Asociación Argentina
de Mecánica Computacional

Mecánica Computacional Vol XXIX, págs. 4235-4245 (artículo completo)
Eduardo Dvorkin, Marcela Goldschmit, Mario Storti (Eds.)
Buenos Aires, Argentina, 15-18 Noviembre 2010

ON AN OBJECTED ORIENTED IMPLEMENTATION OF PLASTICITY MODELS

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Keywords: object oriented, constitutive modelling , elastoplasticity , invariants differentiation

Abstract.

This study reviews the elements of incremental plasticity theory in an elastoplastic formulation of the behaviour at the material point and its implementation. It is also computed the analytical derivatives of the principal invariants and eigenvalues, due its importance in plastic flow rules. The constitutive models of Drucker-Prager and Mohr-Coulomb are presented. It is also exposed the class framework behind the object oriented code, and, finally, some uniaxial tests results are presented.

1 INTRODUCTION

The plasticity theory describes the behaviour of solid materials that, after being subjected to a loading cycle, maintains permanent deformation even after fully discharged (Souza Neto et al. (2008)). Current approach to the development and implementation of elastoplastic material models according to the theory of incremental plasticity in computational mechanics relies in the definition of yield functions, flow potential rules and evolution laws.

The yield functions are usually based on a set of already existing stress envelope shapes. When it comes to frictional materials such as soil, concrete and rock, the shear yielding envelopes are confining-dependent. This paper mentions two of the most prominent examples of yield functions for frictional materials: Drucker–Prager and Mohr–Coulomb.

The object oriented philosophy has been recently extensively used in computational mechanics. To this end we mention early experimental developments and implementations of (Devloo (1997)),(Devloo and Santos (2003)),(Jeremic and Yang (2001)),(Jeremic et al. (2008)),(Devloo et al. (2006)). In this paper the authors take advantage of these computational tools to implement an elastoplastic strain decomposition code and to evaluate the corresponding consistent stress-strain constitutive tensor.

The paper is organized as follows: Section 2 presents an overview of the basis of elastoplastic mathematical theory. In Section 3 it is exposed the numerical approximation of the elastoplastic problem. The models of Mohr-Coulomb and Drucker-Prager are presented in section 4. In Section 5 it is reviewed the definition of the principal invariants, Lode Angle and the analytical calculation of the eigenvalues. Section 6 introduces the object-oriented design and implementation. The results are presented in Section 7 and the conclusions in Section 8.

2 ELASTO-PLASTICITY

The plasticity theory represents a mathematical modelling for materials which may present permanent deformations after a loading cycle. The material response is time-independent and thus the strain rate is not considered in the formulation. This mechanical behaviour may be applied to a wide range of materials to some extent.

The basic components of an elastoplastic constitutive model are:

- Decomposition of the strain tensor;
- Elastic stress-strain law;
- Definition of a yield function;
- Definition of plastic flow rule;
- Definition of a hardening rule.

2.1 Decomposition of deformation tensor

The total strain ε can be splited into two parts, one linear and elastic ε^e, and the other, non-linear and plastic ε^p. The plastic part is a consequence of the history of irreversible dissipative processes which the material underwent and measured under a unstressed condition. The mathematical representation is,

$$\varepsilon = \varepsilon^e + \varepsilon^p.$$

2.2 An elastic law

The stress-strain isotropic infinitesimal relation is function of the elastic strain tensor ε^e only:

$$\sigma = \lambda tr\,(\varepsilon^e) I + 2\mu\varepsilon^e,$$

where the scalars λ and μ are the Lamé constants and are defined as:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$\mu = \frac{E}{2(1+\nu)}.$$

2.3 Yield criterion

The yield criterion is a potential function Ψ in the stress space that defines the elastic and the plastic domains. By definition, it assumes non-positives values in elastic regime and null values under plastic straining. A is defined as a thermodynamic force related to the hardening/softening behaviour.

$$\Psi(\sigma,\, A)$$

2.4 Plastic flow rule

In a loading cycle the stress state shall not violate the elastoplastic domain. When the stress state reaches a plastic regime, the material yields and the plastic strain evolutes according to the plastic flow rule or yield direction $N(\sigma, A)$.

$$N = \frac{\partial \Psi}{\partial \sigma},$$

The plastic strain increment is then ruled by:

$$\dot{\varepsilon^p} = \dot{\gamma} N,$$

where $\dot{\gamma}$ is a scalar.

