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MODELING IN OPTION PRICING WITH MEMORY IN ASSETS AND STOCHASTIC VOLATILITY OF HOBSON AND ROGERS

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Abstract. This paper presents a new class of models for continuous time price process of financial assets. The dynamics of asset returns traded on spot markets are based on Jump-Telegraph-Diffusion-Drift processes (JTDD-process) and also on the Stochastic Volatility of Hobson and Rogers. Thus, the main contribution of this paper is the inclusion of memory not only into the price but also in the volatility of underlying assets of European and American options. In this framework, the models are formulated as a differential system. The variational formulations, their respective finite element approximations, as well as the numerical results via simulations are presented.

1 INTRODUCTION

The model proposed by Black and Scholes (1973) is the most important landmark research in financial economics. This is based upon the common assumption that the proportional price changes of the asset form a Gaussian process with stationary independent increments. This assumption has been the subject of much attention over the years. A survey of several papers in the literature has focused primarily on the most important parameter the volatility. Empirical analysis of stock volatility has shown that it is not constant. Not to mention that the prices of the derivatives that are traded are inconsistent with a constant volatility assumption.

For this reason a number of authors have suggested variants of the Black-Scholes model, As Hobson and Rogers (1998) proposes a original class of models for the price process of continuous time of a financial security with non-constant volatility, and volatility defined in terms of exponentially weighted moments of historic log-price. Thus, the instantaneous volatility is therefore driven by the same stochastic factors as the of price process unlike many other models of stochastic volatility, it is not necessary to introduce new sources of randomness.

Over the years, several papers argue that have the Black and Scholes model cannot describe option prices dynamics in real markets. This finding is presented in Francesco and Pascucci (2004), this paper presents a complete model with stochastic volatility by Hobson and Rogers, in which the price of the options are the solutions to degenerate partial differential equations obtained by the inclusion of other variables describing the state of dependence the past prices of the underlying asset.

On the other hand, Foschi and Pascucci (2009) presents empirical tests with an option pricing model assuming the volatility by Hobson and Rogers in the complete market, where reproduce the "smile" and observed the expression patterns of implied volatility structure. A calibration procedure based on ad-hoc numerical schemes for "hypoelliptic PDEs" is proposed and used for quantitative investigation into the performance of the price model, based on numerical results of "$S\&P500$ option prices". Hobson and Rogers consider in Hobson and Rogers (1998) a volatility function of the form

$$\sigma(D) = \min\left\{\eta\sqrt{1+\varepsilon D^2}, N\right\} ,$$

for some large constant N and positive parameters ε, η then they show that the model can indeed exhibit smiles and skews of different directions.

Another fact is that recent research in option pricing has taken into account some features of models with memory, they are more realistic for the phenomena to dynamic asset in the financial market. Recent paper of Ratanov (2007, 2008), presents a new class of models of financial markets, these being based on *Generealized Telegraph Processes*. The model in question presented is free of arbitrage and complete, if directions of jumps in asset prices are in a certain correspondence with the speed and with the behavior of interest rates, i.e. the model can be complete without adding another asset that is based on the same sources of randomness. The two articles of N. Ratanov present detailed descriptions of the *Telegraph Processes*, and the last Ratanov (2008) has developed an approach to the problem of European option pricing with the derivation of explicit formulas. These recent articles following the ideas originally in 2002 presented by Crescenzo and Pellerey (2002), where *Geometric Telegraph Process* is initially proposed as a model to describe the dynamics of the prices of risky assets.

The idea here is that working with the two modeling concepts by Hobson and Rogers (1998) and Crescenzo and Pellerey (2002) resulting in a new class of model that takes into consideration the past of price and volatility of the underlying asset.

1.1 Modeling Financial Assets With Memory

The telegraph process describes a random motion with finite velocity and it is usually proposed as an alternative to classical diffusion models (see Goldstein (1951) and Kac (1974)). Crescenzo and Pellerey (2002) proposed the geometric telegraph process as a model to describe the dynamics of the price of risky assets where $X(t)$ replaces the standard Brownian motion of the original Black-Scholes-Merton model. Conversely to the geometric Brownian motion, given that $X(t)$ is of bounded variation, so is the geometric telegraph process. This seems a realistic way to model paths of assets in the financial markets. Ratanov (2007, 2008) proposed to model financial markets using a telegraph process with two intensities $\lambda_\pm$ and two velocities $\nu_\pm$.

The dynamics of asset returns cannot be adequately described by geometric Brownian motion with constant volatility by Black and Scholes (1973). Here we admit that the model for dynamics of underlying asset returns given by the stochastic differential equation (SDE),

$$dS_t = \mu(D_t^\pm)S_t dt + \sigma(D_t^\pm)S_t dW_t + \sigma S_t dX_\pm(t) + S(t-)dJ_\pm(t), \tag{1}$$

where σ^2 is the volatility of the price of asset (S) and μ is the expected return of asset (S). We assume that the assets pricing process is continuous to the right.

The dynamics of underlying asset in Eq. (1) incorporates a pure Jump process $J_\pm = \{J_\pm(t)\}_{t\geq 0}$ with alternating jumps of sizes $h_\pm \in (-1, \infty)$, a Telegraph process $X_\pm = \{X_\pm(t)\}_{t\geq 0}$ with velocity $\nu_\pm$ and a pure Diffusion process (Wiener's process) for $W_t = \{W_t\}_{t\geq 0}$. Let $r_\pm \geq 0$ be the riskless interest rate which is in the initial state $\pm$. The riskless asset is given by the exponential of the process

$$Y_\pm = \{Y_\pm\}_{t\geq 0} = \left\{\int_0^t r_\pm d\tau\right\}_{t\geq 0}.$$

In view of such trajectories, the market is set up as a continuous process that evolves with velocities ν_+ or ν_-, changes the direction of movement from $\nu_\pm$ to $\nu_\mp$, and exhibits jumps of sizes $h_\pm$ whenever velocity changes. The different parameters for up and down movements lead to a gain/loss asymmetry.

Let $\nu_\pm$, $r_\pm$, $h_\pm$ be real numbers such that $\nu_+ > \nu_-$, $r_\pm \geq 0$ and $h_\pm > -1$. Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a complete probability space, and let $\lambda_\pm$ be positive numbers. We consider two counting Poisson processes $N_+ = \{N_+(t)\}_{t\geq 0}$ and $N_- = \{N_-(t)\}_{t\geq 0}$ with alternating intensities λ_+, λ_-, λ_+, $\cdots$ and λ_-, λ_+, λ_-, $\cdots$, respectively, that is, as $\Delta t \to 0$,

$$\begin{aligned}
\mathbb{P}(N_+(t+\Delta t) = 2n+1 | N_+(t) = 2n) &= \lambda_+ \Delta t + O(\Delta t),\\
\mathbb{P}(N_+(t+\Delta t) = 2n+2 | N_+(t) = 2n+1) &= \lambda_- \Delta t + O(\Delta t),\\
\mathbb{P}(N_-(t+\Delta t) = 2n+1 | N_-(t) = 2n) &= \lambda_- \Delta t + O(\Delta t),\\
\mathbb{P}(N_-(t+\Delta t) = 2n+2 | N_-(t) = 2n+1) &= \lambda_+ \Delta t + O(\Delta t).
\end{aligned}$$

where $n = 0, 1, \cdots$.

Denoting $g_+(t) = (-1)^{N_+(t)}$ and $g_-(t) = -(-1)^{N_+(t)}$, considering all stochastic processes subscribed by $+$ or $-$ are adapted to the filtrations generated by N_+ and N_-, respectively. Consider the (right continuous) processes.

