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A FINITE ELEMENT FORMULATION SATISFYING THE DISCRETE GEOMETRIC CONSERVATION LAW BASED ON AVERAGED JACOBIANS.

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Abstract. In this article a new methodology for developing DGCL (for Discrete Geometric Conservation Law) compliant formulations is presented. It is carried out in the context of the Finite Element Method (FEM) for general advective-diffusive systems on moving domains using an Arbitrary Lagrangian Eulerian (ALE) scheme.

There is an extensive literature about the impact of DGCL compliance on the stability and precision of time integration methods. In those articles it has been proved that satisfying the DGCL is a necessary and sufficient condition for any ALE scheme to maintain on moving grids the nonlinear stability properties of its fixed-grid counterpart. However, only a few works propose a methodology for obtaining a compliant scheme.

In this work, a DGCL compliant scheme based on an Averaged ALE Jacobians Formulation (AJF) is obtained. This new formulation is applied to the theta-family of time integration methods. In addition, an extension to the three-point Backward Difference Formula (BDF) is given. With the aim to validate the AJF formulation a set of numerical tests are performed. These tests include 2D and 3D diffusion problems with different mesh movements, and the 2D Euler flow over a pitching NACA0012 airfoil.

1 INTRODUCTION

When dealing with partial differential equations that need to be solved on moving domains, like problems in the Fluid-Structure Interaction area (FSI) (Storti et al., 2009; Garelli et al., 2010), one of the most used technique is the so-called *Arbitrary Lagrangian Eulerian* (ALE). The idea behind the ALE formulation is the introduction of a computational mesh which moves with a velocity independent of the speed of the material particles. The ALE method was first proposed in the context of finite differences (Noh, 1964; Hirt et al., 1974), then it was extended to finite elements (Donea, 1983; Hughes et al., 1978) and to finite volumes (Trepanier et al., 1991).

When an ALE formulation is used, the governing equations must be rewritten and additional terms related to the mesh velocity and position, are introduced. The reformulated equations must be integrated in time. The common way to proceed is to use a classical time advancing scheme like the θ-family or the BDF's family. In this context the DGCL arises and it is directly related to the evolution of the mesh velocity and the elements volume change. This law was introduced by Thomas and Lombard (1979) and it is a consistency criterion in which the numerical method must be able to reproduce exactly a constant solution on a moving domain.

As noted by Étienne et al. (2009) the effect of the DGCL on the stability of ALE schemes is still unclear and somewhat contradictory. In the work by Guillard and Farhat (2000), it has been observed that the movement of the domain can degrade the accuracy and stability of the numerical scheme with respect to their counterpart on fixed domains. In this direction, many researchers have been working with the aim of linking the accuracy and the stability of numerical schemes on an ALE framework with the discrete version of the Geometric Conservation Law (Guillard and Farhat, 2000; Boffi and Gastaldi, 2004; Formaggia and Nobile, 2004; Étienne et al., 2009). In the article by Geuzaine et al. (2003) it has been shown that satisfying the DGCL is neither necessary nor sufficient condition for an ALE scheme to preserve on moving grids its time-accuracy established on fixed grids. In the work presented by Farhat et al. (2001) it was proved that for nonlinear scalar problems the DGCL requirement is a necessary and sufficient condition for an ALE time-integrator to preserve the nonlinear stability properties of its fixed-grid counterpart. Meanwhile, Boffi and Gastaldi (2004) and Formaggia and Nobile (2004) have shown that it is neither necessary nor sufficient condition for stability, except for the Backward Euler scheme. While the impact of the DGCL on the stability and precision of the time integration methods is controversial, there is a general consensus in the development of schemes that satisfy the DGCL, in particular for FSI problems (Ahn and Kallinderis, 2006; Lesoinne and C.Farhat, 1996; Nobile, 2001; Mavriplis and Yang, 2005).

A straightforward way to satisfy the DGCL is to use a time integration rule with degree of precision $n_d \cdot s - 1$, where n_d is the spatial dimension and s is the order of the polynomial used to represent the time evolution of the nodal displacement within each time step. For example, in 3D problems with a linear in time reconstruction a rule with degree of precision 2 should be used. Alternatively, the methodology proposed by Farhat and Geuzaine (2004) to obtain an ALE extension for a given time-integrator in fixed meshes, could be used.

In this work a new methodology, which is based on averaged ALE Jacobians is proposed to obtain DGCL compliant FEM formulations. It is applied to the θ-family of time integration methods in general nonlinear advective-diffusive problems. In addition, an extension to the three-point Backward Difference Formula (BDF) is given.

In previous work (Farhat and Geuzaine, 2004) averaged coefficients are obtained by starting with a general integration scheme with a series of unknown parameters, which are then adjusted in order to preserve DGCL compliance, and the temporal accuracy of the fixed mesh counterpart. In contrast, in this work the geometric coefficients are obtained by averaging them over the time step. So that, precision is preserved and the DGCL is satisfied in a natural way.

Finally, to validate the Averaged Jacobians Formulation (AJF) a set of numerical tests are performed. This includes 2D/3D diffusion problems on moving meshes and 2D Euler equations for a pitching NACA0012 airfoil.

2 VARIATIONAL FORMULATION FOR ADVECTIVE DIFFUSIVE SYSTEM FOR MOVING MESHES USING ALE

Let us start with the derivation of the ALE formulation for a general advective-diffusive system (Donea, 1983; Lesoinne and C.Farhat, 1996). The governing equation is

$$\frac{\partial U_j}{\partial t} + \left(\mathcal{F}^c_{jk}(\mathbf{U}) - \mathcal{F}^d_{jk}(\mathbf{U}, \nabla\mathbf{U})\right)_{,k} = 0, \quad \text{in } \Omega^t \tag{1}$$

where $1 \leq k \leq n_d$, n_d is the number of spatial dimensions, $1 \leq j \leq m$, m is the dimension of the state vector (e.g. $m = n_d + 2$ for compressible flow), t is time, $(\,)_{,j}$ denotes derivative with respect to the j-th spatial dimension, $\mathbf{U} \in \mathbb{R}^n$ is the state vector, and $\mathcal{F}^{c,d}_{jk} \in \mathbb{R}^{n\times n_d}$ are the convective and diffusive fluxes, respectively. Appropriate Dirichlet and Neumann conditions are imposed at the boundary.

As the problem is posed in a time-dependent domain Ω^t, it can not be solved with standard fixed-domain methods, so that it is assumed that there is an inversible and continuously differentiable map $\mathbf{x} = \boldsymbol{\chi}(\boldsymbol{\xi}, t)$ between the current domain Ω^t and a reference domain Ω^ξ, which can be for instance the initial domain $\Omega^\xi = \Omega^{t=0}$, and $\boldsymbol{\xi}$ is the coordinate in the reference domain. The Jacobian of the transformation is

$$J = \left|\frac{\partial x_j}{\partial \xi_k}\right|, \tag{2}$$

and satisfies the following volume balance equation

$$\left.\frac{\partial J}{\partial t}\right|_{\boldsymbol{\xi}} = J\frac{\partial v^*_k}{\partial x_k}, \tag{3}$$

where

$$v^*_k = \left.\frac{\partial x_k}{\partial t}\right|_{\boldsymbol{\xi}}, \tag{4}$$

are the components of the mesh velocity.

