grouped in three categories: electrothermal, electrostatic and electromagnetic. Electromagnetic plasma propulsion systems offer significantly higher exhaust velocities than chemical propulsion systems and is in our days a competitive alternative to chemical propulsion, and presently it is being used for satellite orbit raising and station-keeping. A pulsed plasma thruster is being developed in Córdoba, and it is necessary to develop software to simulate the flow inside the device.

Magnetogasdynamics (MGD) flows have applications in aerospace technologies, astrophysics, geophysics, interstellar gas masses dynamics, etc. A MGD model is generally based on the assumption that plasma can be regarded as a continuum and thus may be characterized by relatively few macroscopic quantities. A revision about the physical models used in aerospace applications is given in (D'Ambrosio and Giordano, 2004). The real magnetogasdynamics equations (MGDR) are represented by a hyperbolic-parabolic system of equations. A revision about the physical models used in aerospace applications is given in Ref. [5]. The real MGD equations constitute a parabolic-hyperbolic partial differential system. The parabolic part represents the non-ideal effects and it includes transport effects such as viscous and resistive diffusion and heat transfer. The hyperbolic or ideal part of the MGD equations presents nonconvex singularities and the wave structure is more complicated than for the Euler equations (Kantrowitz and Petschek, 1966). The nonlinear coupling of these waves plays an important role in determining physical phenomena and in the numerical solution (Leveque *et al.*, 1998).

In ideal MGD the numerical simulations are a very important tool, by reducing expensive, and sometimes unviable, experimental parametric studies. However, the numerical simulations always are limited by the ability to analyze and to solve accurately the hyperbolic non-linear differential equations system.

To solve the ideal MGD equations system is convenient to use a conservative form because it allows to obtain the correct jump conditions at discontinuities and shocks (Leveque, 1992; Toro, 2009). The utilization of the numerical conservative scheme is desirable because ensures that mass, momentum, and energy are indeed conserved. Several schemes has been proposed and implemented to solve the ideal MGD equations (Balbas *et al.*, 2004; Myong and Roe, 1998; Udrea, 1999); in this work, the Harten and Yee TVD technique is used (Yee *et al.*, 1985). It has proven to be accurate and reliable for the simulation of supersonic flows of gases (Yee, 1989; Elaskar, *et al.*, 2000; Falcinelli, *et al,* 2008). This technique is implemented here, with a modification to numerically solve ideal MGD flows.

Between the difficulties to reach accurate numerical solutions for the ideal MGD equations is important to note the acoustic causality points where a new wave structure can be produced by the non-linear wave interaction (Courant and Friedrich, 1999). In ideal MGD the sonic points and points where non-convexity appears are points of acoustic causality (Serna, 2009). In theses points is necessary to apply a corrector entropy scheme introducing the necessary artificial viscosity.

The main objective of this work is to present a modification of the original Harten and Yee's TVD scheme by incorporating a new sonic fix for the acoustic causality points. The proposed sonic fix is implemented by means of software specifically developed to solve the transient, two-dimensional ideal MGD equations.

The numerical approach is based on an approximate Riemann solver with a high resolution TVD technique. The eight-wave technique introduced by Powell is used (Powell, 1995) and the eigenvectors are normalized according to Zarachay, *et al.* (1994) and Roe (1996). The parabolic numerical fluxes are evaluated maintaining a second order approach using central finite differences.

The code is used to simulate the coplanar MGD Riemann problem introduced for Brio and Wu (1988) where results using the new sonic fix are compared with those given by the traditional Harten-Yee scheme. In a near future, the objective is to achieve the capability of simulating flows for plasma propulsion in complex geometries. Finally, to check the robustness of the new sonic fix, we simulate the Hartmann flow. The Hartmann flow is steady-state, incompressible and the diffusive effects are important.

2 MAGNETOGASDYNAMICS EQUATIONS

The equations of non-dimensional transient real MGD in conservative form are given by (Goldston and Rutherford, 2003; D'Ambrosio and Giordano, 2004).

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho\boldsymbol{u}\\ \boldsymbol{B}\\ e\end{bmatrix}+\nabla\cdot\begin{bmatrix}\rho\boldsymbol{u}\\ \rho\boldsymbol{uu}-\boldsymbol{BB}+\boldsymbol{I}\left(p+\frac{1}{2}B^2\right)\\ \boldsymbol{uB}-\boldsymbol{Bu}\\ \left(e+p+\frac{B^2}{2}\right)\boldsymbol{u}-\left(\boldsymbol{B}\cdot\boldsymbol{u}\right)\boldsymbol{B}\end{bmatrix}=\nabla\cdot\begin{bmatrix}0\\ \frac{\boldsymbol{\tau}}{R_eA_l}\\ \frac{\boldsymbol{E}_r}{L_uA_l}\\ \frac{\boldsymbol{u}\cdot\tau}{R_eA_l}-\frac{\left[\boldsymbol{\eta}\cdot\left(\nabla\times\boldsymbol{B}\right)\right]\times\boldsymbol{B}}{L_uA_l}+\frac{\boldsymbol{k}\cdot\nabla T}{P_eA_l}\end{bmatrix} \tag{1}$$

where ρ, $\boldsymbol{u}$,e ,p, T are the density, velocity, total energy, pressure and temperature of plasma respectively. B is the magnetic field, K thermal conductivity, η electrical resistive and τ viscous stress. R_e, A_l, L_u, P_e are the Reynolds, Alfvén, Lundquist and Peclet numbers.