Defining the Telegraph process with states (ν_+, λ_+) and (ν_-, λ_-) as

$$X_+(t) \triangleq \int_0^t \nu_{g_+(\tau)} d\tau \quad \text{and} \quad X_-(t) \triangleq \int_0^t \nu_{g_-(\tau)} d\tau, \tag{2}$$

the pure Jump process with jumps at the Poisson times τ_j, $j = 1, 2, \cdots$ as

$$J_+(t) \triangleq \int_0^t h_{g+(\tau)} dN_+(\tau) = \sum_{j=1}^{N_+(t)} h_{g+(\tau_j-)} \quad \text{and} \quad J_-(t) \triangleq \int_0^t h_{g-(\tau)} dN_-(\tau) = \sum_{j=1}^{N_-(t)} h_{g-(\tau_j-)}, \tag{3}$$

the Diffusion process (Wiener's process)

$$W_t = W(t) = \int_0^t \xi_v dv, \tag{4}$$

where ξ_t is a white noise, where the integral in Eq. (4) is indefinite and symbolically presented as

$$dW_t = \xi_t dt.$$

Integrating Eq. (1),

$$S(t) = S_0 \; EXP\left\{\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right\} \varepsilon_t \Big\{X_\pm(t) + J_\pm(t)\Big\}, \tag{5}$$

where $S_0 = S(0)$ and $\varepsilon_t\{\cdot\}$ denote the stochastic exponential. Therefore,

$$\varepsilon_t \Big\{X_\pm(t) + J_\pm(t)\Big\} = e^{X_\pm(t)} K_\pm(t),$$

$$K_\pm(t) = \prod_{\tau \le t} (1 + \Delta J_\pm(t)) = \prod_{j=1}^{N_\pm(t)} (1 + h_{g\pm(\tau_j-)}).$$

Here the τ_j , $j \ge 1$, are the jumping times of the Poisson processes $N_\pm$.

The Jump-Telegraph process (JT-process) is defined as $Z \triangleq X_\pm + J_\pm$

$$\varepsilon_t \Big\{Z\Big\} = e^{Z(t) - \frac{1}{2}\langle Z\rangle^{cont}(t)} \prod_{0<\tau\le t} (1 + \Delta Z(\tau)) e^{-\Delta Z(\tau)}.$$

The Telegraph process without jumps cannot be a martingale. Here

$$\langle Z\rangle^{cont} = \langle X_\pm + J_\pm\rangle^{cont} = 0$$

and

$$\begin{aligned} \varepsilon_t \Big\{X_\pm + J_\pm\Big\} &= e^{X_\pm \, + \, J_\pm} \prod_{0<\tau\le t} (1 + \Delta J_\pm(\tau)) e^{-\Delta J_\pm(\tau)} = \\ &= e^{X_\pm \, + \, J_\pm} e^{-J_\pm(t)} \prod_{0<\tau\le t} (1 + \Delta J_\pm(\tau)) = e^{X_\pm} \prod_{0<\tau\le t} (1 + \Delta J_\pm(\tau)) \; . \end{aligned}$$

Therefore Eq. (5) is expressed by

$$S(t) = S_0 \; EXP\left\{\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t + X_\pm(t)\right\} K_\pm(t), \tag{6}$$

where

$$K_\pm(t) = \prod_{j=1}^{N_\pm(t)} (1 + h_{g\pm(\tau_j-)}).$$

On the other hand, The riskless asset price has the following form

$$B(t) = e^{Y_\pm}, \qquad Y_\pm = \int_0^t r_\pm d\tau,$$

where the interest rates $r_\pm > 0$ and $\pm$ indicate the initial market state. Here again $Y_\pm = \{Y_\pm(t)\}_{t\geq 0}$ is a Telegraph process with velocity values $r_\pm$.

In Ratanov (2008) presents the following theorem: Jump-Telegraph-Diffusion process (JTD-process) is a martingale if and only if

$$\lambda_\pm h_\pm = -\nu_\pm. \tag{7}$$

2 EUROPEAN OPTION PRICING MODEL

In Hobson and Rogers (1998) introduce a new class of stock-price models. Specify local volatility in terms of weighted moments of past returns, i.e. specification of instantaneous volatility in terms of exponentially weighted moments of the historic log-price. This introduces a feedback effect into the volatility process: presents shocks in the asset price result in hight future uncertainty.

We define the discounted log-price process Z_t at time t as

$$Z_t^\pm = log(e^{-r_\pm t} S_t) \ ,$$

where $r_\pm$ is the (constant) risk-free interest rate, and the offset function of order d, denoted by $D_t^{(d),\pm}$, by

$$D_t^{(d),\pm} = \theta \int_0^{+\infty} e^{-\theta v} \left[Z_t^\pm - Z_{t-v}^\pm \right]^d dv, \quad \theta > 0, \tag{8}$$

where the parameter θ describes the rate at which past information is discounted, describes the weight of historic observations. Stock prices are driven by the stochastic differential equation

$$dZ_t^\pm = \mu(t, Z_t, D_t^{(1),\pm}, \cdots, D_t^{(d),\pm})dt + \sigma(t, Z_t, D_t^{(1),\pm}, \cdots, D_t^{(d),\pm})dW_t \tag{9}$$

for some smooth functions $\sigma(\cdot) > 0$ and $\mu(\cdot)$ are *Lipschitz* functions.

In Platania and Rogers (2003) say that $\sigma(\cdot)$ can eventually depend on S_t, the model includes as a subclass the case when the volatility rate is a deterministic function of the underlying. Furthermore the hypotheses preserve completeness, allowing for preference independent option pricing. This last feature constitutes an advantage over fully stochastic volatility processes, where arbitrage considerations are not sufficient to identify "*risk premia*" uniquely.

In the following, we will assume the instantaneous volatility is a function of the first order offset $D_t^\pm = D_T^{(1),\pm}$ only, since we want to obtain a tractable PDE and to solve it with reliable precision. Hobson and Rogers showed that even in this case the model has the potential to explain volatility smiles and skews, and our simulation studies seem to suggest that including higher order offset functions does not improve the results significantly.

We readily decompose $D_t^\pm$ as the deviation of the current price from an exponentially weighted average of past records

$$D_t^\pm = Z_t^\pm - \int_0^{+\infty} \theta e^{-\theta v} \, Z_{t-v}^\pm dv, \quad \theta > 0 \ . \tag{10}$$

The latter says that θ determines the horizon of the "*moving time window*" of the integral on the right. For bigger values of this parameter, S_t is more dependent on the recent past, while small values almost identify the offset increments with price changes. Obviously in this case a level dependent volatility assumption would be numerically more convenient.

We denote by $V = V(S, D, t)$ the price at time "t" of an European option with maturity "T", expressed as a function of the state variables: time "t", asset price "S" and deviation "D". We have $(S, D, t) \in \mathbb{R}_+ \times \mathbb{R} \times [0, T]$.

We assume that the asset price is governed by the stochastic differential equation (Jump-Telegraph-Diffusion-Drift process), positive past information,

$$dS_t = \mu(D_t^+)S_t dt + \sigma(D_t^+)S_t dW_1(t) + \sigma(D_t^+)S_t dX_+(t) + S(t-)dJ_+(t) \tag{11}$$

and negative past information,

$$dS_t = \mu(D_t^-)S_t dt + \sigma(D_t^-)S_t dW_1(t) + \sigma(D_t^-)S_t dX_-(t) + S(t-)dJ_-(t). \tag{12}$$

and the deviation (cf. formula (11) in Hobson and Rogers (1998)) is governed by

$$dD_t^\pm = -\left[\frac{1}{2}\sigma^2(D_t^\pm) + \theta D_t^\pm\right] dt + \sigma(D_t^\pm)\, dW_2(t) \ . \tag{13}$$

We define option pricing with past information from the initial value of the price of high (positive past information) as $V^+(S, D, t)$ and option pricing with past information from the initial value of the price of down (negative past information) as $V^-(S, D, t)$. Observe that in Eq. (11) or Eq. (12) and Eq. (13) the Wiener process have been given subscripts. This is because we are allowing "S" and "D" to be governed by two different random variables, this is a two-factor model. Thus, although "dW_1" and "dW_2" are both draw from Normal distributions with zero mean and variance "dt", they are not necessarily the same random variable. They are, however, correlated by $\varepsilon[dW_1 dW_2] = \rho dt$ with $-1 \leq \rho(S, D, t) \leq 1$. The "$\rho$" is equal 1 by definition of "D". Then, $\varepsilon[dW_1 dW_2] = 1dt$. We can still think of Eq. (11) or Eq. (12) and Eq. (13) as formulas for generating random walks for "S" and "D", but now at each time-step we must draw two random numbers.