The variational formulation of (1) is obtained multiplying with a weighting function $w(\mathbf{x}, t) = w(\boldsymbol{\chi}(\mathbf{x}, t))$ and integrating over the current domain Ω^t

$$\int_{\Omega^t} w\,\frac{\partial U_j}{\partial t}\,\mathrm{d}\Omega^t + \int_{\Omega^t} \left[\mathcal{F}^c_{jk} - \mathcal{F}^d_{jk}\right]_{,k} w\,\mathrm{d}\Omega^t = 0. \tag{5}$$

The integrals are brought to the reference domain Ω^ξ

$$\int_{\Omega^\xi} w\,\frac{\partial U_j}{\partial t} J\,\mathrm{d}\Omega^\xi + \int_{\Omega^\xi} \left[\mathcal{F}^c_{jk} - \mathcal{F}^d_{jk}\right]_{,k} wJ\,\mathrm{d}\Omega^\xi = 0, \tag{6}$$

and the temporal derivative term can be converted to the reference mesh by noting that the partial derivative of U_j is in fact a partial derivative at $\mathbf{x}$ = constant, and then can be converted to a partial derivative at $\boldsymbol{\xi}$ = constant with the relation

$$\left.\frac{\partial U_j}{\partial t}\right|_{\mathbf{x}} = \left.\frac{\partial U_j}{\partial t}\right|_{\boldsymbol{\xi}} - v_k^* \frac{\partial U_j}{\partial x_k}. \tag{7}$$

So the temporal derivative term in (6) can be transformed, using (3), as follows

$$\begin{aligned} J \left.\frac{\partial U_j}{\partial t}\right|_{\mathbf{x}} &= J \left.\frac{\partial U_j}{\partial t}\right|_{\boldsymbol{\xi}} - J v_k^* \frac{\partial U_j}{\partial x_k}, \\ &= \left.\frac{\partial (J U_j)}{\partial t}\right|_{\boldsymbol{\xi}} - J U_j \frac{\partial v_k^*}{\partial x_k} - J v_k^* \frac{\partial U_j}{\partial x_k}, \\ &= \left.\frac{\partial (J U_j)}{\partial t}\right|_{\boldsymbol{\xi}} - J \frac{\partial (U_j v_k^*)}{\partial x_k}. \end{aligned} \tag{8}$$

Replacing (8) in (6),

$$\int_{\Omega^\xi} w(\boldsymbol{\xi}) \left.\frac{\partial}{\partial t}(J U_j)\right|_{\boldsymbol{\xi}} \mathrm{d}\Omega^\xi + \int_{\Omega^\xi} \left(\mathcal{F}^c_{jk} - v_k^* U_j - \mathcal{F}^d_{jk}\right)_{,k} w(\boldsymbol{\xi}) J \,\mathrm{d}\Omega^\xi = 0. \tag{9}$$

The temporal derivative can be commuted with the integral and the weighting function since both do not depend on time, so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega^\xi} w\, J U_j \,\mathrm{d}\Omega^\xi\right) + \int_{\Omega^\xi} \left(\mathcal{F}^c_{jk} - v_k^* U_j - \mathcal{F}^d_{jk}\right)_{,k} w J \,\mathrm{d}\Omega^\xi = 0, \tag{10}$$

and the integrals can be brought back to the Ω^t domain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega^t} w\, U_j \,\mathrm{d}\Omega^t\right) + \int_{\Omega^t} \left(\mathcal{F}^c_{jk} - v_k^* U_j - \mathcal{F}^d_{jk}\right)_{,k} w \,\mathrm{d}\Omega^t = 0. \tag{11}$$

The variational formulation can be obtained by integrating by parts, so that

$$\frac{\mathrm{d}}{\mathrm{d}t}(H(w,U)) + F(w,U) = 0, \tag{12}$$

where

$$\begin{aligned} H(w,U) &= \int_{\Omega^t} w\, U_j \,\mathrm{d}\Omega^t, \\ F(w,U) &= A(w,U) + B(w,U) + S(w,U), \\ A(w,U) &= -\int_{\Omega^t} \left(\mathcal{F}^c_{jk} - v_k^* U_j - \mathcal{F}^d_{jk}\right) w_{,k} \,\mathrm{d}\Omega^t, \\ B(w,U) &= \int_{\Gamma^t} \left(\mathcal{F}^c_{jk} - v_k^* U_j - \mathcal{F}^d_{jk}\right) n_k w \,\mathrm{d}\Gamma, \end{aligned} \tag{13}$$

Γ^t is the boundary of Ω^t, and n_k is its unit normal vector pointing to the exterior of Ω. Also, a consistent stabilization term $S(w,U)$ is included in order to avoid numerical problems for advection dominated problems (Franca et al., 1992).

Finally (11) is discretized in time with the *trapezoidal rule* (application to the *Backward Differentiation Formula (BDF)* will be described later)

$$\begin{aligned} H(w, U^{n+1}) - H(w, U^n) &= -\int_{t^n}^{t^{n+1}} F(w, U^{t'})\,\mathrm{d}t', \\ &\approx -\Delta t\, F(w, U^{n+\theta}). \end{aligned} \tag{14}$$

with $0 \le \theta \le 1$. During the time step it is assumed that the nodal points move with constant velocity, i.e.

$$\left. \begin{aligned} v_k^*(\boldsymbol{\xi}) &= \frac{x_k(\boldsymbol{\xi}, t^{n+1}) - x_k(\boldsymbol{\xi}, t^n)}{\Delta t}, \\ x_k(\boldsymbol{\xi}, t) &= x_k(\boldsymbol{\xi}, t^n) + (t - t^n) v_k^*(\boldsymbol{\xi}), \end{aligned} \right\}, \qquad \text{for } t^n \le t \le t^{n+1}. \tag{15}$$

2.1 The Discrete Geometric Conservation Law Condition

A discrete formulation is said to satisfy the DGCL condition if it solves exactly a constant state regime, i.e. not depending on space or time, for a general mesh movement $\mathbf{x}(\boldsymbol{\xi}, t)$. As was mentioned in §1 the effect of the DGCL in the precision and numerical stability of the scheme is an open discussion, but in several works (Guillard and Farhat, 2000; Formaggia and Nobile, 2004) it is recommended to employ numerical schemes that satisfy the DGCL. This may help in improve the precision and the stability.

By replacing U_j = constant and after some manipulations it can be shown that the DGCL is satisfied if

$$\int_{\Omega^{n+1}} w\,\mathrm{d}\Omega - \int_{\Omega^n} w\,\mathrm{d}\Omega = \Delta t \int_{\Omega^{n+\theta}} v_k^*\, w_{,k}\,\mathrm{d}\Omega. \tag{16}$$

A similar restriction holds for the boundary term. The stabilization term $S(w, U)$ normally satisfies automatically the DGCL since it involves gradients of the state, and then it is null for a constant state.