The ideal MGD equations accurately describe the macroscopic dynamics of perfectly conducting plasma. This system expresses conservation of mass, momentum, energy, and magnetic flux and conform a nonlinear conservative system of eight partial differential equations. The equations of non-dimensional ideal one-fluid MGD in conservative form are given by (D'Ambrosio and Giordano, 2004);

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho\boldsymbol{u}\\ \boldsymbol{B}\\ e\end{bmatrix}+\nabla\cdot\begin{bmatrix}\rho\boldsymbol{u}\\ \rho\boldsymbol{uu}-\boldsymbol{BB}+\boldsymbol{I}\left(p+\frac{1}{2}B^2\right)\\ \boldsymbol{uB}-\boldsymbol{Bu}\\ \left(e+p+\frac{1}{2}B^2\right)\boldsymbol{u}-\left(\boldsymbol{B}\cdot\boldsymbol{u}\right)\boldsymbol{B}\end{bmatrix}=\boldsymbol{0} \tag{2}$$

To close de system, is introduced perfect gas state equation, so the specific internal energy depends on temperature only. Then for the total energy results as,

$$e=\frac{p}{\gamma-1}+\rho\ \frac{\boldsymbol{u}\cdot\boldsymbol{u}}{2}+\frac{\boldsymbol{B}\cdot\boldsymbol{B}}{2} \tag{3}$$

Using a Cartesian coordinate system the Eq.(2) can be written, for two dimensions in quasi-linear form, as

$$\frac{\partial \boldsymbol{U}}{\partial t}+\left[A_c\right]\frac{\partial \boldsymbol{U}}{\partial x}+\left[B_c\right]\frac{\partial \boldsymbol{U}}{\partial y}=\boldsymbol{0} \tag{4}$$

with the state vector

$$\boldsymbol{U}=\left(\rho,\rho u_x,\rho u_y,\rho u_z,B_x,B_y,B_z,e\right)^T \tag{5}$$

where $[A_c]$ y $[B_c]$ are the Jacobian matrices. The evaluation of the eigenvalues and the eigenvectors is simpler using the conservative variables:

$$\boldsymbol{W}=\left(\rho,u_x,u_y,u_z,B_x,B_y,B_z,p\right)^T \tag{6}$$

To overcome the difficulties introduced by the null eigenvalue of the Jacobian matrices, the eight-wave technique introduced by Powell (1995) is used in this work. The modified Jacobian matrix $[A_p]$ (using primitive variables) is:

$$\left[A_p\right]=\begin{bmatrix} u_x & \rho & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & u_x & 0 & 0 & 0 & \frac{B_y}{\rho} & \frac{B_z}{\rho} & \frac{1}{\rho} \\ 0 & 0 & u_x & 0 & 0 & -\frac{B_x}{\rho} & 0 & 0 \\ 0 & 0 & 0 & u_x & 0 & 0 & -\frac{B_x}{\rho} & 0 \\ 0 & 0 & 0 & 0 & u_x & 0 & 0 & 0 \\ 0 & B_y & -B_x & 0 & 0 & u_x & 0 & 0 \\ 0 & B_z & 0 & -B_x & 0 & 0 & u_x & 0 \\ 0 & \gamma p & 0 & 0 & 0 & 0 & 0 & u_x \end{bmatrix} \tag{7}$$

The eigenvectors are normalized according to Zarachay *et al.* (1994) and Roe (1996). The resulting eigenvalues representing MGD waves are: “entropy wave”, “Alfvén waves”, “fast magneto-acoustic waves”, “slow magneto-acoustic waves” and “magnetic flux wave”. The expressions for these are:

-Entropy wave: $\lambda_e = u_x$.

$$
\underset{\sim}{r}_e = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \underset{\sim}{l}_e = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\dfrac{1}{c^2} \end{Bmatrix} \tag{8}
$$

-Alfvén waves*:* $\lambda_a = u_x \pm c_a$

$$
\underset{\sim}{r}_a^{\pm} = \frac{1}{\sqrt{2}} \begin{Bmatrix} 0 \\ 0 \\ -\beta_z \\ \beta_y \\ 0 \\ \pm\sqrt{\rho}\beta_z \\ \mp\sqrt{\rho}\beta_y \\ 0 \end{Bmatrix} \qquad \underset{\sim}{l}_a^{\pm} = \frac{1}{\sqrt{2}} \begin{Bmatrix} 0 \\ 0 \\ -\beta_z \\ \beta_y \\ 0 \\ \pm\dfrac{\beta_z}{\sqrt{\rho}} \\ \mp\dfrac{\beta_y}{\sqrt{\rho}} \\ 0 \end{Bmatrix} \tag{9}
$$

-Fast magneto-acoustic waves: $\lambda_f = u_x \pm c_f$

$$
\underset{\sim}{r}_f^{\pm} = \begin{Bmatrix} \rho\alpha_f \\ \pm\alpha_f c_f \\ \mp\alpha_s c_s \beta_y \operatorname{sgn}(B_x) \\ \mp\alpha_s c_s \beta_z \operatorname{sgn}(B_x) \\ 0 \\ \alpha_s\sqrt{\rho}c\beta_y \\ \alpha_s\sqrt{\rho}c\beta_z \\ \alpha_f \gamma p \end{Bmatrix} \qquad \underset{\sim}{l}_f^{\pm} = \begin{Bmatrix} 0 \\ \pm\dfrac{\alpha_f c_f}{2c^2} \\ \mp\dfrac{\alpha_s}{2c^2} c_s \beta_y \operatorname{sgn}(B_x) \\ \mp\dfrac{\alpha_s}{2c^2} c_s \beta_z \operatorname{sgn}(B_x) \\ 0 \\ \dfrac{\alpha_s}{2\sqrt{\rho}c}\beta_y \\ \mp\dfrac{\alpha_s}{2\sqrt{\rho}c}\beta_z \\ \dfrac{\alpha_f}{2\rho c^2} \end{Bmatrix} \tag{10}
$$