In order to manipulate $V^+(S, D, t)$ or $V^-(S, D, t)$ we need to know Itô's Lemma applies to functions of two random variables. As might be expected, the usual Taylor series expansion together with a few rules of thumb results in the correct expression for small change in any function of both "S" and "D". These rules of thumb are

$$dW_1^2 = dt \ , \quad dW_2^2 = dt \quad \text{and} \quad dW_1 dW_2 = \rho dt = dt.$$

Applying Taylor's formula to the positive past information, $V^+(S + dS, D + dD, t + dt)$ we find that

$$\begin{aligned} dV^+ &= \frac{\partial V^+}{\partial t}dt + \frac{\partial V^+}{\partial S}dS + \frac{\partial V^+}{\partial D}dD + \frac{1}{2}\frac{\partial^2 V^+}{\partial S^2}dS^2 + \frac{\partial^2 V^+}{\partial S \partial D}dS dD + \\ &+ \frac{1}{2}\frac{\partial^2 V^+}{\partial D^2}dD^2 + \cdots + \left[V^-(S + h_+ S, D, t) - V^+(S, D, t)\right] dJ_+, \end{aligned}$$

where

$$\begin{aligned} dS^2 &= \sigma^2(D_t^\pm)\ S_t^2 dW_1^2(t) = \sigma^2(D_t^\pm)\ S_t^2 dt \ , \\ dD^2 &= \sigma^2(D_t^\pm)\ dW_2^2(t) = \sigma^2(D_t^\pm)\ dt \ , \\ dS dD &= \sigma(D_t^\pm)\ S_t dW_1\ \sigma(D_t^\pm)\ dW_2 = \sigma^2(D_t^\pm)\ S_t dt \ . \end{aligned}$$

Thus, Itô's Lemma for the two random variables governed by Eq. (11) and Eq. (13) becomes

$$\begin{aligned} dV^+ &= \frac{\partial V^+}{\partial t}dt + \frac{\partial V^+}{\partial S}dS + \frac{\partial V^+}{\partial D}dD + \frac{1}{2}\sigma^2(D_t^+)\ S_t^2\frac{\partial^2 V^+}{\partial S^2}dt + \sigma^2(D_t^+)\ S_t\frac{\partial^2 V^+}{\partial S \partial D}dt + \\ &+ \frac{1}{2}\sigma^2(D_t^+)\ \frac{\partial^2 V^+}{\partial D^2}dt + \Big[V^-(S+h_+S,D,t) - V^+(S,D,t)\Big]dJ_+. \end{aligned} \tag{14}$$

Let us construct a portfolio Π^+ consisting of one option with maturity T_1, $-\Delta_2^+$ option with maturity date T_2 and $-\Delta_1^+$ of the underlying asset, i.e. $\Pi^+ = V_1^+ - \Delta_2^+ V_2^+ - \Delta_1^+ S$. Considering that the increasing in the value of the portfolio in a space-time, keeping the Δ_1^+ and Δ_2^+ rates, is given

$$d\Pi^+ = dV_1^+ - \Delta_2^+ dV_2^+ - \Delta_1^+ dS. \tag{15}$$

Using Eq. (14) in Eq. (15) it can be shown that

$$\begin{aligned} d\Pi^+ &= \left\{\frac{\partial V_1^+}{\partial t} - \Delta_2^+\frac{\partial V_2^+}{\partial t} + \frac{1}{2}\sigma^2(D_t^+)S_t^2\left[\frac{\partial^2 V_1^+}{\partial S^2} - \Delta_2^+\frac{\partial^2 V_2^+}{\partial S^2}\right] + \sigma^2(D_t^+)S_t\left[\frac{\partial^2 V_1^+}{\partial S\partial D} - \right.\right. \\ &- \left.\left. \Delta_2^+\frac{\partial^2 V_2^+}{\partial S\partial D}\right] + \frac{1}{2}\sigma^2(D_t^+)\left[\frac{\partial^2 V_1^+}{\partial D^2} - \Delta_2^+\frac{\partial^2 V_2^+}{\partial D^2}\right]\right\}dt + \left\{\frac{\partial V_1^+}{\partial S} - \Delta_1^+ - \right. \\ &- \left. \Delta_2^+\frac{\partial V_2^+}{\partial S}\right\}dS + \left\{\frac{\partial V_1^+}{\partial D} - \Delta_2^+\frac{\partial V_2^+}{\partial D}\right\}dD + \left\{\Big[V_1^-(S+h_+S,D,t) - \right. \\ &- \left. V_1^+(S,D,t)\Big] - \Delta_2^+\Big[V_2^-(S+h_+S,D,t) - V_2^+(S,D,t)\Big]\right\}dJ_+. \end{aligned}$$

Fixing Δ_1^+ and Δ_2^+ according to

$$\Delta_2^+ = \frac{\partial V_1^+/\partial D}{\partial V_2^+/\partial D} \quad \text{and} \quad \Delta_1^+ = \frac{\partial V_1^+}{\partial S} - \Delta_2^+\frac{\partial V_2^+}{\partial S} = \frac{\partial V_1^+}{\partial S} - \frac{\partial V_1^+/\partial D}{\partial V_2^+/\partial D}\frac{\partial V_2^+}{\partial S},$$

the risk from the portfolio is removed. Then,

$$\begin{aligned} \frac{d\Pi^+}{dt} &= \left\{\frac{\partial V_1^+}{\partial t} - \frac{\partial V_1^+/\partial D}{\partial V_2^+/\partial D}\frac{\partial V_2^+}{\partial t} + \frac{1}{2}\sigma^2(D_t^+)S_t^2\left[\frac{\partial^2 V_1^+}{\partial S^2} - \frac{\partial V_1^+/\partial D}{\partial V_2^+/\partial D}\frac{\partial^2 V_2^+}{\partial S^2}\right] + \right. \\ &+ \sigma^2(D_t^+)\ S_t\left[\frac{\partial^2 V_1^+}{\partial S\partial D} - \frac{\partial V_1^+/\partial D}{\partial V_2^+/\partial D}\frac{\partial^2 V_2^+}{\partial S\partial D}\right] + \frac{1}{2}\sigma^2(D_t^+)\left[\frac{\partial^2 V_1^+}{\partial D^2} - \right. \\ &- \left.\left. \frac{\partial V_1^+/\partial D}{\partial V_2^+/\partial D}\frac{\partial^2 V_2^+}{\partial D^2}\right]\right\} + \left\{\Big[V_1^-(S+h_+S,D,t) - V_1^+(S,D,t)\Big] - \right. \\ &- \left. \frac{\partial V_1^+/\partial D}{\partial V_2^+/\partial D}\Big[V_2^-(S+h_+S,D,t) - V_2^+(S,D,t)\Big]\right\}\frac{dJ_+}{dt}. \end{aligned} \tag{16}$$

We remark that holds the following

$$\frac{dJ_+}{dt} = \frac{d}{dt}\left\{\sum_{j=1}^{N_+(\tau)} h_{g_+(\tau_j -)}\right\} = \lambda_+. \tag{17}$$

Considering the arbitrage arguments we can state the return of the portfolio

$$\frac{d\Pi^+}{dt} = r_+\Pi^+ = r_+V_1^+ - r_+\frac{\partial V_1^+/\partial D}{\partial V_2^+/\partial D}V_2^+ - r_+S\frac{\partial V_1^+}{\partial S} + r_+S\frac{\partial V_1^+/\partial D}{\partial V_2^+/\partial D}\frac{\partial V_2^+}{\partial S}. \tag{18}$$