Note that this previous equation holds if the right hand side is evaluated as an integral instead of being evaluated at $t^{n+\theta}$, i.e. the DGCL error comes from the approximation that was made in (14), i.e. it is always true that

$$\int_{\Omega^{n+1}} w\,\mathrm{d}\Omega - \int_{\Omega^n} w\,\mathrm{d}\Omega = \int_{t^n}^{t^{n+1}} \left\{ \int_{\Omega^t} v_k^*\, w_{,k}\,\mathrm{d}\Omega \right\} \mathrm{d}t. \tag{17}$$

Consider the integrand in the right hand side. Transforming to the reference domain Ω^ξ we obtain

$$\begin{aligned} \int_{t^n}^{t^{n+1}} \left\{ \int_{\Omega^t} v_k^*\, w_{,k}\,\mathrm{d}\Omega \right\} \mathrm{d}t &= \int_{t^n}^{t^{n+1}} \left\{ \int_{\Omega^\xi} v_k^* \frac{\partial w}{\partial \xi_l} \frac{\partial \xi_l}{\partial x_k} J\,\mathrm{d}\Omega^\xi \right\} \mathrm{d}t, \\ &= \int_{\Omega^\xi} v_k^* \frac{\partial w}{\partial \xi_l} \int_{t^n}^{t^{n+1}} \left(\frac{\partial \xi_l}{\partial x_k} J \right)^t \mathrm{d}t\,\mathrm{d}\Omega^\xi, \\ &= \int_{\Omega^\xi} v_k^* g_k^{n+\theta} J^{n+\theta}\,\mathrm{d}\Omega^\xi. \\ &= \int_{\Omega^{n+\theta}} v_k^* g_k^{n+\theta}\,\mathrm{d}\Omega, \end{aligned} \tag{18}$$

where g_k is an *averaged interpolation function gradient*

$$
\begin{aligned}
g_k^{n+\theta} &= (J^{n+\theta})^{-1} \bar{Q}_{lk}^{n+1/2} \frac{\partial w}{\partial \xi_l},\\
\bar{Q}_{lk}^{n+1/2} &= \int_{t^n}^{t^{n+1}} Q_{lk}^t \,\mathrm{d}t,\\
Q_{lk}^t &= \left(J \frac{\partial \xi_l}{\partial x_k} \right)^t.
\end{aligned}
\tag{19}
$$

The proposed scheme is then to replace the $A(w, U^{n+\theta})$ operator in (14) by

$$
A^{\mathrm{GCL}}(w, U^{n+\theta}) = -\int_{\Omega^{n+\theta}} \left[\mathcal{F}_{jk}^c - v_k^* U_j - \mathcal{F}_{jk}^d \right]\Big|_{t^{n+\theta}} g_k^{n+\theta} \,\mathrm{d}\Omega, \tag{20}
$$

A similar modification must be introduced in the boundary term $B(w, U)$, this will be explained later in Section §2.3 . It is easy to check that with this modification the scheme is DGCL compliant for all θ.

2.2 Evaluation of the average interpolation function gradient

Due to (15) each component x_k is a linear function of time inside the time step, then the spatial derivatives $(\partial x_k / \partial \xi_l)$ are also linear functions, and the determinant J is a polynomial of degree n_d. Also, the components of the inverse transformation $\boldsymbol{\xi} \to \mathbf{x}$ can be determined from the inverse of the direct transformation $\mathbf{x} \to \boldsymbol{\xi}$ as

$$
\begin{aligned}
\frac{\partial \xi_l}{\partial x_k} &= \left(\frac{\partial \mathbf{x}}{\partial \boldsymbol{\xi}} \right)_{lk}^{-1},\\
J \frac{\partial \xi_l}{\partial x_k} &= (-1)^{k+l} \,\mathrm{minor} \left(\frac{\partial \mathbf{x}}{\partial \boldsymbol{\xi}} \right)_{kl},
\end{aligned}
\tag{21}
$$

where $\mathrm{minor}(\mathbf{A})_{ij}$ is the determinant of the submatrix of $\mathbf{A}$ when row i and column j have been eliminated. Then, the minors are polynomials of order $n_d - 1$ and so are the entries of $J \frac{\partial \xi_l}{\partial x_k}$ that are the integrands in (19).

As a check, well known results about the compliance of the DGCL with the trapezoidal rule will be verified. The DGCL is satisfied if the integration rule used to approximate the time integral in (19) is exact, for instance $\theta = 1/2$ satisfies the DGCL in 2D, since the integrand is linear and the trapezoidal rule reduces to the midpoint rule. In addition, DGCL is satisfied in 1D for any $0 \leq \theta \leq 1$, and for none in 3D. The point is that using $\theta = 1/2$ (Crank-Nicolson) is restrictive, and there is no θ that satisfies the DGCL in 3D, so that the method proposed here uses a higher order time integration for (19) so that the DGCL is satisfied for an arbitrary θ in any dimension. The method can be extended easily to other temporal integration schemes (see section §2.4).

For instance, the Gauss integration method can be used. Normally the Jacobians and determinants are known at t^n and t^{n+1} since they are needed for the computation of the temporal term (the right hand side in (14)), so perhaps it is better to use the Gauss-Lobatto version which includes the extremes of the interval. The Gauss-Lobatto method integrates exactly polynomials of up to degree $2n - 3$ where n is the number of integration points, so that it suffices to use the extreme points for simplices in $n_d = 2$ and to add a point at the center of the interval for $n_d = 3$, i.e.

$$g_k^{n+\theta} = \begin{cases} \dfrac{\Delta t}{2J^{n+\theta}} \left[Q_{lk}^n + Q_{lk}^{n+1}\right] \dfrac{\partial w}{\partial \xi_l}, & \text{in 2D,} \\ \dfrac{\Delta t}{6J^{n+\theta}} \left[Q_{lk}^n + 4Q_{lk}^{n+1/2} + Q_{lk}^{n+1}\right] \dfrac{\partial w}{\partial \xi_l}, & \text{in 3D,} \end{cases} \tag{22}$$

being Q_{lk}^t defined in (19).

2.3 The boundary term

The boundary term in (13) can be brought to the reference domain as follows

$$\begin{aligned} B(w,U) &= \int_{\partial\Gamma^t} \left[\mathcal{F}_{jk}^c - v_k^* U_j - \mathcal{F}_{jk}^d\right] w n_k \,\mathrm{d}\Gamma, \\ &= \int_{\partial\Gamma^\xi} \left[\mathcal{F}_{jk}^c - v_k^* U_j - \mathcal{F}_{jk}^d\right] w n_k J_\Gamma \,\mathrm{d}\Gamma^\xi, \end{aligned} \tag{23}$$

where J_Γ is the Jacobian of the transformation between a surface element in Γ^t and Γ^ξ. The DGCL is satisfied if the averaged normal vector is used, i.e.

$$\begin{aligned} B^{\mathrm{GCL}}(w,U) &= \int_{\partial\Gamma^t} \left[\mathcal{F}_{jk}^c - v_k^* U_j - \mathcal{F}_{jk}^d\right] w \bar{n}_k \,\mathrm{d}\Gamma, \\ \bar{n}_k &= \frac{1}{J_\Gamma^\theta} \eta_k, \\ \eta_k &= \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} n_k J_\Gamma \,\mathrm{d}t. \end{aligned} \tag{24}$$

Regarding the evaluation of the integral for computing η_k, the considerations are very similar to those given in §2.2. The components of $n_k J_\Gamma$ are also polynomials of degree $n_d - 1$ in time. For instance in 3D, if $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are the nodes at the vertices of a triangle element (ordered counter-clockwise when viewed from the exterior of the fluid) on the surface Γ^t, then

$$\mathbf{n} J_\Gamma = \frac{(\mathbf{x}_2 - \mathbf{x}_1) \times (\mathbf{x}_3 - \mathbf{x}_1)}{2|\Gamma^\xi|}, \tag{25}$$

where $\times$ denotes the vector cross product and $|\Gamma^\xi|$ is the area of the triangle in the reference coordinates. As the coordinates of the nodes are linear in time and $|\Gamma^\xi|$ is constant, the components of $n_k J_\Gamma$ are quadratic polynomials.