-Slow magneto-acoustic waves: $\lambda_s = u_x \pm c_s$

$$\underset{\sim}{r}_s^{\pm} = \left\{ \begin{array}{c} \rho \alpha_s \\ \pm \alpha_s c_s \\ \pm \alpha_f c_f \beta_y \operatorname{sgn}(B_x) \\ \pm \alpha_f c_f \beta_z \operatorname{sgn}(B_x) \\ 0 \\ -\alpha_f \sqrt{\rho} c \beta_y \\ -\alpha_f \sqrt{\rho} c \beta_z \\ \alpha_s \gamma p \end{array} \right\} \qquad \underset{\sim}{l}_s^{\pm} = \left\{ \begin{array}{c} 0 \\ \pm \dfrac{\alpha_s c_s}{2c^2} \\ \pm \dfrac{\alpha_f}{2c^2} c_f \beta_y \operatorname{sgn}(B_x) \\ \pm \dfrac{\alpha_f}{2c^2} c_f \beta_z \operatorname{sgn}(B_x) \\ 0 \\ -\dfrac{\alpha_f}{2\sqrt{\rho} c} \beta_y \\ \dfrac{\alpha_f}{2\sqrt{\rho} c} \beta_z \\ \dfrac{\alpha_s}{2\rho c^2} \end{array} \right\} \tag{11}$$

-Magnetic flux wave: $\lambda_d = u_x$

$$\underset{\sim}{r}_d = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{array} \right\} \qquad \underset{\sim}{l}_d = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{array} \right\} \tag{12}$$

Where: $c_{A,n} = \dfrac{|B_x|}{\sqrt{\rho}}$;

$$c_{f,s}^2 = \frac{1}{2}\left(\frac{\gamma p + B^2}{\rho} \pm \sqrt{\left(\frac{\gamma p + B^2}{\rho} \right)^2 - 4 \frac{\gamma p B_x^2}{\rho^2}} \right)$$

$$\beta_y = \begin{cases} \dfrac{B_y}{B_\perp} & B_\perp \neq 0 \\ \dfrac{1}{\sqrt{2}} & B_\perp = 0 \end{cases}$$

$$\beta_z = \begin{cases} \dfrac{B_z}{B_\perp} & B_\perp \neq 0 \\ \dfrac{1}{\sqrt{2}} & B_\perp = 0 \end{cases}$$

$$\beta_\perp = \sqrt{B_y^2 + B_z^2}$$

The Alfvén, entropy wave and magnetic flux waves, are linearly degenerate; hence the flow velocity is constant throughout the wave. The magneto-acoustic waves are nonlinear and can be shock or rarefaction waves. However, under particular relations between the magnetic field and the sound velocity theses waves may be locally non-convex (Serna, 2009).

3 FINITE VOLUME FORMULATION

To obtain the numerical solution of the system described by Eq.(2), a finite volume scheme has been implemented using a structured mesh, together an approximate Riemann solver to calculate the fluxes with an explicit finite-differences scheme for the evaluation of the time evolution.

The numerical flows are evaluated by means of the Harten-Yee TVD technique, which allows the capturing of discontinuities, simultaneously achieving a second order approach (Yee, 1989).

The explicit TVD-finite volume scheme can be expressed as, see Fig.(1),

$$\boldsymbol{U}_{ij}^{n+1} = \boldsymbol{U}_{ij}^{n} - \Delta t \left[\frac{\overline{\boldsymbol{F}}_{i+\frac{1}{2};j}^{n} - \overline{\boldsymbol{F}}_{i-\frac{1}{2};j}^{n}}{\Delta x} + \frac{\overline{G}_{i;j+\frac{1}{2}}^{n} - \overline{G}_{i;j-\frac{1}{2}}^{n}}{\Delta y} \right] \tag{13}$$

where the function that determines the second-order numerical flux is defined as

$$\overline{\boldsymbol{F}}_{i+\frac{1}{2};j}^{n} = \frac{1}{2}\left(\boldsymbol{F}_{i+1}^{n} + \boldsymbol{F}_{i}^{n} + \left(\sum_m \boldsymbol{R}_{i+\frac{1}{2}}^{m} \Phi_{i+\frac{1}{2}}^{m} \right)^{(n)} \right) \tag{14}$$

The limiter function used is one of minmod type,

$$\Phi_{i+\frac{1}{2}}^{m} = \left(g_{i+1}^{m} + g_{i}^{m} \right) - \psi\left(\lambda_{i+\frac{1}{2}}^{m} + \gamma_{i+\frac{1}{2}}^{m} \right) \alpha_{i+\frac{1}{2}}^{m}$$