Replacing Eq. (18) and Eq. (17) in Eq. (16), we have

$$\begin{aligned} r_+V_1^+ &- r_+\frac{\partial V_1^+/\partial D}{\partial V_2^+/\partial D}V_2^+ - r_+S\frac{\partial V_1^+}{\partial S} + r_+S\frac{\partial V_1^+/\partial D}{\partial V_2^+/\partial D}\frac{\partial V_2^+}{\partial S} = \frac{\partial V_1^+}{\partial t} - \frac{\partial V_1^+/\partial D}{\partial V_2^+/\partial D}\frac{\partial V_2^+}{\partial t} + \\ &+ \frac{1}{2}\sigma^2(D_t^+)\, S_t^2\left[\frac{\partial^2 V_1^+}{\partial S^2} - \frac{\partial V_1^+/\partial D}{\partial V_2^+/\partial D}\frac{\partial^2 V_2^+}{\partial S^2}\right] + \sigma^2(D_t^+)\, S_t\left[\frac{\partial^2 V_1^+}{\partial S\partial D} - \frac{\partial V_1^+/\partial D}{\partial V_2^+/\partial D}\frac{\partial^2 V_2^+}{\partial S\partial D}\right] + \\ &+ \frac{1}{2}\sigma^2(D_t^+)\left[\frac{\partial^2 V_1^+}{\partial D^2} - \frac{\partial V_1^+/\partial D}{\partial V_2^+/\partial D}\frac{\partial^2 V_2^+}{\partial D^2}\right] + \left\{\left[V_1^-(S+h_+S,D,t) - V_1^+(S,D,t)\right] - \right. \\ &- \left.\frac{\partial V_1^+/\partial D}{\partial V_2^+/\partial D}\left[V_2^-(S+h_+S,D,t) - V_2^+(S,D,t)\right]\right\}\lambda_+. \end{aligned}$$

Gathering together all "V_1" terms on the left-hand side and all "V_2" terms on the right-hand side, we find that

$$\begin{aligned} \left\{\frac{\partial V_1^+}{\partial t}\right. &+ \frac{1}{2}\sigma^2(D_t^+)\, S_t^2\frac{\partial^2 V_1^+}{\partial S^2} + \sigma^2(D_t^+)\, S_t\frac{\partial^2 V_1^+}{\partial S\partial D} + \frac{1}{2}\sigma^2(D_t^+)\, \frac{\partial^2 V_1^+}{\partial D^2} - r_+V_1^+ + \\ &+ \left. r_+S\frac{\partial V_1^+}{\partial S} + \left[V_1^-(S+h_+S,D,t) - V_1^+(S,D,t)\right]\lambda_+\right\}/\frac{\partial V_1^+}{\partial D} = \\ &= \left\{\frac{\partial V_2^+}{\partial t} + \frac{1}{2}\sigma^2(D_t^+)\, S_t^2\frac{\partial^2 V_2^+}{\partial S^2} + \sigma^2(D_t^+)\, S_t\frac{\partial^2 V_2^+}{\partial S\partial D} + \frac{1}{2}\sigma^2(D_t^+)\, \frac{\partial^2 V_2^+}{\partial D^2} - \right. \\ &- \left. r_+V_2^+ + r_+S\frac{\partial V_2^+}{\partial S} + \left[V_2^-(S+h_+S,D,t) - V_2^+(S,D,t)\right]\lambda_+\right\}/\frac{\partial V_2^+}{\partial D}. \end{aligned}$$

The last equation presents two unknowns. However, the left-hand side is a function of "T_1" and the right-hand side is a function of "T_2". The only way for this to be possible is for both sides to be independent of the maturity date. Thus, dropping the subscript from "V",

$$\begin{aligned} \left\{\frac{\partial V^+}{\partial t}\right. &+ \frac{1}{2}\sigma^2(D_t^+)\, S_t^2\frac{\partial^2 V^+}{\partial S^2} + \sigma^2(D_t^+)\, S_t\frac{\partial^2 V^+}{\partial S\partial D} + \frac{1}{2}\sigma^2(D_t^+)\, \frac{\partial^2 V^+}{\partial D^2} - r_+V^+ + \\ &+ \left. r_+S\frac{\partial V^+}{\partial S} + \left[V^-(S+h_+S,D,t) - V^+(S,D,t)\right]\lambda_+\right\}/\frac{\partial V^+}{\partial D} = A^+(S,D,t), \end{aligned}$$

for some function $A^+(S,D,t)$. In view of later development it is convenient to write

$$A^+(S,D,t) = \sigma(D_t^+)\, \gamma^+(S,D,t) + \left[\frac{1}{2}\sigma^2(D_t^+) + \theta D_t^+\right]. \tag{19}$$

In General $\gamma^+(S,D,t)$ is the market price of risk. Here in this work we assume that $\gamma^+(S,D,t)$ is equal to zero. Thus Eq. (19) reduces to

$$A^+(S,D,t) = \frac{1}{2}\sigma^2(D_t^+) + \theta D_t^+. \tag{20}$$

We have

$$\begin{aligned} \frac{\partial V^+}{\partial t} &+ \frac{1}{2}\sigma^2(D_t^+)\, S_t^2\frac{\partial^2 V^+}{\partial S^2} + \sigma^2(D_t^+)\, S_t\frac{\partial^2 V^+}{\partial S\partial D} + \frac{1}{2}\sigma^2(D_t^+)\frac{\partial^2 V^+}{\partial D^2} - r_+V^+ + \\ &+ r_+S\frac{\partial V^+}{\partial S} + \left[V^-(S+h_+S,D,t) - V^+(S,D,t)\right]\lambda_+ = \left[\frac{1}{2}\sigma^2(D_t^+) + \theta D_t^+\right]\frac{\partial V^+}{\partial D}, \end{aligned}$$

we find the equation for the **positive past information** with stochastic volatility of Hobson and Rogers

$$\begin{aligned}\frac{\partial V^+}{\partial t} &+ \frac{1}{2}\sigma^2(D_t^+)\, S_t^2 \frac{\partial^2 V^+}{\partial S^2} + r_+ S\frac{\partial V^+}{\partial S} + \sigma^2(D_t^+)\, S_t \frac{\partial^2 V^+}{\partial S \partial D} - \left[\frac{1}{2}\sigma^2(D_t^+) + \theta D_t^+\right]\frac{\partial V^+}{\partial D} + \\ &+ \frac{1}{2}\sigma^2(D_t^+)\frac{\partial^2 V^+}{\partial D^2} = \left[r_+ + \lambda_+\right]V^+ - \lambda_+ V^-(S + h_+S, D, t). \end{aligned} \tag{21}$$

Similarly to the case of positive past information, the following equation for the **negative past information** can be deduced

$$\begin{aligned}\frac{\partial V^-}{\partial t} &+ \frac{1}{2}\sigma^2(D_t^-)\, S_t^2 \frac{\partial^2 V^-}{\partial S^2} + r_- S\frac{\partial V^-}{\partial S} + \sigma^2(D_t^-)\, S_t \frac{\partial^2 V^-}{\partial S \partial D} - \left[\frac{1}{2}\sigma^2(D_t^-) + \theta D_t^-\right]\frac{\partial V^-}{\partial D} + \\ &+ \frac{1}{2}\sigma^2(D_t^-)\frac{\partial^2 V^-}{\partial D^2} = \left[r_- + \lambda_-\right]V^- - \lambda_- V^+(S + h_-S, D, t). \end{aligned} \tag{22}$$

Thus, Eq. (21) and Eq. (22) define the differential system given by

$$\begin{cases} \partial V^+/\partial t + \frac{1}{2}\sigma^2(D_t^+)\, S_t^2\left[\partial^2 V^+/\partial S^2\right] + r_+ S\left[\partial V^+/\partial S\right] + \\ +\sigma^2(D_t^+)\, S_t\left[\partial^2 V^+/\partial S\partial D\right] - \left[\frac{1}{2}\sigma^2(D_t^+) + \theta D_t^+\right]\left[\partial V^+/\partial D\right] + \\ +\frac{1}{2}\sigma^2(D_t^+)\left[\partial^2 V^+/\partial D^2\right] = \left[r_+ + \lambda_+\right]V^+ - \lambda_+ V^-(S + h_+S, D, t) \\ \\ \partial V^-/\partial t + \frac{1}{2}\sigma^2(D_t^-)\, S_t^2\left[\partial^2 V^-/\partial S^2\right] + r_- S\left[\partial V^-/\partial S\right] + \\ +\sigma^2(D_t^-)\, S_t\left[\partial^2 V^-/\partial S\partial D\right] - \left[\frac{1}{2}\sigma^2(D_t^-) + \theta D_t^-\right]\left[\partial V^-/\partial D\right] + \\ +\frac{1}{2}\sigma^2(D_t^-)\left[\partial^2 V^-/\partial D^2\right] = \left[r_- + \lambda_-\right]V^- - \lambda_- V^+(S + h_-S, D, t) \end{cases} \tag{23}$$

2.1 Formulation of the Problems

The mathematical modeling associated with memory in asset and volatility for pricing options is defined in an unbounded domain, $\Omega_\infty = [0,\infty] \times [-\infty,\infty]$. To construct the approximate solution, it is necessary to truncate Ω_∞ obtaining a bounded domain $\Omega = [0, S^{max}] \times [D^{min}, D^{max}]$, where S^{max} is the maximum value the underlying asset, D^{max} is the maximum deviation and D^{min} is the minimum deviation.