Then, the considerations about the number of points for the Gauss-Lobatto integration are the same as discussed before, i.e. two integration points are enough to compute the integral in (24), and three are needed in 3D.

2.4 Application to the BDF

The Backward Differentiation Formula is another popular method for the integration of the system of ordinary differential equations. Applied to (12) gives

$$\frac{1}{\Delta t} \left(3/2 H^{n+1} - 2H^n + 1/2 H^{n-1}\right) = F(w, U^{n+1}). \tag{26}$$

In order to apply the Averaged Jacobian Formulation, the right hand side of (26) must be rewritten as an integral over time. For this, note that, for any differentiable function $X(t)$ we have

$$\begin{aligned} {}^{3}\!/_{2} X^{n+1} - 2X^n + {}^{1}\!/_{2} X^{n-1} &= {}^{3}\!/_{2}(X^{n+1} - X^n) - {}^{1}\!/_{2}(X^n - X^{n-1}), \\ &= {}^{3}\!/_{2} \int_{t^n}^{t^{n+1}} \dot{X}\,\mathrm{d}t - {}^{1}\!/_{2} \int_{t^{n-1}}^{t^n} \dot{X}\,\mathrm{d}t. \end{aligned} \tag{27}$$

If this relation is applied with the semidiscrete equations (12) with $X = H$ and $\dot{X} = F$, then the following relation is obtained

$${}^{3}\!/_{2} H^{n+1} - 2H^n + {}^{1}\!/_{2} H^{n-1} = -{}^{3}\!/_{2} \int_{t^n}^{t^{n+1}} F(w, U^{t'})\,\mathrm{d}t' + {}^{1}\!/_{2} \int_{t^{n-1}}^{t^n} F(w, U^{t'})\,\mathrm{d}t'. \tag{28}$$

The BDF integration method is obtained if the right hand side in (28) is replaced by the value of the integrand at t^{n+1}. The proposed method in order to satisfy the DGCL is to assume that the state in (28) remains constant but the geometric quantities v_k^* and $w_{,k}$ not, so that

$${}^{3}\!/_{2} H^{n+1} - 2H^n + {}^{1}\!/_{2} H^{n-1} = -\Delta t F^{\mathrm{BDF}}(w, U^{t^{n+1}}), \tag{29}$$

where

$$\begin{aligned} F^{\mathrm{BDF}}(w, U^{n+1}) &= A^{\mathrm{BDF}}(w, U^{n+1}) + B^{\mathrm{BDF}}(w, U^{n+1}) + S(w, U^{n+1}), \\ A^{\mathrm{BDF}}(w, U^{n+1}) &= -\int_{\Omega^{n+1}} \left[(\mathcal{F}_{jk}^c - \mathcal{F}_{jk}^d)^{n+1} g_k^{n+1} - U_j^{n+1} r^{n+1} \right] \mathrm{d}\Omega, \\ B^{\mathrm{BDF}}(w, U^{n+1}) &= \int_{\partial\Gamma^t} \left[(\mathcal{F}_{jk}^c - \mathcal{F}_{jk}^d) \beta_k^{n+1} - U_j^{n+1} s^{n+1} \right] w\,\mathrm{d}\Gamma, \end{aligned} \tag{30}$$

and g_k, r, β_k, and s are time averaged geometric quantities given by

$$\begin{aligned} g_k^{n+1} &= \frac{1}{J^{n+1}} \left({}^{3}\!/_{2} Q_{lk}^{n+1/2} - {}^{1}\!/_{2} Q_{lk}^{n-1/2} \right) \frac{\partial w}{\partial \xi_l}, \\ r^{n+1} &= \frac{1}{J^{n+1}} \left({}^{3}\!/_{2} Q_{lk}^{n+1/2}\, v_k^{*n+1/2} - {}^{1}\!/_{2} Q_{lk}^{n-1/2}\, v_k^{*n-1/2} \right) \frac{\partial w}{\partial \xi_l}, \\ \beta_k^{n+1} &= \frac{1}{J_\Gamma^{n+1}} ({}^{3}\!/_{2} \eta_k^{n+1/2} - {}^{1}\!/_{2} \eta_k^{n-1/2}), \\ s^{n+1} &= \frac{1}{J_\Gamma^{n+1}} ({}^{3}\!/_{2} \eta_k^{n+1/2} v_k^{*n+1/2} - {}^{1}\!/_{2} \eta_k^{n-1/2} v_k^{*n-1/2}), \\ \eta_k^{n+1/2} &= \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} n_k J_\Gamma\,\mathrm{d}t, \end{aligned} \tag{31}$$

and $v_k^{*n+1/2}$ is the (constant) velocity in time step $[t^n, t^{n+1}]$. Regarding the computation of the averaged Jacobians $Q_{lk}^{n+1/2}$ and $\eta_k^{n+1/2}$ the rules are the same as before, since their entries are polynomials of degree $n_d - 1$ within the time interval.

3 NUMERICAL TESTS

In this section a set of numerical tests are performed in order to validate the Averaged Jacobian Formulation (AJF) proposed in section (§2).

3.1 DGCL validation for 2D scalar diffusion problem with internal node movement

For the sake of clarity, let us consider, the scalar diffusion version of the equation (1).

$$\begin{aligned} \frac{\partial u}{\partial t} - \mu\Delta u = 0 \quad & \text{for } \mathbf{x} \in \Omega^{\mathrm{t}},\ \mathrm{t} \in (0, \mathrm{T}] \\ u = u_0 \quad & \text{for } \mathbf{x} \in \Omega^0,\ \mathrm{t} = 0 \\ u = u_D \quad & \text{for } \mathbf{x} \in \partial\Omega^{\mathrm{t}},\ \mathrm{t} \in [0, \mathrm{T}] \end{aligned} \tag{32}$$

where μ is the constant diffusivity and Δ is the Laplacian operator.

To carry out the DGCL compliance test, the problem (32) is solved on an unit square domain with $\mu = 0.01$, so that

$$\begin{aligned} u_t - 0.01\Delta u = 0 \quad & \text{for } \mathbf{x} \in \Omega^{\mathrm{t}},\ \mathrm{t} \in (0, \mathrm{T}], \\ u_0 = 1 \quad & \text{for } \mathbf{x} \in \Omega^0,\ \mathrm{t} = 0, \\ u = 1 \quad & \text{for } \mathbf{x} \in \partial\Omega^{\mathrm{t}},\ \mathrm{t} \in [0, \mathrm{T}], \end{aligned} \tag{33}$$

being the mesh deformed according to the following rule

$$\begin{aligned} \boldsymbol{\chi}_1(\xi, t) = x = \xi + 0.125\ \sin(\pi\ t)\ \sin(2\pi\ \xi). \\ \boldsymbol{\chi}_2(\eta, t) = y = \eta + 0.125\ \sin(\pi\ t)\ \sin(2\pi\ \eta). \end{aligned} \tag{34}$$

Figure (1) shows the reference domain and the deformed mesh for $t = 0.5\ [s]$ where the maximum deformation occurs.