(15)

$$g_i^m = \mathrm{s\,gn}\left(\lambda_{i+\frac{1}{2}}^{m} \right) \max \left\{ \begin{matrix} 0 \\ \min \left[\begin{matrix} \sigma_{i+\frac{1}{2}}^{m} \left| \alpha_{i-\frac{1}{2}}^{m} \right| \\ \sigma_{i-\frac{1}{2}}^{m} \dfrac{\mathrm{sgn}\left(\lambda_{i+\frac{1}{2}}^{m} \right)}{2} \alpha_{i-\frac{1}{2}}^{m} \end{matrix} \right] \end{matrix} \right\}; \quad \sigma_{i+\frac{1}{2}}^{m} = \sigma\left(\lambda_{i+\frac{1}{2}}^{m} \right) \tag{16}$$

$$\gamma_{i+\frac{1}{2}}^{m}=\begin{cases}\dfrac{1}{\alpha_{i+\frac{1}{2}}^{m}}\left(g_{i+1}^{m}-g_{i}^{m}\right) & \alpha_{i+\frac{1}{2}}^{m}\neq 0\\[2ex] 0 & \alpha_{i+\frac{1}{2}}^{m}=0\end{cases} \tag{17}$$

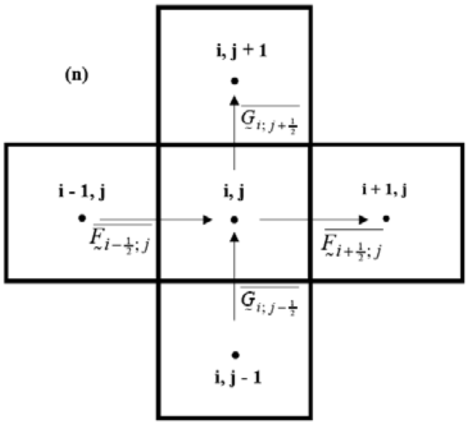

Fig. 1 Adjacent cells of the two-dimensional domain.

Approximate Roe-type Riemann solver produces only shock waves so a physically correct smooth rarefaction wave is replaced by a rarefaction shock wave that violates the entropy condition. An alternative to correct this non-physical solution is using a "sonic entropy fix" that smoothes out eigenvalues in the vicinity around zero. Harten (1982) suggested an entropy fix for Roe's method, which has widespread use:

$$\psi(z)=\begin{cases}|z| & |z|\geq\delta\\[1ex] \dfrac{1}{2\delta}\left(z^{2}+\delta^{2}\right) & |z|<\delta\end{cases} \tag{18}$$

The function ψ in Eq.(15) is an entropy correction to z, whereas δ is generally a small and constant value that needs to be calibrated for each problem. A proper choice of the entropy parameter δ for higher Mach number flows not only helps in preventing nonphysical solutions but can act, in some sense, as a control in the convergence rate and in the sharpness of shocks (Yee, 1989).

For time-accurate calculations in explicit numerical algorithms

$$\sigma(z)=\frac{1}{2}\left[\psi(z)-\frac{\Delta t}{\Delta x}z^{2}\right] \tag{18}$$

and the wave strength of the m-th wave is

$$\alpha^{m}=\boldsymbol{L}^{m}\cdot\left(\boldsymbol{W}_{i+1}-\boldsymbol{W}_{i}\right) \tag{19}$$

where $\boldsymbol{L}^{m}$ is the left eigenvector for the m-th wave and $\boldsymbol{W}$ represents the primitive variable vector.

4 RESULTS

To assess accuracy and reliability in computational simulations at first stage of research, after development and implementation the code was subjected to test with emphasis in the results verification (Oberkampf and Truncano, 2002).

The software ability to solve 2-D magnetogasdynamics problems was evaluated using the Riemann problem introduced by Brio and Wu (1988). The proposed MGD Riemann problem has not analytical solutions but numerical solutions were published (Brio and Wu, 1988; Udrea, 1999; Elaskar *et al*., 2001; Sankaran, 2001)

Numerical results of the MGD Riemann problem using different alternatives are presented in this section. For MGD flows, this benchmark is called the coplanar Riemann problem. This problem initially bears a discontinuity that separates two constant states, a leftward one and a rightward one. These states are defined by the corresponding initials conditions. With the objective of verifying the correct operation of the 2D code being presented here, the mesh is rotated with respect to the longitudinal axis of the flow forcing the code to simulate a 2-D flow. The initial conditions used in the simulation are:

$$\begin{aligned} W_l &= \left(1.0, 0.0, 0.0, 0.0, 0.75, 1.0, 0.0, 1.0\right)^T \\ W_r &= \left(0.125, 0.0, 0.0, 0.0, 0.75, -1.0, 0.0, 0.1\right)^T \end{aligned} \tag{20}$$

If we apply to Eq.(2) the traditional Harten-Yee scheme, developed for gas dynamics equations, the sonic fix, given by Eq.(15), acts only on sonic point, but it does not act on non-convex point; because the gasdynamics flows do not present non-convex points. Fig. (2) shows the results by the Brio and Wu benchmark applying the Harten-Yee scheme with δ = 0.001. In Figures. (2a, 2b and 2c) are indicated the density, normal velocity and transverse magnetic field distribution respectively. In these pictures it is possible to observe large oscillations around the compound wave. Theses oscillations destabilize the numerical simulations for relatively long times.

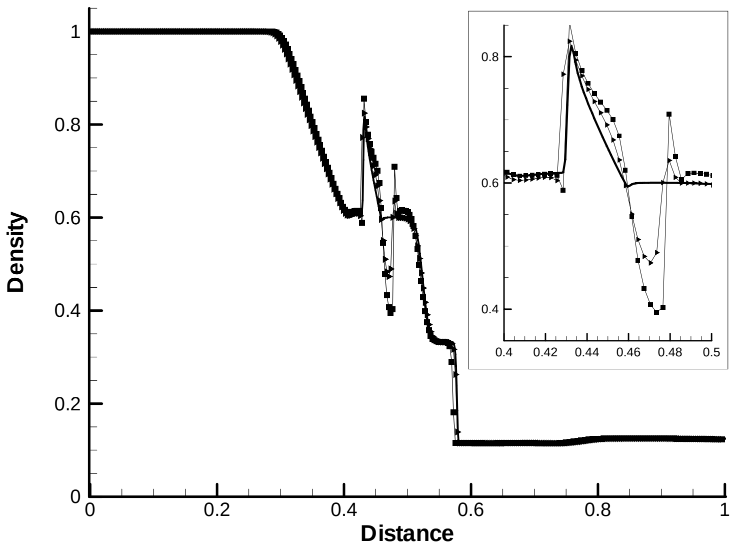

Fig. 2a Density for 2D MGD Riemann problem. Solid line: Benchmark 1D, Square: Harten-Yee technique with δ = *1E-3*, Right Triangle: Harten-Yee and Van Leer technique.