2.5 Hardening rule

When the material yields, the yield criterion shape and therefore the elastoplastic domain may change reflecting a hardening or softening material behaviour. In this paper the special case of an isotropic hardening rule is handled. The isotropic hardening rule is characterized by an expansion of the yield surface as the material yield and is represented by the rate of change of the potential function with respect to the thermodynamic force,

$$H = \frac{\partial \Psi}{\partial A}.$$

The evolution of the damage internal variables is given by:

$$\dot{\alpha} = \dot{\gamma} H$$

where $\alpha(\Psi, A)$ is the set of damage internal variables.

3 ELASTOPLASTIC PROBLEM

The numerical approximation of the elastoplastic problem can be given by computing the stress increment as the solution of system of nonlinear equations. Given a deformation state ε_n, and the corresponding plastic strain ε_n^p, and internal variables α_n, the plastic strain and internal variables corresponding to ε_{n+1} are obtained as the solution system of nonlinear equations:

$$\left\{ \begin{array}{rcl} \Delta\gamma \geq 0, \Phi(\sigma_{n+1}, A_{n+1}) \leq 0, \Delta\gamma\Phi(\sigma_{n+1}, A_{n+1}) & = & 0 \\ \varepsilon_{n+1}^p - \varepsilon_n^p & = & \Delta\gamma N(\sigma_{n+1}, A_{n+1}) \\ \alpha_{n+1} - \alpha_n & = & \Delta\gamma H(\sigma_{n+1}, A_{n+1}) \end{array} \right. \quad (1)$$

The number of equations of this system is equal to the number of yield functions plus the number of components of the plastic strain 6, plus the number of internal variables. The dependent variables of the system are the values of $\Delta\gamma$, the value of the plastic strain ε_{n+1}^p and the internal variables α_{n+1}. The residual of the system of equations is computed as

$$Res(\varepsilon_{n+1}^p, \alpha_{n+1}, \Delta\gamma) = \left\{ \begin{array}{c} \Delta\gamma\Phi(\sigma_{n+1}, A_{n+1}) \\ \varepsilon_{n+1}^p - \varepsilon_n^p - \Delta\gamma N(\sigma_{n+1}, A_{n+1}) \\ \alpha_{n+1} - \alpha_n - \Delta\gamma H(\sigma_{n+1}, A_{n+1}) \end{array} \right. \quad (2)$$

If any function $\Phi(\sigma_{n+1}, A_{n+1}) < 0$, then the corresponding $\Delta\gamma = 0$ and the corresponding residual is set to zero. This is the particular case where the material is still in the elastic domain with $\varepsilon_{n+1}^p = \varepsilon_n^p$ and $\alpha_{n+1} = \alpha_n$. On the other hand, if this condition is not met at the elastic predictor the material may present plastic behavior and the above system is solved using the Newton method for the variable:

$$(\varepsilon_{n+1}^p, \alpha_{n+1}, \Delta\gamma).$$

The tangent matrix $T(\varepsilon_{n+1}^p, \alpha_{n+1}, \Delta\gamma)$. is computed using numerical differentiation as

$$T(\varepsilon_{n+1}^p, \alpha_{n+1}, \Delta\gamma) = \left(\begin{array}{c} \frac{\partial Res(\varepsilon_{n+1}^p, \alpha_{n+1}, \Delta\gamma)}{\partial \varepsilon_{n+1}^p} \\ \frac{\partial Res(\varepsilon_{n+1}^p, \alpha_{n+1}, \Delta\gamma)}{\partial \alpha_{n+1}} \\ \frac{\partial Res(\varepsilon_{n+1}^p, \alpha_{n+1}, \Delta\gamma)}{\partial \Delta\gamma_{n+1}} \end{array} \right)^T \quad (3)$$

There are some elastoplastic models for which analytic elastoplastic decompostition are available. However, the proposed code aims to implement a generic code able to handle a wide variety of constitutive models and thus the numerical approach is preferred. In order to assemble a Newton's scheme for the solution of this initial value problem in a versatile manner, automatic differentiation tools are employed to evaluated the residual derivatives.