Usually Dirichlet boundary conditions are assumed in a bounded set Ω

$$p^{\pm}(0, D, t) = Ee^{-r_{\pm}(T-t)} \quad \text{and} \quad p^{\pm}(S_{max}, D, t) = 0\ , \tag{24}$$

resulting from classical theory. In the case of a call option, the boundary conditions of Dirichlet type in Ω come from $c(0, D, t) = 0$, the parity formula with the boundary condition for the put options $\lim_{S\to\infty} c(S, D, t) = \lim_{S\to\infty} c_\infty(S, D, t) \approx S$ for $S = S_{max}$ large enough, resulting

$$c^{\pm}(0, D, t) = 0 \quad \text{and} \quad c^{\pm}(S_{max}, D, t) = S_{max} - Ee^{-r_{\pm}(T-t)}\ . \tag{25}$$

The problem for European put option pricing is formulated as follows

PROBLEM EPHR: For $t \in [0,T]$, we find $p(S,D,t) = \Big[p^+(S,D,t) + p^-(S,D,t)\Big]/2$, such that

$$\left\{\begin{array}{l} \partial p^+/\partial t + \frac{1}{2}\sigma^2(D_t^+)\ S_t^2\Big[\partial^2 p^+/\partial S^2\Big] + r_+ S\Big[\partial p^+/\partial S\Big] + \\ +\sigma^2(D_t^+)\ S_t\Big[\partial^2 p^+/\partial S\partial D\Big] - \Big[\frac{1}{2}\sigma^2(D_t^+) + \theta D_t^+\Big]\Big[\partial p^+/\partial D\Big] + \\ +\frac{1}{2}\sigma^2(D_t^+)\Big[\partial^2 p^+/\partial D^2\Big] = \Big[r_+ + \lambda_+\Big]p^+ - \lambda_+ p^-(S + h_+S, D, t) \\ \\ \partial p^-/\partial t + \frac{1}{2}\sigma^2(D_t^-)\ S_t^2\Big[\partial^2 p^-/\partial S^2\Big] + r_- S\Big[\partial p^-/\partial S\Big] + \\ +\sigma^2(D_t^-)\ S_t\Big[\partial^2 p^-/\partial S\partial D\Big] - \Big[\frac{1}{2}\sigma^2(D_t^-) + \theta D_t^-\Big]\Big[\partial p^-/\partial D\Big] + \\ +\frac{1}{2}\sigma^2(D_t^-)\Big[\partial^2 p^-/\partial D^2\Big] = \Big[r_- + \lambda_-\Big]p^- - \lambda_- p^+(S + h_-S, D, t) \end{array}\right. \tag{26}$$

with Payoff condition,

$$p^+(S,D,T) = (E-S)^+ \quad \text{and} \quad p^-(S,D,T) = (E-S)^+,$$

boundary condition,

$$p^+(0,D,t) = Ee^{-r_+[T-t]} \quad , \quad p^+(S_{max},D,t) = 0,$$

$$p^-(0,D,t) = Ee^{-r_-[T-t]} \quad \text{and} \quad p^-(S_{max},D,t) = 0.$$

And the problem of European call option pricing as

PROBLEM ECHR: For $t \in [0,T]$, we find $c(S,D,t) = \Big[c^+(S,D,t) + c^-(S,D,t)\Big]/2$, such that

$$\left\{\begin{array}{l} \partial c^+/\partial t + \frac{1}{2}\sigma^2(D_t^+)\ S_t^2\Big[\partial^2 c^+/\partial S^2\Big] + r_+ S\Big[\partial c^+/\partial S\Big] + \\ +\sigma^2(D_t^+)\ S_t\Big[\partial^2 c^+/\partial S\partial D\Big] - \Big[\frac{1}{2}\sigma^2(D_t^+) + \theta D_t^+\Big]\Big[\partial c^+/\partial D\Big] + \\ +\frac{1}{2}\sigma^2(D_t^+)\Big[\partial^2 c^+/\partial D^2\Big] = \Big[r_+ + \lambda_+\Big]c^+ - \lambda_+ c^-(S + h_+S, D, t) \\ \\ \partial c^-/\partial t + \frac{1}{2}\sigma^2(D_t^-)\ S_t^2\Big[\partial^2 c^-/\partial S^2\Big] + r_- S\Big[\partial c^-/\partial S\Big] + \\ +\sigma^2(D_t^-)\ S_t\Big[\partial^2 c^-/\partial S\partial D\Big] - \Big[\frac{1}{2}\sigma^2(D_t^-) + \theta D_t^-\Big]\Big[\partial c^-/\partial D\Big] + \\ +\frac{1}{2}\sigma^2(D_t^-)\Big[\partial^2 c^-/\partial D^2\Big] = \Big[r_- + \lambda_-\Big]c^- - \lambda_- c^+(S + h_-S, D, t) \end{array}\right. \tag{27}$$

with Payoff condition,

$$c^+(S,D,T) = (S-E)^+ \quad \text{and} \quad c^-(S,D,T) = (S-E)^+,$$

boundary condition,

$$c^+(0,D,t) = 0 \quad , \quad c^+(S_{max},D,t) = S_{max} - Ee^{-r_+[T-t]},$$

$$c^-(0,D,t) = 0 \quad \text{and} \quad c^-(S_{max},D,t) = S_{max} - Ee^{-r_-[T-t]}.$$

2.2 Variational Formulation

In order to get an initial value problem we need to consider the transformation in the time variable $\tau = T - t$. The following problem for the European put option pricing is then deduced ***PROBLEM IEPHR***: For all $\tau \in [0,T]$, we find $p(S,D,\tau) = \Big[p^+(S,D,\tau) + p^-(S,D,\tau)\Big]/2$, such that

$$\left\{\begin{array}{l} \partial p^+/\partial\tau - \frac{1}{2}\sigma^2(D_\tau^+)\; S_t^2\Big[\partial^2 p^+/\partial S^2\Big] - r_+ S\Big[\partial p^+/\partial S\Big] - \\ -\sigma^2(D_\tau^+)\; S_t\Big[\partial^2 p^+/\partial S\partial D\Big] + \Big[\frac{1}{2}\sigma^2(D_\tau^+) + \theta D_\tau^+\Big]\Big[\partial p^+/\partial D\Big] - \\ -\frac{1}{2}\sigma^2(D_\tau^+)\Big[\partial^2 p^+/\partial D^2\Big] = -\Big[r_+ + \lambda_+\Big]p^+ + \lambda_+ p^-(S + h_+ S, D, t) \\ \\ \partial p^-/\partial\tau - \frac{1}{2}\sigma^2(D_\tau^-)\; S_t^2\Big[\partial^2 p^-/\partial S^2\Big] - r_- S\Big[\partial p^-/\partial S\Big] - \\ -\sigma^2(D_\tau^-)\; S_t\Big[\partial^2 p^-/\partial S\partial D\Big] + \Big[\frac{1}{2}\sigma^2(D_\tau^-) + \theta D_\tau^-\Big]\Big[\partial p^-/\partial D\Big] - \\ -\frac{1}{2}\sigma^2(D_\tau^-)\Big[\partial^2 p^-/\partial D^2\Big] = -\Big[r_- + \lambda_-\Big]p^- + \lambda_- p^+(S + h_- S, D, t) \end{array}\right. \tag{28}$$

with initial condition,

$$p^+(S,D,0) = (E-S)^+ \quad \text{and} \quad p^-(S,D,0) = (E-S)^+,$$

boundary condition,

$$p^+(0,D,T-\tau) = Ee^{-r_+[\tau]} \quad , \quad p^+(S_{max},D,T-\tau) = 0,$$
$$p^-(0,D,T-\tau) = Ee^{-r_-[\tau]} \quad \text{and} \quad p^-(S_{max},D,T-\tau) = 0.$$