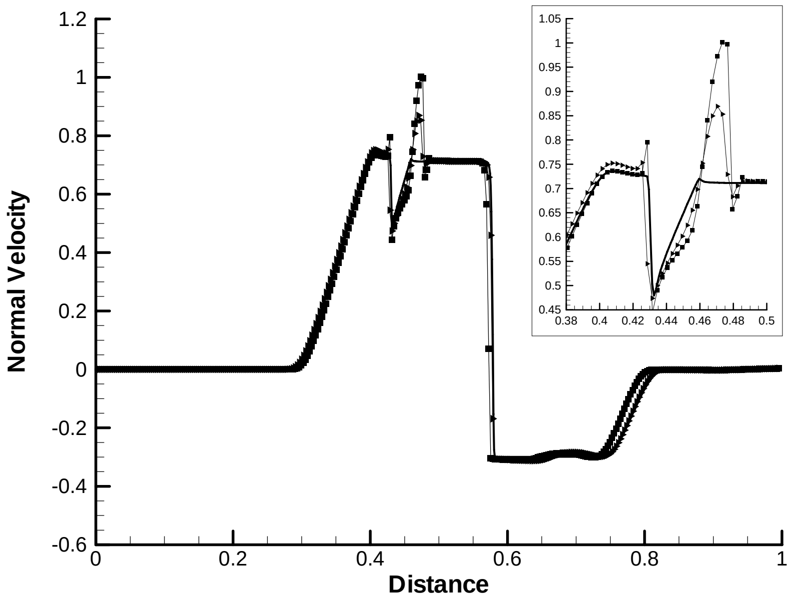

Fig. 2b Normal Velocity for 2D MGD Riemann problem. Solid line: Benchmark 1D, Square: H-Y technique with δ = 1E-3, Right Triangle: H-Y and Van Leer technique.

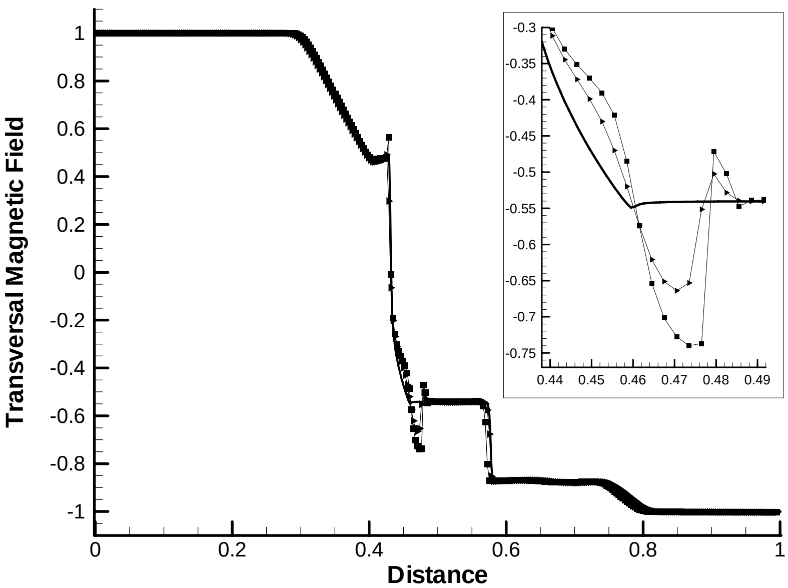

Fig. 2c Transversal Magnetic Field for 2D MGD Riemann problem. Solid line: Benchmark 1D, Square: H-Y technique with δ = 0.001, Right Triangle: H-Y and Van Leer technique.

To obtain "proper" numerical results for the Brio and Wu two dimensional MGD problem, the entropy correction of Harten scheme, Eq.(18), needs to be calibrated with relatively big values of δ (Maglione *et al.*, 2003). For gasdynamics hypersonic flows, a variable δ depending on the spectral radius of the Jacobian matrices of fluxes is very helpful in terms of stability and convergence rate (Yee, 1989). However, numerical tests show that this technique does not provide satisfactory results on the coplanar Riemann MGD problem.

The use of a constant value, for 2-D simulations, equal to the average in absolute value of the eigenvalues of the Jacobian matrices of fluxes show satisfactory results for short time only (Maglione et al., 2007), also this technique introduces too much numerical viscosity around a large vicinity of the sonic point. As a result of this scheme the solutions are not particularly satisfactory for long computation time, especially in capturing the fast shock wave moving to the right, see Fig.(3). It is possible to show that for short times the right fast shock wave is captured correctly, however for long times this wave is destabilized.