4 ELASTOPLASTIC CONSTITUTIVE MODELS OF MOHR-COULOMB AND DRUCKER-PRAGER

4.1 Mohr-Coulomb

4.1.1 Mohr-Coulomb yield function

This criterion is for frictional materials sensible to the hydrostatic pressure, such as soil, rock and concrete. The Mohr-Coulomb yield criterion consider that the phenomenon of plastic yield occurs due to shear forces in internal interfaces. Generalizing the friction law of Coulomb, for any direction in continuum media the plastic flow begin when, in a arbitrary plane of the body, the shear stress and the normal stress reach a critical combination.

The Mohr-Coulomb yield criterion in the principal stress space is represented in terms of the eigenvalues and assumes the shape of Figure 1.

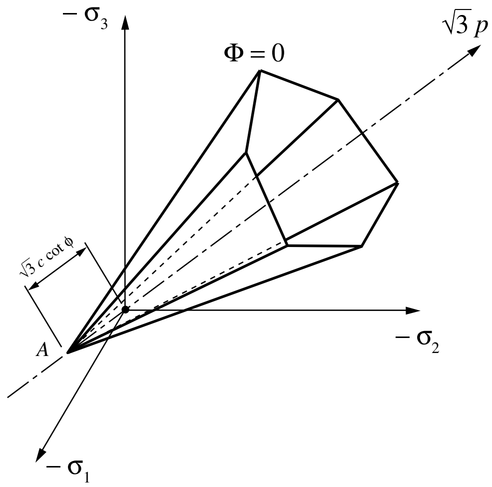

Figure 1: The Mohr-Coulomb yield surface in principal stress space Souza Neto et al. (2008)

As the surface of these criterion has some singularities, it is usual the use of multiple yield functions to represent it(Chen and Han (1988)). By ordering the principal stresses as $\sigma_1 \geq \sigma_2 \geq \sigma 3$ the criterion simplifies to the following three expressions:

$$\Phi_1(\sigma) = -\left(\frac{(\sigma_1+\sigma_3)}{2} + \frac{(\sigma_1-\sigma_3)}{2}\right)\tan(\phi) - \frac{1}{2}\cos(\phi)(\sigma_1-\sigma_3) + c,$$

$$\Phi_2(\sigma) = -\left(\frac{(\sigma_1+\sigma_2)}{2} + \frac{(\sigma_1-\sigma_2)}{2}\right)\tan(\phi) - \frac{1}{2}\cos(\phi)(\sigma_1-\sigma_2) + c,$$

$$\Phi_3(\sigma) = -\left(\frac{(\sigma_2+\sigma_3)}{2} + \frac{(\sigma_2-\sigma_3)}{2}\right)\tan(\phi) - \frac{1}{2}\cos(\phi)(\sigma_2-\sigma_3) + c,$$

where c is the material cohesion and ϕ is internal friction angle.

4.1.2 Mohr-Coulomb flow rule

The plastic flow rule considering associativity, is defined as,

$$\frac{\partial \Phi_1}{\partial \sigma} = N_1 = -\left(\frac{\sin(\phi)(\nabla\sigma_1 - \nabla\sigma_3)}{2} + \frac{(\nabla\sigma_1 + \nabla\sigma_3)}{2}\right)\tan(\phi) - \frac{1}{2}\cos(\phi)(\nabla\sigma_1 - \nabla\sigma_3),$$

$$\frac{\partial \Phi_2}{\partial \sigma} = N_2 = -\left(\frac{\sin(\phi)(\nabla\sigma_1 - \nabla\sigma_2)}{2} + \frac{(\nabla\sigma_1 + \nabla\sigma_2)}{2}\right)\tan(\phi) - \frac{1}{2}\cos(\phi)(\nabla\sigma_1 - \nabla\sigma_2),$$

$$\frac{\partial \Phi_3}{\partial \sigma} = N_3 = -\left(\frac{\sin(\phi)(\nabla\sigma_2 - \nabla\sigma_3)}{2} + \frac{(\nabla\sigma_2 + \nabla\sigma_3)}{2}\right)\tan(\phi) - \frac{1}{2}\cos(\phi)(\nabla\sigma_2 - \nabla\sigma_3).$$