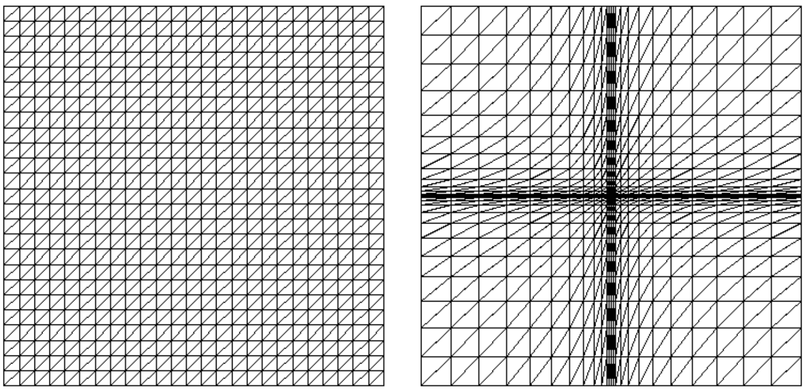

Figure 1: Reference and deformed mesh.

The problem is solved using piecewise linear triangles for the spatial discretization, a piecewise linear interpolation of the mesh movement and for the time integration the *Backward Euler* ($\theta = 1$), *Crank-Nicolson* ($\theta = 0.5$) and *Galerkin* ($\theta = 2/3$) schemes are considered with Δt ={0.15, 0.1, 0.05, 0.025}. Figure (2) reports the error $||u_h - u||_{L^2(\Omega^n)}$ for three periods of oscillation, which must be null to machine precision over time for a DGCL compliant scheme.

An error is introduced when using the *Backward Euler* or *Garlerkin* scheme due to lack in DGCL compliance. In Figure (3) the solution for times $t = \{0.1, 2.4, 5.4\}\ [s]$ is shown for the three different integration schemes. The error related to the constant solution is located in the zones of the domain where the element deformation is higher, as in the center and the corners.

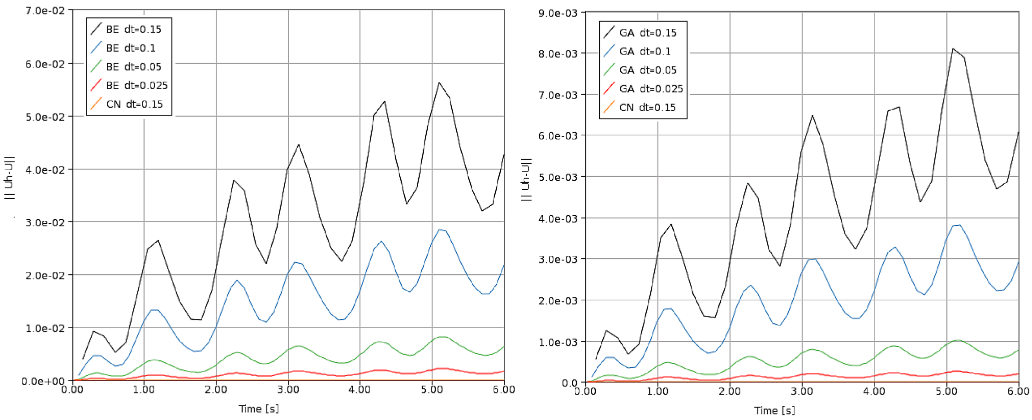

Figure 2: $||u_h - u||_{L^2(\Omega^n)}$ for *Garlerkin* (GA) and *Backward Euler* (BE) schemes compared with *Crank-Nicolson* (CN).

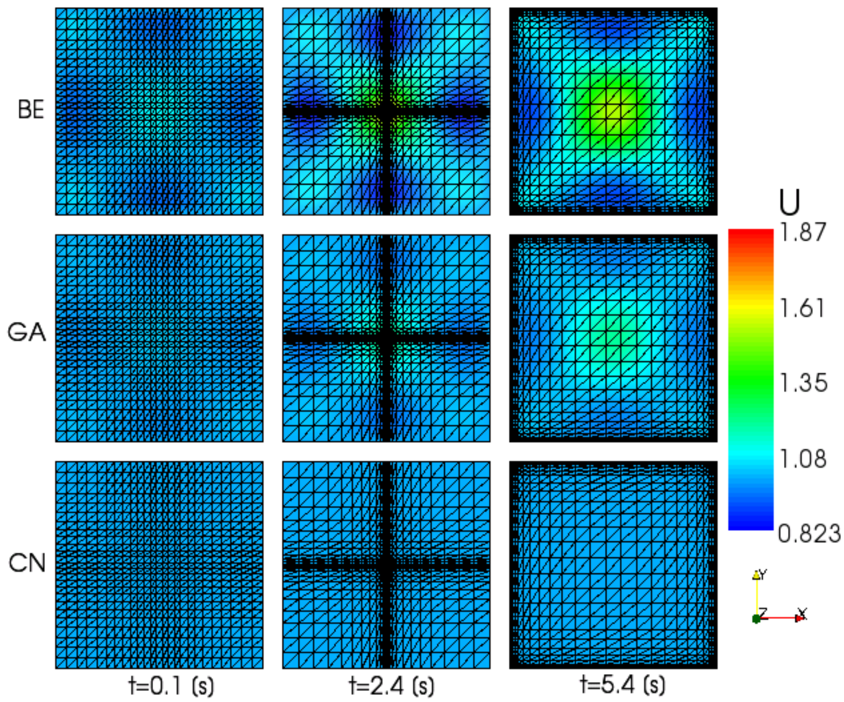

Figure 3: Solution for the *Backward Euler* (BE), *Galerkin* (GA) and *Crank-Nicolson* (CN) schemes.

Now, if the Averaged Jacobian Formulation is used, the scheme is DGCL compliant for all the time integration schemes and no error is introduced.

In Figure (4) it is shown that using the Averaged Jacobian Formulation all these three time integration schemes are DGCL compliant.

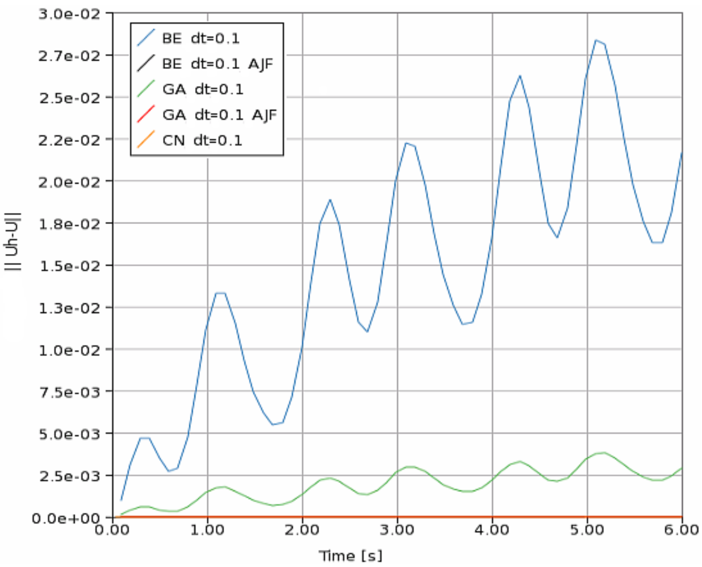

Figure 4: Errors for Averaged Jacobian Formulation (AJF) and No-Averaged Jacobian Formulation.