In order to obtain a method that does not need δ calibration for each MGD problem, it is convenient to improve the Van Leer technique (Van Leer *et al.*, 1989), vastly applied for gases.

$$\delta_{GD} = \max\left[\left|\lambda^m_{i+\frac{1}{2}} - \lambda^m_{i-\frac{1}{2}}\right|, 0\right] \tag{21}$$

$$\delta_k^{MGD} = \begin{cases} \max\left[\left|\lambda^m_{i+\frac{1}{2}} - \lambda^m_{i-\frac{1}{2}}\right|\right] & \text{If } \lambda^m_{i+\frac{1}{2}} \text{ cuts across zero} \\ \min\left|\lambda^m_{i+\frac{1}{2}}\right| & \text{Otherwise} \end{cases} \qquad k = 1,..,8 \tag{22}$$

As an alternative procedure it was implemented the traditional Harten-Yee technique only in sonic points, but δ defined by Eq.(22). The results are shown in Fig.(2) and they have more accuracy and the oscillations have been reduced. To include the treatment of non-convex points in the numerical method, it is implemented the Eq.(18) together with δ as given by Eq.(22) for all acoustic points; however, convergence difficulties was found.

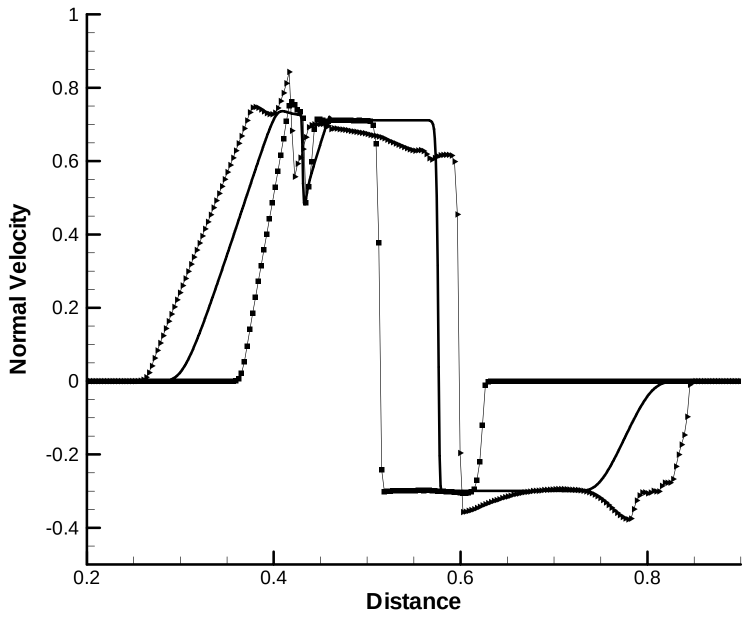

Fig. 3 Normal Velocity for 2D MGD Riemann problem. Solid line: Benchmark 1D, Square and Right Triangle: H-Y technique with average δ for different times.

For increasing the accuracy of the previous schemes and to avoid the spurious oscillations, a new entropy correction function is proposed. The new entropy correction function introduces high numerical viscosity only restricted to the proximity of the acoustic points,

$$\psi(z)=\begin{cases}|z| & |z|\geq\delta \\ \dfrac{z^2}{\delta^2}+\dfrac{\delta-2}{\delta}|z|+1 & |z|<\delta\end{cases} \tag{23}$$

A comparison between Harten´s original sonic entropy fix, Eq.(18) and the new proposed fix Eq.(22), is shown in Fig.(4). This new function is a continuously differentiable approximation to |z|, fulfilling,

$$\begin{aligned} &\psi(\delta)_{\delta^-}=\psi(\delta)_{\delta^+} \\ &\psi(0)_{\delta^-}=1 \\ &\left.\frac{d\psi(z)_{\delta^-}}{dz}\right|_{z=\delta}=\left.\frac{d\psi(z)_{\delta^+}}{dz}\right|_{z=\delta} \end{aligned} \tag{24}$$

Firstly the new function was implemented only for non-convex points, whereas for sonic points the Eq.(18) was used; for both types of points δ was evaluated according to Eq.(22). Under these conditions was possible to reach convergence and the results are presented in Fig.(5), where the oscillations show a slight reduction.

The new function Eq.(23), together with δ calculated by Eq.(22), was applied in all acoustic points. The results are shown in Fig.(5). This figure shows that the oscillations were significantly reduced around the compound wave.

At the present stage of development, the entropy parameter seems still highly geometric and problem dependent. A universal method is yet to be discovered. However, we note that the joint implementation of Eqs.(22 and 23) has proven to be an effective methodology, almost, for the magnetogasdynamic Riemann problem.

The necessity to introduce a new sonic fix for 2-D MGD flow and not for the 1-D MGD occurs because the number of the eigenvalues crossing over zero, when the modified Van Leer's technique is used, is increasing for the two-dimensional test with respect the one-dimensional case. This effects it is specially note for the compound wave (Maglione *et al.*, 2010).