4.2 Drucker-Prager

4.2.1 Drucker-Prager yield criterion

This criterion is a smooth approach of the Mohr-Coulomb criterion. It may be understood as a modification of the Von-Mises yield criterion, where the pressure sensitivity is introduced. Due to the material isotropy, it may be represented by the principal invariants.

$$\sqrt{J_2} + \eta p = \bar{c},$$

where η and $\bar{c}$ are material parameters. For $\eta = 0$ the Von-Mises cilinder is recovered. The Drucker-Prager cone is shown in Figure 2.

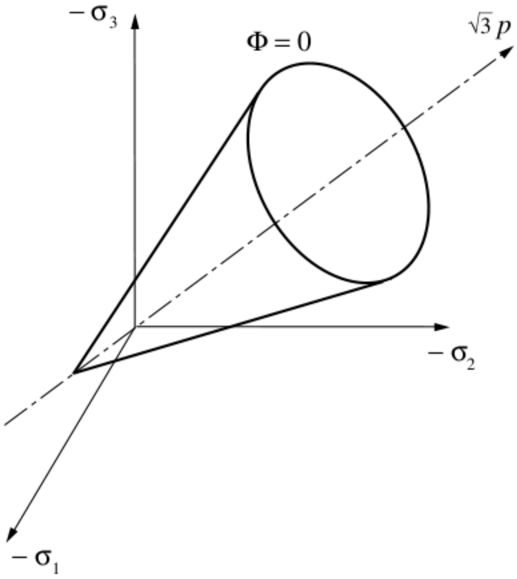

Figure 2: Drucker-Prager yield function Souza Neto et al. (2008)

A convenient way to define the Drucker-Prager yield criterion as a potential function:

$$\Phi(\sigma, c) = \sqrt{J_2} + \eta p - \xi c,$$

where c is the cohesion. The parameters ξ and η may be related to the equivalent Mohr-Coulomb parameters as given below:

To circumscribe Mohr-Coulomb yield function,

$$\eta = \frac{6\,sen\,\phi}{\sqrt{3}(3 - sen\,\phi)}, \qquad \xi = \frac{6\,cos\,\phi}{\sqrt{3}(3 - sen\,\phi)},$$

and to inscribe,

$$\eta = \frac{6\,sen\,\phi}{\sqrt{3}(3 + sen\,\phi)}, \qquad \xi = \frac{6\,cos\,\phi}{\sqrt{3}(3 + sen\,\phi)}.$$

4.2.2 Drucker-Prager flow rule

The plastic flow rule in case of associativity is,

$$N = \frac{1}{2\sqrt{J_2}}\nabla J_2 + \frac{\eta}{3}\nabla I_1$$

5 INVARIANTS, LODE ANGLE AND EIGENVALUES

The matrix representation of a second order tensor depends on the chosen basis. A more objective manner (and therefore observer-independent) of measuring stress and strain relies on the tensor invariants:

$$I_1(A) = tr\,(A),$$
$$I_2(A) = \frac{1}{2}[(tr\,A)^2 - tr\,(A^2)],$$
$$I_3(A) = det\,(A).$$

The deviatoric tensor is defined by:

$$S(A) = A - \frac{1}{3}tr(A)I$$

and its respective invariants are:

$$J_1 = 0$$
$$J_2 = \frac{1}{2}[(tr\,S)^2 - tr\,(S^2)]$$
$$J_3 = det(S)$$

This work requires the evaluation of the derivatives of the invariants I_1, J_2 and J_3:

$$\nabla I_1 = I,$$

$$\nabla J_2 = S(A),$$

$$\nabla J_3 = det(A)A^{-T} - \frac{1}{3}det(A)tr(A^{-1})I.$$

The tree principal stresses or the eigenvalues of the stress tensor can be obtained analytically in function of the principal invariants and of the Lode angle(Chen and Han (1988)). The Lode angle is equal to:

$$\theta = \frac{1}{3}\arccos\left(\frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{\frac{3}{2}}}\right), (0 \leq \theta \leq \frac{\pi}{3}).$$