3.2 DGCL validation for 2D scalar diffusion problem with a periodic expansion and contraction of the domain

In this test case the problem (32) is solved in an unit square domain with $\mu = 0.1$, so that

$$\begin{aligned} u_t - 0.1\Delta u &= 0 \quad \text{for } \mathbf{x} \in \Omega^{\mathrm{t}},\ \mathrm{t} \in (0, \mathrm{T}], \\ u_0 &= 1 \quad \text{for } \mathbf{x} \in \Omega^{0},\ \mathrm{t} = 0, \\ u &= 1 \quad \text{for } \mathbf{x} \in \partial\Omega^{\mathrm{t}},\ \mathrm{t} \in [0, \mathrm{T}], \end{aligned} \tag{35}$$

being the domain deformed according to the following rule

$$\boldsymbol{\chi}(\boldsymbol{\xi}, t) = (2 - \cos(20\pi t))\boldsymbol{\xi}. \tag{36}$$

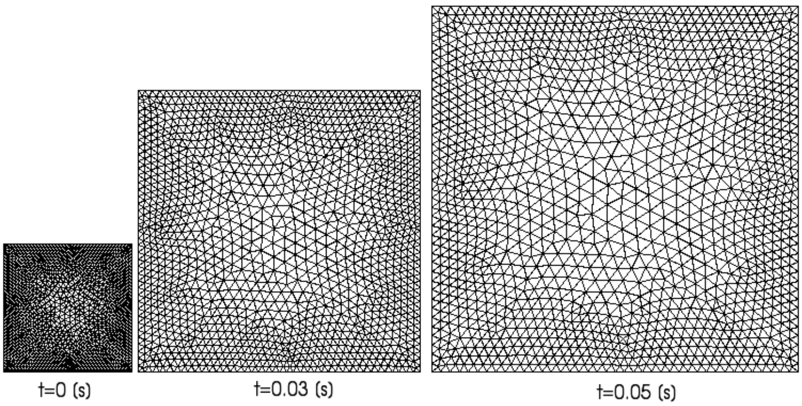

Figure 5: Deformed domain.

Figure (5) shows the deformed domain for $t = \{0, 0.03, 0.05\}$ $[s]$.
As in the previous case an error is introduced when using the *Backward Euler* or *Garlerkin* scheme due to lack in DGCL compliance, but when the Averaged Jacobian Formulation is used all the time integration schemes are DGCL compliant.
In Figure (6) the error $||u_h - u||_{L^2(\Omega^n)}$ in the solution is reported.

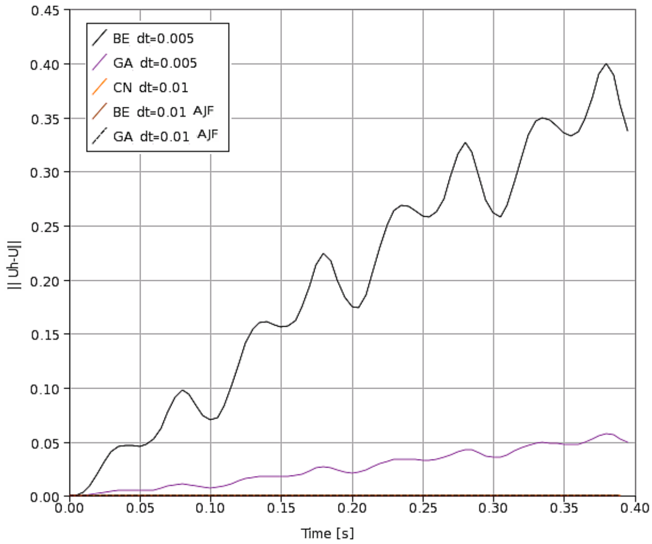

Figure 6: $||u_h - u||_{L^2(\Omega^n)}$ for *Backward Euler* (BE), *Galerkin* (GA) and *Crank-Nicolson* (CN) schemes.

3.3 DGCL validation for 3D scalar diffusion problem with a periodic expansion and contraction of the domain

In this section the Averaged Jacobian Formulation is validated for 3D problems. The initial test is the extension to 3D of the problem (35) and the mesh moving rule (36). It is solved using piecewise linear tetrahedra for the spatial discretization, a piecewise linear interpolation of the mesh movement and for the time integration the *Backward Euler* ($\theta = 1$), *Crank-Nicolson* ($\theta = 0.5$) and *Garlerkin* ($\theta = 2/3$) schemes.

Figure (7) shows the deformed domain for $t = \{0, 0.03, 0.05\}$ $[s]$ and Figure (8) reports the error $||u_h - u||_{L^2(\Omega^n)}$ for four periods of oscillation, which must remain null to machine precision over time for a DGCL compliant scheme.

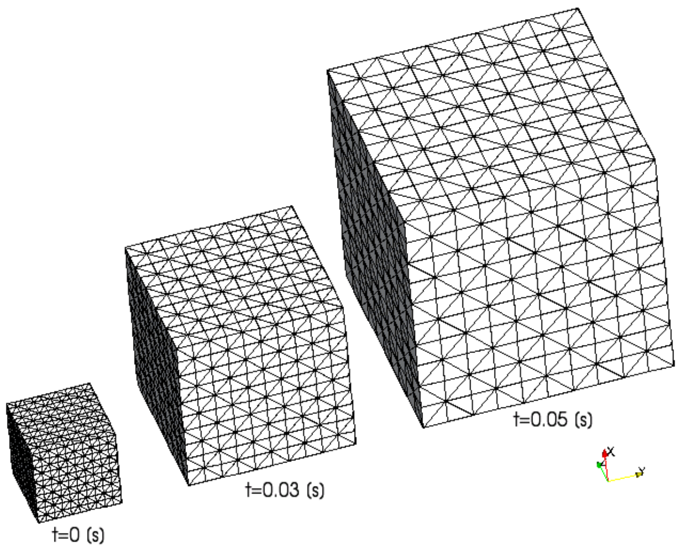

Figure 7: Deformed domain.

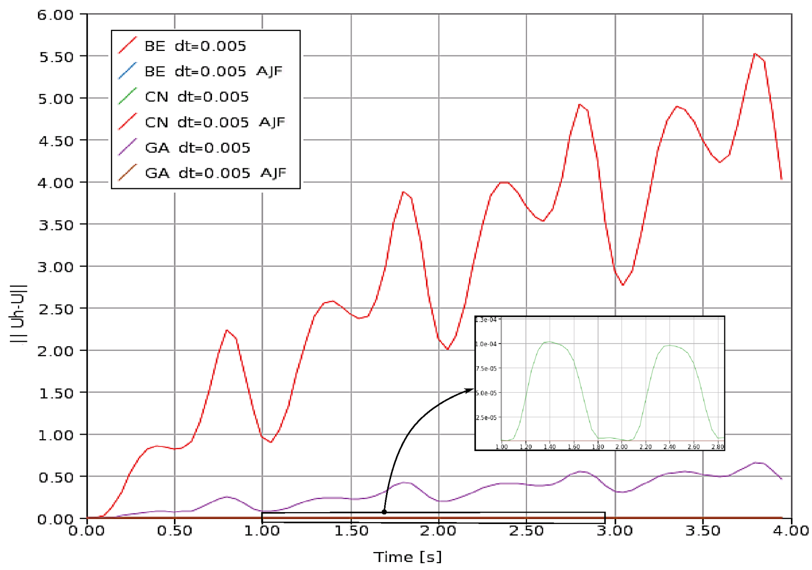

Figure 8: Errors for Averaged Jacobian Formulation (AJF) and No-Averaged Jacobian Formulation.