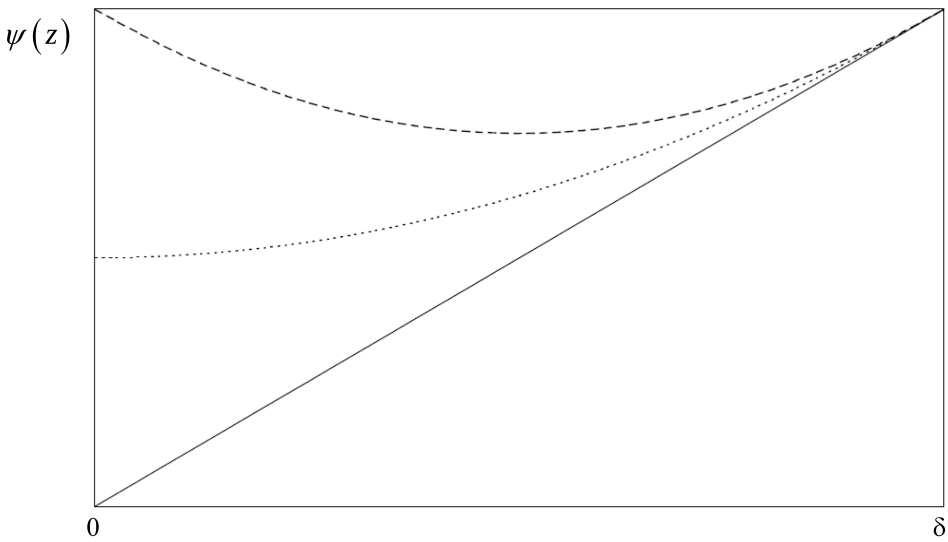

Fig. 4 Comparison between the new sonic fix and Harten´s original (Dotted line: Original sonic fix, Long Dash line: Proposed Sonic fix).

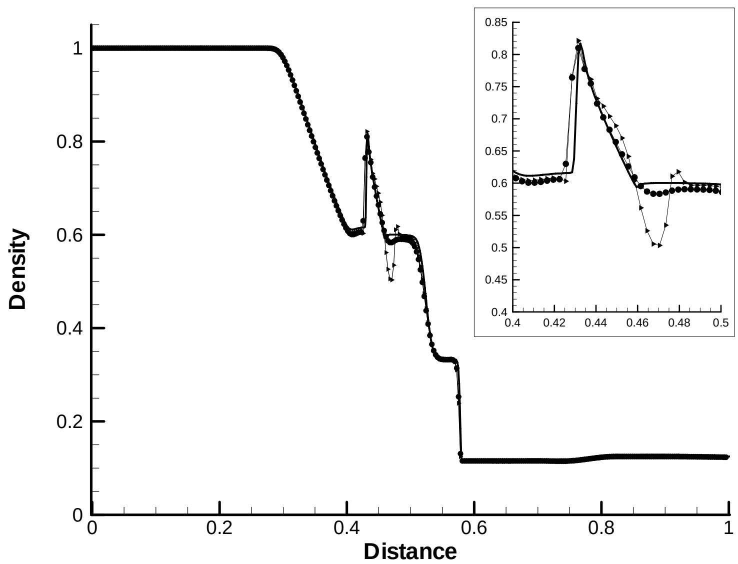

Fig. 5a Density for 2D MGD Riemann problem. Solid line: Benchmark 1D, Square: New technique implemented in all acoustic points, Right Triangle: New technique implemented only in non-convex points.

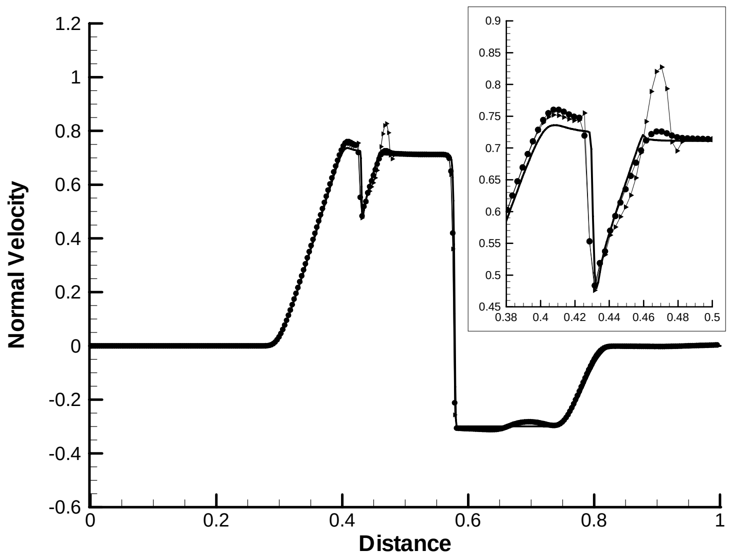

Fig. 5b Normal Velocity for 2D MGD Riemann problem. Solid line: Benchmark 1D, Square: New technique implemented in all acoustic points, Right Triangle: New technique implemented only in non-convex points.

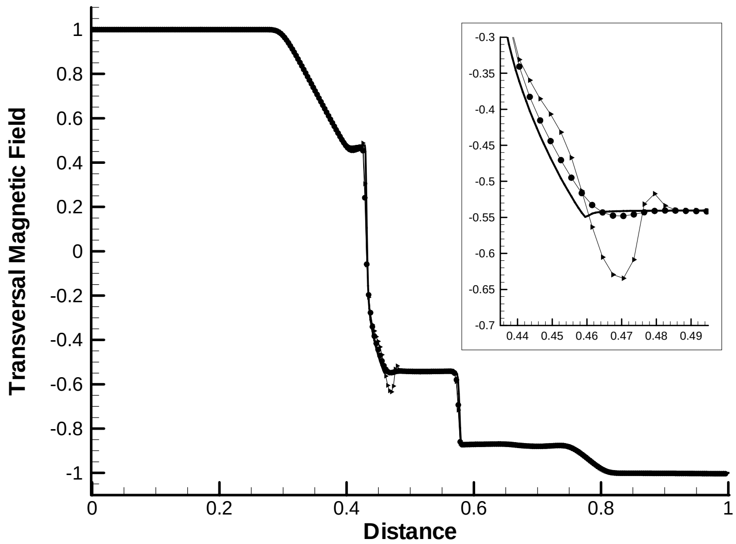

Fig. 5c Transversal Magnetic Field for 2D MGD Riemann problem. Solid line: Benchmark 1D, Square: New technique implemented in all acoustic points, Right Triangle: New technique implemented only in non-convex points.