The eigenvalues are equal to:

$$\sigma_1 = \frac{I_1}{3} + \frac{2}{\sqrt{3}}\sqrt{J_2}\cos(\theta),$$

$$\sigma_2 = \frac{I_1}{3} + \frac{2}{\sqrt{3}}\sqrt{J_2}\cos(\theta - \frac{2\pi}{3}),$$

$$\sigma_3 = \frac{I_1}{3} + \frac{2}{\sqrt{3}}\sqrt{J_2}\cos(\theta + \frac{2\pi}{3}).$$

6 IMPLEMENTATION

In this section is described the object oriented implementation of the template elastoplastic framework.

6.1 Class TPZPlasticStep

This is the main class and is designed to be as general as possible. It gathers the hole logic of the elastoplastic problem and perform the advancement of one step of plastification using the Newton's Method to solve the non-linear elastoplastic problem and its constructor has the shape TPZPlasticStep<YieldCriterion , TPZElasticResponse , TPZThermoForceA>.

It receives as parameter the following template objects:

- TPZThermoForceA: is a generic argument to inform the evolution model of the isotropic thermodynamic force according whit the evolution of the damage variable α during the yield whit hardening or softening.
- TPZElasticResponse: is a generic argument for the aggregation of a behaviour of a elastic regime or elasticity law.
- YieldCriterion: template argument that implements the yield criterion $\Phi(\sigma, A)$ and the flow direction $N(\sigma, A)$. In case that the model require the implementation of a yield surface, rupture or beginning of behaviour of material softening, the implementations has to be done on this class. This class may also be generic, which facilitated the assemble of associative or non-associative models.

6.2 Class TPZTensor

The constructor is TPZTensor<T>. These class implements a second order symmetric tensor 3x3. The generic argument tha can be, for example, a number. These class store the tensor in its vector shape. The class implements some basic functionalities of a tensor, as the calculation of the principal invariants and its derivatives.

6.3 Class TPZPlasticState

The constructor is TPZPlasticState<T>. These class implements a concept of plastic state variables, useful for the algorithms of the main class. The class represents one aggregation of:

- $fEpsT$: that is a tensor of the type TPZTensor<T> to store the state of total deformation ε;
- $fEpsP$: that is a tensor of the type TPZTensor<T> to store the state of plastic deformation ε^p;
- $fAlpha$: that is a scalar of type T to store the damage variable k.

6.4 Class TPZPlasticIntegrMem

During the integration of the elastoplastic problem can be necessary the sub incrementation of the total imposed deformation. It is necessary store the history of the intermediate solutions of the sub incrementation steps, this task is realised by a object vector of the class TPZPlasticIntegrMem. This is not a generic class because is known the type will be used will always be a real type. This class aggregates the following data:

- fPlasticState: an object of the type TPZPlasticState<REAL> to store the yield state of the material;
- fk : a REAL to store the step k of the incrementation, being $0 < k < 1$;
- fLambda: a REAL to store the result λls of a analysis of line search;
- fDelGamma: a REAL vector to store the plastic yield multipliers $\Delta\lambda$, to each yield function;

7 RESULTS

In this section the results of this work are exposed. The results will be divided in two sections. The first section will show the results of a uni-axial loading cycle in a material point, and, in the second part will be shown the results of the integration of the plastic models whit the Finite Element Method is shown.

1. A loading-unloading cycle applied to a material point with the elastoplastic model of Drucker-Prager. A uni-axial load is applied in each step, and its corresponding strain is computed. The result is shown in Figure 3.
2. In this test a Finite Element analysis is performed. A equilibrated distributed forces is applied to the faces of the hexagonal domain. A rotation is applied to the domain to verify the consistency of the code. The forces applied exceeded the elastic range so that was possible to verify the elastoplastic integration as well. As shown in Figure 4, the original domain is compared with the rotated in order to check the invariance of the observer.