The variable τ is the time remaining to the maturity of the option. While System Eq. (28) does not require the condition $\tau < T$, the model assumes that the price p is computed to the limit time T. We define the set of functions,

$$\begin{aligned} \mathcal{W}_\tau \;\; = \;\; & \Big\{\mathbf{p} \in L^2(0,T;H^1(\Omega)) \mid \partial\mathbf{p}/\partial\tau \in L^2(\Omega) \text{ a.e. in } [0,T]\;;\; \mathbf{p}(0,D,\tau) = Ee^{-r\tau}, \\ & \mathbf{p}(S_{max},D,\tau) = 0, \text{ a.e. } [D^{min},D^{max}] \times [0,T]\Big\}, \end{aligned}$$

and the variations space

$$\mathcal{W}_0 = \Big\{\mathbf{v} \in H_0^1(\Omega)\Big\}\;.$$

Defining the following inner product in the $L^2(\Omega)$,

$$(\mathbf{u},\mathbf{v}) \triangleq \int_\Omega \mathbf{u}(x)\mathbf{v}(x)\; dx,$$

the system Eq. (28) can be written in vector form

$$\frac{\partial\mathbf{p}}{\partial\tau} - \frac{1}{2}\mathbf{Volt}S^2\frac{\partial^2\mathbf{p}}{\partial S^2} - \mathbf{R}S\frac{\partial\mathbf{p}}{\partial S} - \mathbf{Volt}S\frac{\partial^2\mathbf{p}}{\partial S\partial D} + \mathbf{\Phi}\frac{\partial\mathbf{p}}{\partial D} - \frac{1}{2}\mathbf{Volt}\frac{\partial^2\mathbf{p}}{\partial D^2} + \mathbf{\Lambda}\mathbf{p} = \mathbf{F}\;, \tag{29}$$

where

$$\mathbf{p} = \begin{bmatrix} p^+ \\ \\ p^- \end{bmatrix}\;, \quad \mathbf{Volt} = \begin{bmatrix} \sigma^2(D_\tau^+) & 0 \\ \\ 0 & \sigma^2(D_\tau^-) \end{bmatrix}\;, \quad \mathbf{R} = \begin{bmatrix} r_+ & 0 \\ \\ 0 & r_- \end{bmatrix}$$

$$\mathbf{\Phi} = \begin{bmatrix} \frac{1}{2}\sigma^2(D_\tau^+) + \theta D_t^+ & 0 \\ 0 & \frac{1}{2}\sigma^2(D_\tau^-) + \theta D_t^- \end{bmatrix}$$

$$\mathbf{\Lambda} = \begin{bmatrix} r_+ + \lambda_+ & 0 \\ 0 & r_- + \lambda_- \end{bmatrix} \quad , \quad \mathbf{F} = \begin{bmatrix} F_+ \\ F_- \end{bmatrix} = \begin{bmatrix} \lambda_+ p^-(S + h_+S, D, \tau) \\ \lambda_- p^+(S + h_-S, D, \tau) \end{bmatrix}$$

multiplying the Eq. (29) by a test function $\mathbf{v}$ and integrating in Ω, we obtain

$$\begin{aligned} \left(\frac{\partial \mathbf{p}}{\partial \tau}, \mathbf{v}\right) \quad &- \quad \left(\frac{1}{2}\mathbf{Volt}S^2\frac{\partial^2 \mathbf{p}}{\partial S^2}, \mathbf{v}\right) - \left(\mathbf{R}S\frac{\partial \mathbf{p}}{\partial S}, \mathbf{v}\right) - \left(\mathbf{Volt}S\frac{\partial^2 \mathbf{p}}{\partial S\partial D}, \mathbf{v}\right) + \\ &+ \quad \left(\mathbf{\Phi}\frac{\partial \mathbf{p}}{\partial D}, \mathbf{v}\right) - \left(\frac{1}{2}\mathbf{Volt}\frac{\partial^2 \mathbf{p}}{\partial D^2}, \mathbf{v}\right) + \left(\mathbf{\Lambda}\mathbf{p}, \mathbf{v}\right) = \left(\mathbf{F}, \mathbf{v}\right) , \end{aligned} \tag{30}$$

$\forall\ \mathbf{v} \in \mathcal{W}_0(\Omega)$. Considering that

$$\frac{1}{2}\mathbf{Volt}S^2\frac{\partial^2 \mathbf{p}}{\partial S^2} = \frac{1}{2}\mathbf{Volt}\frac{\partial}{\partial S}\left\{S^2\frac{\partial \mathbf{p}}{\partial S}\right\} - \mathbf{Volt}S\frac{\partial \mathbf{p}}{\partial S} , \tag{31}$$

as also

$$\frac{1}{2}\mathbf{Volt}\frac{\partial^2 \mathbf{p}}{\partial D^2} = \frac{1}{2}\frac{\partial}{\partial D}\left\{\mathbf{Volt}\frac{\partial \mathbf{p}}{\partial D}\right\} - \frac{\partial}{\partial D}\left\{\mathbf{Volt}\right\}\frac{\partial \mathbf{p}}{\partial D} \tag{32}$$

and finally

$$\mathbf{Volt}S\frac{\partial^2 \mathbf{p}}{\partial S\partial D} = \mathbf{Volt}\frac{\partial}{\partial S}\left\{S\frac{\partial \mathbf{p}}{\partial D}\right\} - \mathbf{Volt}\frac{\partial \mathbf{p}}{\partial D} . \tag{33}$$

So substituting Eq. (31), Eq. (32) and Eq. (33) in Eq. (30), we get

$$\begin{aligned} \left(\frac{\partial \mathbf{p}}{\partial \tau}, \mathbf{v}\right) \quad &- \quad \left(\frac{1}{2}\mathbf{Volt}\frac{\partial}{\partial S}\left\{S^2\frac{\partial \mathbf{p}}{\partial S}\right\}, \mathbf{v}\right) - \left(\mathbf{Volt}\frac{\partial}{\partial S}\left\{S\frac{\partial \mathbf{p}}{\partial D}\right\}, \mathbf{v}\right) - \\ &- \quad \left(\frac{1}{2}\frac{\partial}{\partial D}\left\{\mathbf{Volt}\frac{\partial \mathbf{p}}{\partial D}\right\}, \mathbf{v}\right) - \left(\Big[\mathbf{R} - \mathbf{Volt}\Big]S\frac{\partial \mathbf{p}}{\partial S}, \mathbf{v}\right) + \left(\mathbf{\Lambda}\ \mathbf{p}, \mathbf{v}\right) + \\ &+ \quad \left(\left[\mathbf{\Phi} + \mathbf{Volt} + \frac{\partial}{\partial D}\left\{\mathbf{Volt}\right\}\right]\frac{\partial \mathbf{p}}{\partial D}, \mathbf{v}\right) = \left(\mathbf{F}, \mathbf{v}\right). \end{aligned} \tag{34}$$

Using the integration by parts in the second, third and fourth terms of the Eq. (34) and $\mathbf{v} \in \mathcal{W}_0(\Omega)$, we obtain

$$\begin{aligned} \left(\frac{\partial \mathbf{p}}{\partial \tau}, \mathbf{v}\right) \quad &+ \quad \left(\frac{1}{2}\mathbf{Volt}S\frac{\partial \mathbf{p}}{\partial S}, S\frac{\partial \mathbf{v}}{\partial S}\right) + \left(\mathbf{Volt}S\frac{\partial \mathbf{p}}{\partial D}, \frac{\partial \mathbf{v}}{\partial S}\right) + \\ &+ \quad \left(\frac{1}{2}\mathbf{Volt}\frac{\partial \mathbf{p}}{\partial D}, \frac{\partial \mathbf{v}}{\partial D}\right) - \left(\Big[\mathbf{R} - \mathbf{Volt}\Big]S\frac{\partial \mathbf{p}}{\partial S}, \mathbf{v}\right) + \\ &+ \quad \left(\left[\mathbf{\Phi} + \mathbf{Volt} + \frac{\partial}{\partial D}\left\{\mathbf{Volt}\right\}\right]\frac{\partial \mathbf{p}}{\partial D}, \mathbf{v}\right) + \left(\mathbf{\Lambda}\ \mathbf{p}, \mathbf{v}\right) = \left(\mathbf{F}, \mathbf{v}\right) . \end{aligned}$$