To check the robustness of the new scheme we simulate the flow of Hartmann. It is an extension of Couette's flow for electrically conductive fluids. In this problem the flow is steady-state, laminar and it develops between two, virtually infinite, parallel moving plates with opposite velocities of equal magnitude. The applied field is normal to the plates and constant. Figs. 6a and b show the results for the Hartmann number $H_a = 1$ and 10 respectively (Maglione, *et al.*, 2007). The benchmark case here simulated can be found in the Sutton and Sherman book (1965).

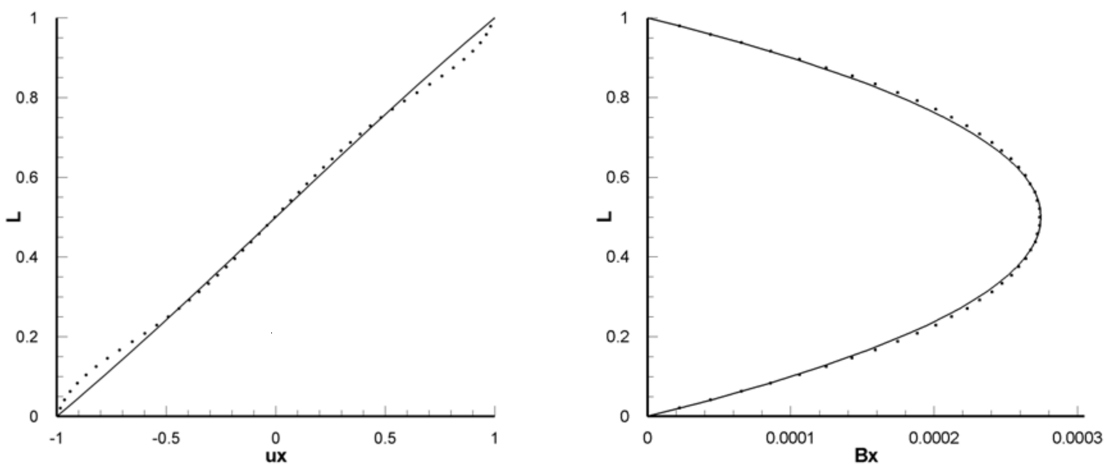

Figure 6a. Hartmann flow. Ha = 1. Left velocity, right magnetic field. Theoretical solution: line. Numerical simulation: points.

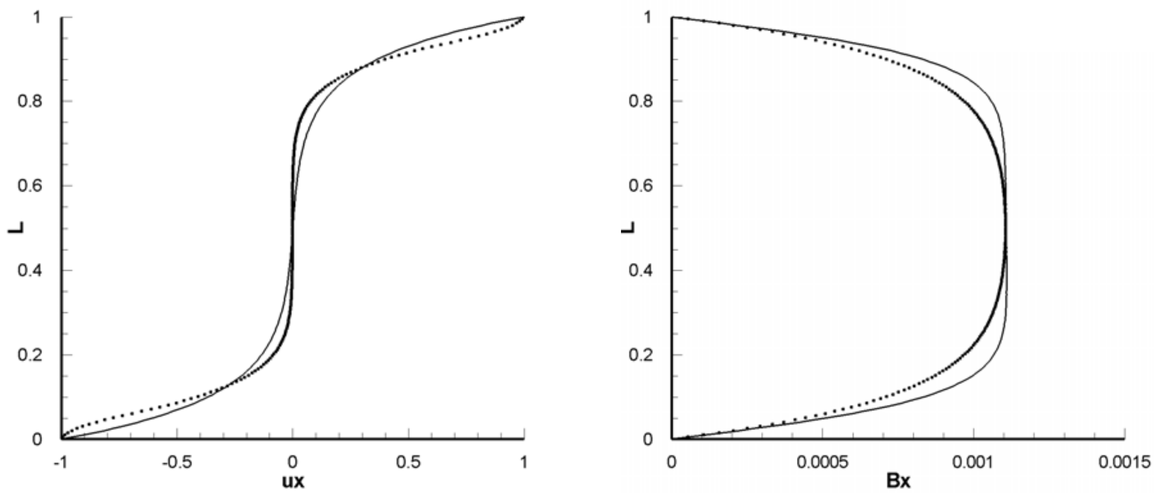

Figure 6b. Hartmann flow. Ha = 10. Left velocity, right magnetic field. Theoretical solution: line. Numerical simulation: points.

5 CONCLUSION

In this paper we present a modification to the Harten and Yee 's original method, incorporating a new sonic fix. The oscillations, present when using Harten-Yee traditional scheme, are notably reduced when the new sonic fix is applied in sonic and non convex points.

This new technique has three important advantages:

1 - The method does not need particular calibration.

2 – The numerical oscillations are reduced.

3 - The method increases the numerical viscosity only in the proximity of the acoustic points.

Finally we note that the method has proven to be robust, we can simulate correctly two benchmarks: the 2-D magnetogasdynamic shock tube and the Hartmann flow. For the 2-D magnetogasdynamic shock tube is represented by the hyperbolic part of the MGD equations; however the Hartmann flow is govern by the diffusive effects.

We have two next objectives: 1- Applied the new scheme to 2-D, non-steady magnetogasdynamics numerical code using non-structured meshes and finite volumes; 2 –To achieve the capability of simulating flows for plasma propulsion in complex geometries.

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