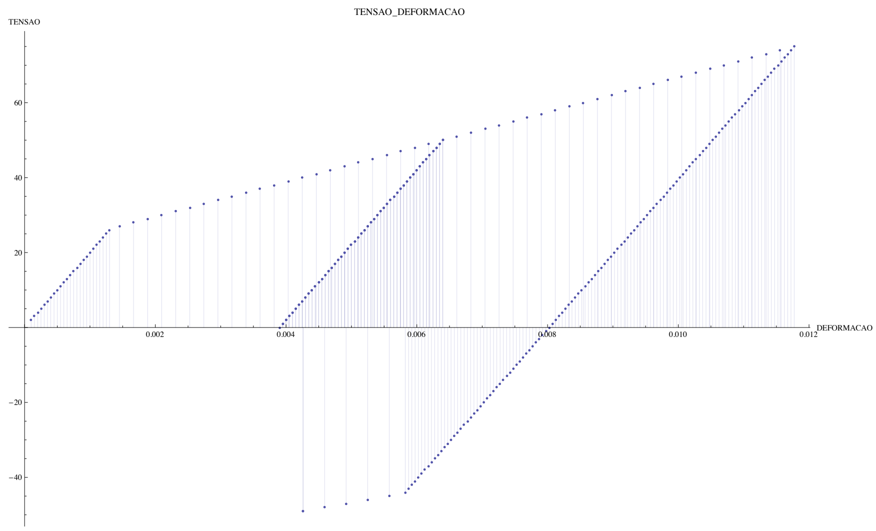

Figure 3: Uniaxial stress-strain test result

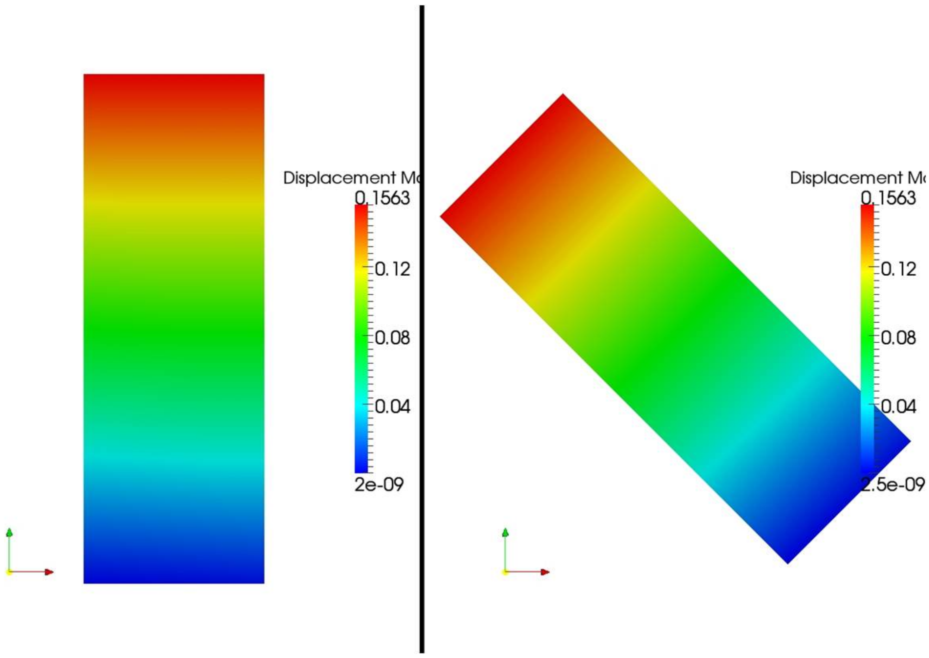

Figure 4: non rotated and rotate elements

8 CONCLUSIONS

In this paper the approach was based on the object oriented design philosophy and observations on similarity of most incremental elastoplastic material models. Based on this approach we have shown that new elastoplastic material models can be created by combining small number of building blocks. This has an added benefit of allowing an easy implementation of other elastoplastic material models based on the object oriented design principles.

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