Thus, the problem of put options pricing can be formulated as follows

PROBLEM IEPHRV: For all $\tau \in [0, T]$, we find $p(S, D, \tau) = \Big[p^+(S, D, \tau) + p^-(S, D, \tau)\Big]/2$, such that

$$\left(\frac{\partial \mathbf{p}}{\partial \tau}, \mathbf{v}\right) + a(\mathbf{p}, \mathbf{v}) = \left(\mathbf{F}, \mathbf{v}\right) , \quad \forall \mathbf{v} \in \mathcal{W}_0,$$

satisfying the initial condition

$$\Big(\mathbf{p}(0),\mathbf{v}\Big) = \begin{bmatrix} \Big(p^{+}(0), v^{+}\Big) \\ \\ \Big(p^{-}(0), v^{-}\Big) \end{bmatrix}, \quad \forall \mathbf{v} \in \mathcal{W}_0,$$

with the bilinear form given by

$$\begin{aligned} a(\mathbf{p},\mathbf{v}) &= \left(\frac{1}{2}\mathbf{Volt}S\frac{\partial \mathbf{p}}{\partial S}, S\frac{\partial \mathbf{v}}{\partial S}\right) + \left(\mathbf{Volt}S\frac{\partial \mathbf{p}}{\partial D}, \frac{\partial \mathbf{v}}{\partial S}\right) + \left(\frac{1}{2}\mathbf{Volt}\frac{\partial \mathbf{p}}{\partial D}, \frac{\partial \mathbf{v}}{\partial D}\right) + \Big(\mathbf{\Lambda}\,\mathbf{p},\mathbf{v}\Big) - \\ &- \left(\Big[\mathbf{R} - \mathbf{Volt}\Big]S\frac{\partial \mathbf{p}}{\partial S}, \mathbf{v}\right) + \left(\Big[\mathbf{\Phi} + \mathbf{Volt} + \frac{\partial}{\partial D}\Big\{\mathbf{Volt}\Big\}\Big]\frac{\partial \mathbf{p}}{\partial D}, \mathbf{v}\right) \end{aligned} \tag{35}$$

On existence and uniqueness of this kind of parabolic variational equality see, for example, Brézis (1984).

Next we present a finite element approximation to *PROBLEM IEPHRV* in the space domain combined with an implicit finite difference approximation in the time domain.

2.3 Finite Element Approximation

For the construction of an approximation through finite elements, we define

$$\mathcal{W}_h^k = \{\mathbf{v}_h \in C^0(\Omega); \mathbf{v}_h|_k \in P_k(K)\}\ ,$$

$$\begin{aligned} \mathcal{W}_h &= \Big\{\mathbf{p}_h \in L^2(0,T;\mathcal{W}_h^k) \mid \partial \mathbf{p}_h/\partial\tau \in L^2(\Omega) \text{ a.e. in } [0,T]\ ;\ \mathbf{p}_h(0,D,\tau) = Ee^{-r\tau}, \\ &\quad \mathbf{p}_h(S_{max},D,\tau) = 0 \text{ a.e. in } [0,T],\ \forall D \in [D^{min},D^{max}]\Big\}, \end{aligned}$$

so that $\mathcal{W}_h^k \subset \mathcal{W}_\tau$ is the space of element's of degree $k \geq 1$, in each element K of triangulation T_h, where $P_k(K)$ is the polynomial set of degree less or equal defined in K.

By using the Galerkin method in *PROBLEM IEPHRV*, we obtain the semi-discrete approximation or the continuous approximation in time.

The fully discrete problem is now defined using the *Euler Implicit method* in the discretisation in time. We split $[0,T]$ in sub-intervals $[\tau_{n-1},\tau_n]$, where $\tau_n = n\Delta t$, $n = 1,\cdots,N$, with $\tau_0 = 0$ and $\tau_N = T$, and we use the notations $p^{n,\pm} = p^{\pm}(\tau_n)$,

$$\partial_\tau p^{n,\pm} = \frac{p^{n+1,\pm} - p^{n,\pm}}{\Delta t}\ .$$

The fully discrete approximation is considered in the following functional space

$$\begin{aligned} \mathcal{W}_{\Delta\tau,h} &= \Big\{\mathbf{p}_h^n,\ n = 1,\ldots,N,\ \mathbf{p}_h^n \in \mathcal{W}_h^k,\ \mathbf{p}_h(0,D,\tau_n) = Ee^{-r\tau_n}, \\ &\quad \mathbf{p}_h(S_{max},D,\tau_n) = 0, n = 1,\ldots,N,\ \forall D \in [D^{min},D^{max}]\Big\}. \end{aligned}$$

The fully discrete approximation in then computed using the following variational problem:
PROBLEM IEPHRhm: Given $n = 1,\cdots,N$, we find $\mathbf{p}_h^n \in \mathcal{W}_{\Delta\tau,h}$ such that

$$\Big(\partial_\tau \mathbf{p}_h^n, \mathbf{v}_h\Big) + a(\mathbf{p}_h^{n+1}, \mathbf{v}_h) = \Big(\mathbf{F}_h, \mathbf{v}_h\Big),\ \mathbf{v}_h \in \mathcal{W}_h^k \cap H_0^1(\Omega) \tag{36}$$

satisfying the initial condition

$$\Big(\mathbf{p}_h^0, \mathbf{v}_h\Big) = \Big(\mathbf{p}(0), \mathbf{v}_h\Big),\ \mathbf{v}_h \in \mathcal{W}_h^k \cap H_0^1(\Omega).$$

3 AMERICAN OPTION PRICING MODEL

Was previously presented a formulation of the problem of European option. Now will be a formulation of American options pricing as free boundary problems. Following the same idea presented by Wilmott et al. (2000) to the case in standard Black and Scholes modeling.

If at any time $t^* < T$ the price of the underlying asset is $S^* < E(1 - e^{-r(T-t^*)})$, the put option must be exercised immediately, because the income L generated by the premature exercise satisfies

$$L = (E - S^*) > E(e^{-r(T-t^*)}) > p^*(S^*, t),$$

where $p^*(S^*, t)$ is the price of put European option with the exercise price E and maturity in $T - t^*$ years. Like any portfolio income is $(E - S)^+$ in $t = T$ and this has the same value that the European option, we have no portfolio is a better alternative to the exercise premature. Noting that in this case the price $P(S^*, t^*)$ should have the value $E - S^*$ not to have arbitrage.

In particular, we have for $S^* = 0$, $P(0, t^*) = E$ using the arbitrage argument, we can see that

$$P(S, D, t) \geq (E - S)^+ = 0, \quad \forall\ S \geq E.$$

It appears that price time t that lead to the premature exercise of forming a range of $[0, S_f]$, whose upper limit is called the **optimal point of exercise**.

The point S^f, divides the dominion in a segment where the option is exercised, and the other in which it should exercise it later. Thus, the American put options pricing problem may be seen as **free boundary problems**, where the free boundary is given by $S_f = S_f(t)$.