3.4 DGCL validation for 3D scalar diffusion problem with internal node movement

This test is the extension to 3D of the problem (33) and the deformation rule (34). It is solved using piecewise linear tetrahedra for the spatial discretization, a piecewise linear interpolation of the mesh movement and for the time integration the *Backward Euler* ($\theta = 1$), *Crank-Nicolson* ($\theta = 0.5$) and *Garlerkin* ($\theta = 2/3$) schemes.

Figure (9) shows the deformed mesh for $t = \{0, 0.5, 1.5\}$ $[s]$ and Figure (10) reports the error $||u_h - u||_{L^2(\Omega^n)}$.

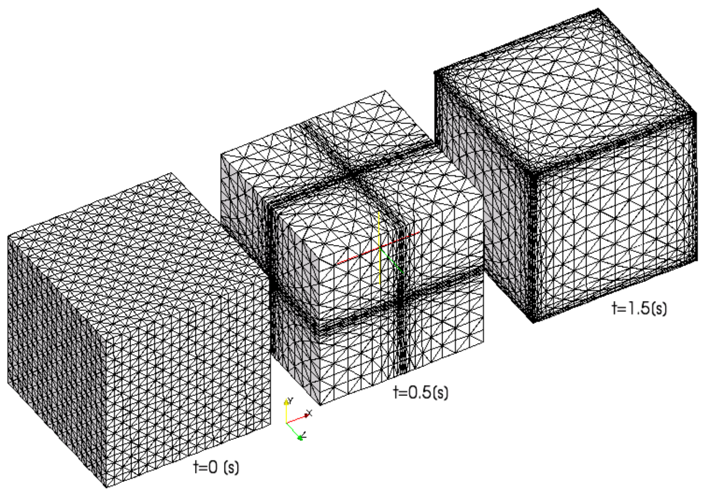

Figure 9: Deformed mesh.

An error is introduced when using any of the θ-family scheme in 3D problems due to lack in GCL compliance. In Figure (11) the solution for times $t = \{0.1, 2.4, 5.4\}$ $[s]$ is shown for the *Backward Euler* scheme. The error with respect to the constant solution are localized in the zones of the domain where the element deformation is higher, as in the center.

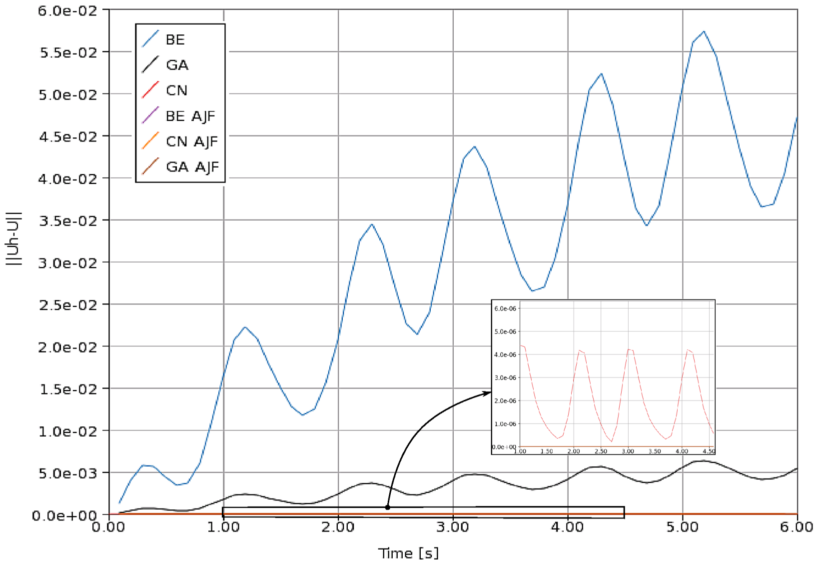

Figure 10: $||u_h - u||_{L^2(\Omega^n)}$ for *Backward Euler* (BE), *Galerkin* (GA) and *Crank-Nicolson* (CN) schemes.

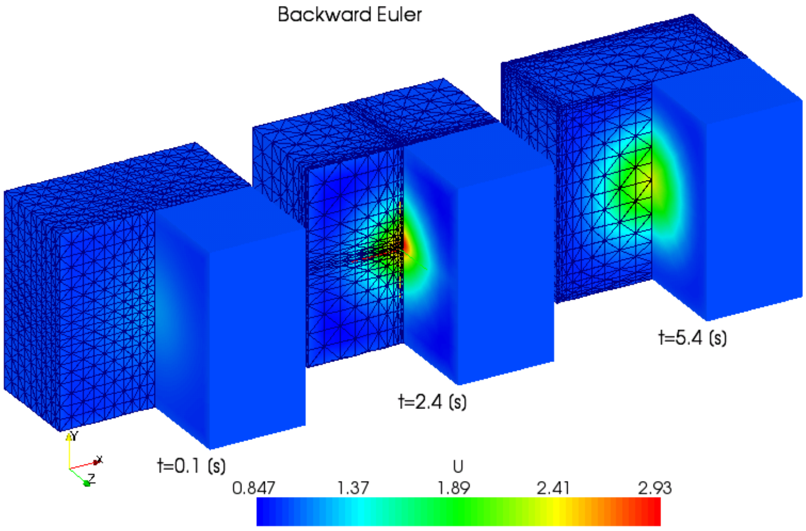

Figure 11: Solution for the *Backward Euler* (BE) scheme.

3.5 Transonic flow over a pitching airfoil.

In the following problem the 2D Euler equations are solved for a pitching NACA0012 airfoil around an axis passing through its quarter-chord point. The angle of attack is prescribed using a sinusoidal function of time.

$$\alpha_{(t)} = \alpha_m + \alpha_0 \cdot \sin(\omega t) \tag{37}$$

The freestream Mach number for this case is 0.755, the mean incidence $\alpha_m = 0.016°$ and the amplitude $\alpha_0 = 2.51°$. The reduced frequency for the airfoil motion is $\frac{f \cdot c}{u} = 0.1628$, where f is the frequency of oscillation, c is the airfoil chord, and u is the freestream flow velocity. The gas properties for this test case are, $\gamma = 1.4$ and $R = 287$ [J/kg K].

The problem is solved using a two-dimensional unstructured mesh with 9165 nodes and 17802 triangles (Fig.(12)).

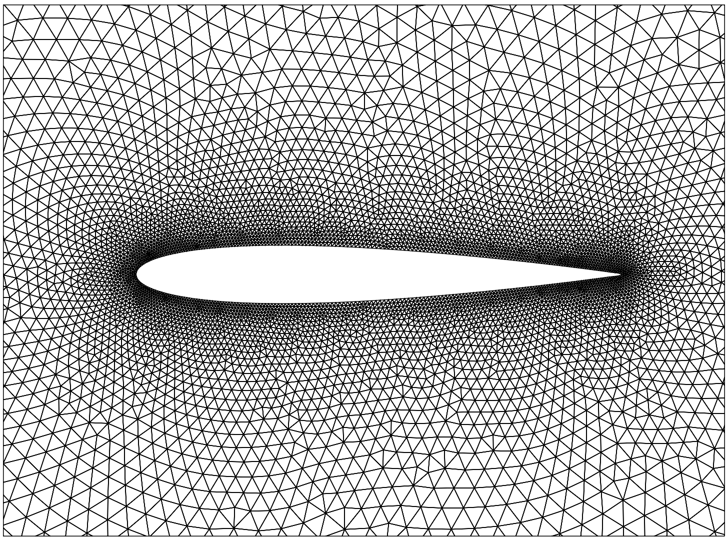

Figure 12: Two-dimensional unstructured mesh.

The nodes on the airfoil skin are moved as a rigid body while the nodes on the boundary of the mesh remanin fixed.

The movement of the interior nodes is performed using a node relocation, keeping the topology unchanged. This is done by using an elastic-like movement strategy (Johnson and Tezduyar, 1994), which is an extra problem solved at each time step.

The problem is solved using piecewise linear triangles for the spatial discretization, piecewise linear interpolation of the mesh movement and for the time integration the *Backward Euler* and *Crank-Nicolson* schemes are considered with Δt ={0.1, 0.05}.

In Figures (13) and (14) the lift coefficient is plotted as a function of the angle of attack for this periodic motion. Note that for two-dimensional problems the *Crank-Nicolson* scheme is DGCL compliant while the *Backward Euler* is not. But, if the Averaged Jacobian Formulation is used both schemes are DGCL compliant.

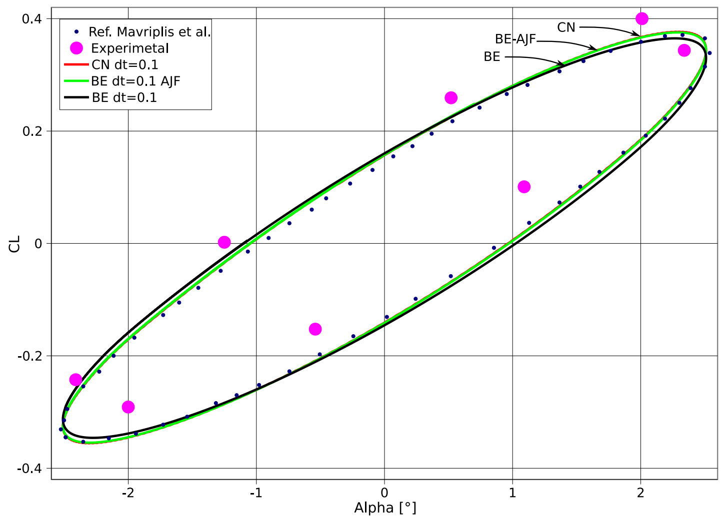

Figure 13: Lift coefficient as a function of the angle of attack for $\Delta t = 0.1[s]$.

The obtained solution is compared with a reference solution from Mavriplis and Yang (2005)

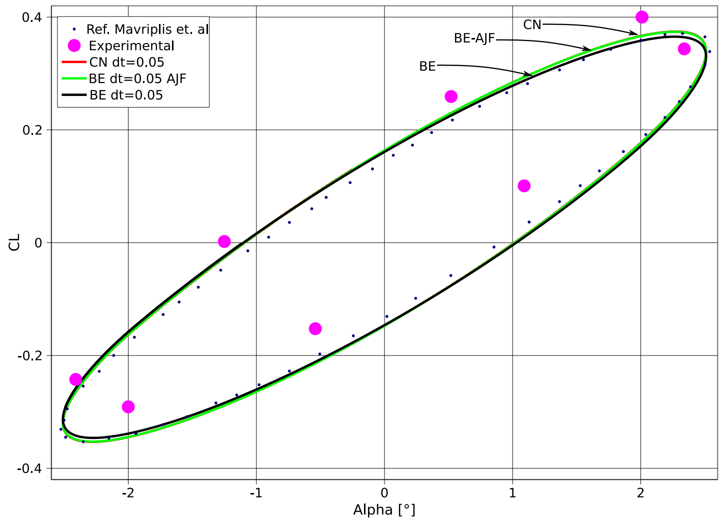

Figure 14: Lift coefficient as a function of the angle of attack for $\Delta t = 0.05[s]$.

and with an experimental solution from NATO (1982). When the Averaged Jacobian Formulation is used the solution obtained with *Backward Euler* is similar to that obtained with *Crank-Nicolson*. While if the Averaged Jacobian Formulation is not used the solution obtained with *Backward Euler* shows a clear difference with the *Crank-Nicolson*. This means that for this conditions the main source of error for BE is the lack of DGCL compliance, rather than the low (first order) precision of the algorithm.

In Figure (15) the comparison of isobars between a DGCL compliant *Backward Euler* scheme and a non DGCL *Backward Euler* compliant scheme for $\alpha = 2.51°$ are shown. There is a difference in the pressure around the airfoil, which is translated in a different lift coefficient.

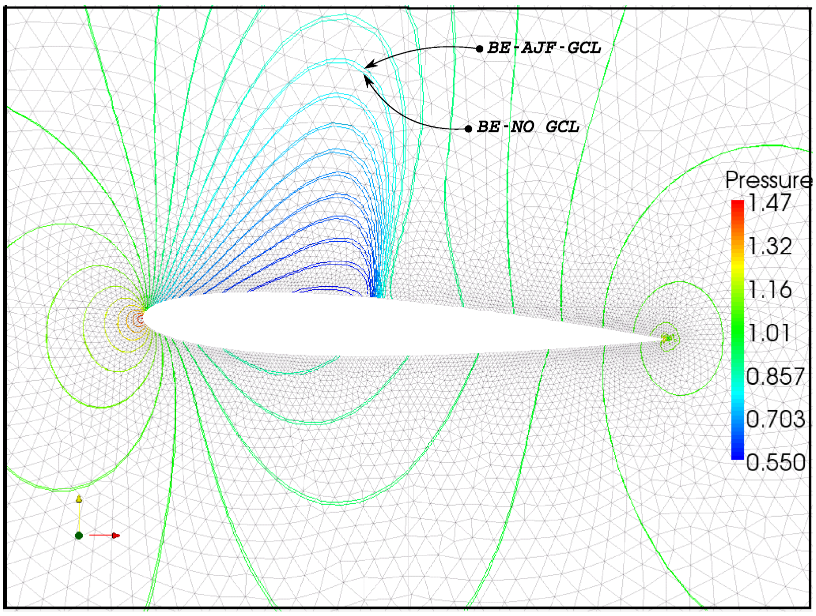

Figure 15: Comparison of isobars for $\alpha = 2.51°$.

4 CONCLUSIONS

The proposed methodology guarantees compliance with the DGCL criterion in the context of the ALE solutions of general advective-diffusive systems using classical temporal integration schemes and simplicial finite elements in 2D and 3D. Detailed expressions for the computation of the averaged Jacobians and its application to the θ-family and the three point BDF schemes were given. Also, in order to validate the AJF a set of typical numerical tests for linear scalar advective-diffusive and Euler models were performed Unlike to previous work, this new methodology is not based on proposing a new temporal integration scheme and computing a set of unknown numerical coefficients in order to achieve compliancy with the DGCL, but rather by averaging some geometrical quantities. These averages are computed exactly using the Gauss-Lobatto numerical quadrature. The averaging process of the Jacobian must be introduced in the volume terms as well as in the boundary terms. The added cost is negligible and only involves a few changes at the elemental routine level.

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