We can formulate the **free boundary problems** in two well defined regions for "positive past information", one where we must exercise the option,

$$P^+(S_t, D, t) = (E - S)^+ \ ,$$

$$\frac{\partial P^+}{\partial t} + \frac{1}{2}\sigma^2(D_t^+)\ S_t^2\frac{\partial^2 P^+}{\partial S^2} + r_+ S\frac{\partial P^+}{\partial S} + \sigma^2(D_t^+)\ S_t\frac{\partial^2 P^+}{\partial S \partial D} - \left[\frac{1}{2}\sigma^2(D_t^+) + \theta D_t^+\right]\frac{\partial P^+}{\partial D} + $$
$$+ \frac{1}{2}\sigma^2(D_t^+)\frac{\partial^2 P^+}{\partial D^2} - \left[r_+ + \lambda_+\right]P^+ \leq -\lambda_+ P^-(S + h_+ S, D, t)\ ,$$
$$0 \leq S \leq S_f.$$

another where is not optimal the exercise of the option

$$P^+(S, D, t) > (E - S)^+,$$

$$\frac{\partial P^+}{\partial t} + \frac{1}{2}\sigma^2(D_t^+)\ S_t^2\frac{\partial^2 P^+}{\partial S^2} + r_+ S\frac{\partial P^+}{\partial S} + \sigma^2(D_t^+)\ S_t\frac{\partial^2 P^+}{\partial S \partial D} - \left[\frac{1}{2}\sigma^2(D_t^+) + \theta D_t^+\right]\frac{\partial P^+}{\partial D} + $$
$$+ \frac{1}{2}\sigma^2(D_t^+)\frac{\partial^2 P^+}{\partial D^2} - \left[r_+ + \lambda_+\right]P^+ = -\lambda_+ P^-(S + h_+ S, D, t)\ ,$$
$$S_f \leq S \leq \infty.$$

The conditions of the interface between two regions of the domain in points S_f, given by

$$P^+(S_f, D, t) = (E - S_f(t)\})^+\ , \quad \frac{\partial}{\partial n}P^+(S_f(t), D, t) = -1.$$

Besides the condition of final time, payoff function, which is

$$P^{+}(S,D,T) = (E-S)^{+},$$

and the boundary condition ($S \in \partial\Omega_\infty$),

$$P^{+}(S=0,D,t) = Ee^{-r_{+}(T-t)}\ , \quad \lim_{S\to\infty} P^{-}(S,D,t) = 0$$

Similarly calculated for “negative past information”, one where we must exercise the option,

$$P^{-}(S_t,D,t) = (E-S)^{+}\ ,$$

$$\begin{aligned}\frac{\partial P^-}{\partial t} &+ \frac{1}{2}\sigma^2(D_t^-)\ S_t^2\frac{\partial^2 P^-}{\partial S^2} + r_- S\frac{\partial P^-}{\partial S} + \sigma^2(D_t^-)\ S_t\frac{\partial^2 P^-}{\partial S\partial D} - \left[\frac{1}{2}\sigma^2(D_t^-) + \theta D_t^-\right]\frac{\partial P^-}{\partial D} + \\ &+ \frac{1}{2}\sigma^2(D_t^-)\frac{\partial^2 P^-}{\partial D^2} - \Big[r_- + \lambda_-\Big]P^- \le -\lambda_- P^+(S+h_- S,D,t)\ ,\end{aligned}$$

$$0 \le S \le S_f.$$

another where is not optimal the exercise of the option

$$P^{-}(S,D,t) > (E-S)^{+},$$

$$\begin{aligned}\frac{\partial P^-}{\partial t} &+ \frac{1}{2}\sigma^2(D_t^-)\ S_t^2\frac{\partial^2 P^-}{\partial S^2} + r_- S\frac{\partial P^-}{\partial S} + \sigma^2(D_t^-)\ S_t\frac{\partial^2 P^-}{\partial S\partial D} - \left[\frac{1}{2}\sigma^2(D_t^-) + \theta D_t^-\right]\frac{\partial P^-}{\partial D} + \\ &+ \frac{1}{2}\sigma^2(D_t^-)\frac{\partial^2 P^-}{\partial D^2} - \Big[r_- + \lambda_-\Big]P^- = -\lambda_- P^+(S+h_- S,D,t)\ ,\end{aligned}$$

$$S_f \le S \le \infty.$$

The conditions of the interface between two regions of the domain in points S_f, given by

$$P^{-}(S_f,D,t) = (E-S_f(t)\})^{+}\ , \quad \frac{\partial}{\partial n}P^{-}(S_f(t),D,t) = -1.$$

Besides the condition of final time, payoff function, which is

$$P^{-}(S,D,T) = (E-S)^{+},$$

and the boundary condition ($S \in \partial\Omega_\infty$),

$$P^{-}(S=0,D,t) = Ee^{-r_{-}(T-t)}\ , \quad \lim_{S\to\infty} P^{-}(S,D,t) = 0$$

3.1 Formulation of the Problem

The problem of American put option pricing as

PROBLEM APHR: For $\tau \in [0,T]$, we find $P(S,D,\tau) = \Big[P^{+}(S,D,\tau) + P^{-}(S,D,\tau)\Big]/2$, such that

- For $P^+(S,D,\tau) = (E-S)^+$, $P^-(S,D,\tau) = (E-S)^+$ and $0 \le S \le S_f$,

$$\begin{cases} \partial P^+/\partial t + \frac{1}{2}\sigma^2(D_t^+)\ S_t^2\Big[\partial^2 P^+/\partial S^2\Big] + r_+ S\Big[\partial P^+/\partial S\Big] + \\ +\sigma^2(D_t^+)\ S_t\Big[\partial^2 P^+/\partial S\partial D\Big] - \Big[\frac{1}{2}\sigma^2(D_t^+) + \theta D_t^+\Big]\Big[\partial P^+/\partial D\Big] + \\ +\frac{1}{2}\sigma^2(D_t^+)\Big[\partial^2 P^+/\partial D^2\Big] \le \Big[r_+ + \lambda_+\Big]p^+ - \lambda_+ P^-(S + h_+ S, D, t) \\ \\ \partial P^-/\partial t + \frac{1}{2}\sigma^2(D_t^-)\ S_t^2\Big[\partial^2 P^-/\partial S^2\Big] + r_- S\Big[\partial P^-/\partial S\Big] + \\ +\sigma^2(D_t^-)\ S_t\Big[\partial^2 P^-/\partial S\partial D\Big] - \Big[\frac{1}{2}\sigma^2(D_t^-) + \theta D_t^-\Big]\Big[\partial P^-/\partial D\Big] + \\ +\frac{1}{2}\sigma^2(D_t^-)\Big[\partial^2 P^-/\partial D^2\Big] \le \Big[r_- + \lambda_-\Big]P^- - \lambda_- P^+(S + h_- S, D, t) \end{cases} \tag{37}$$

- For $P^+(S,D,t) > (E-S)^+$, $P^-(S,D,t) > (E-S)^+$ and $S_f \le S \le \infty$,

$$\begin{cases} \partial p^+/\partial t + \frac{1}{2}\sigma^2(D_t^+)\ S_t^2\Big[\partial^2 p^+/\partial S^2\Big] + r_+ S\Big[\partial p^+/\partial S\Big] + \\ +\sigma^2(D_t^+)\ S_t\Big[\partial^2 p^+/\partial S\partial D\Big] - \Big[\frac{1}{2}\sigma^2(D_t^+) + \theta D_t^+\Big]\Big[\partial p^+/\partial D\Big] + \\ +\frac{1}{2}\sigma^2(D_t^+)\Big[\partial^2 p^+/\partial D^2\Big] = \Big[r_+ + \lambda_+\Big]p^+ - \lambda_+ p^-(S + h_+ S, D, t) \\ \\ \partial p^-/\partial t + \frac{1}{2}\sigma^2(D_t^-)\ S_t^2\Big[\partial^2 p^-/\partial S^2\Big] + r_- S\Big[\partial p^-/\partial S\Big] + \\ +\sigma^2(D_t^-)\ S_t\Big[\partial^2 p^-/\partial S\partial D\Big] - \Big[\frac{1}{2}\sigma^2(D_t^-) + \theta D_t^-\Big]\Big[\partial p^-/\partial D\Big] + \\ +\frac{1}{2}\sigma^2(D_t^-)\Big[\partial^2 p^-/\partial D^2\Big] = \Big[r_- + \lambda_-\Big]p^- - \lambda_- p^+(S + h_- S, D, t) \end{cases} \tag{38}$$

The conditions of the interface between two regions

$$P^+(S_f, D, t) = (E - S_f(t)\})^+ \ , \quad \frac{\partial}{\partial n}P^+(S_f(t), D, t) = -1 \ ,$$

and

$$P^-(S_f, D, t) = (E - S_f(t)\})^+ \ , \quad \frac{\partial}{\partial n}P^-(S_f(t), D, t) = -1 \ ,$$

with payoff condition,

$$P^+(S, D, T) = (E-S)^+ \quad \text{and} \quad P^-(S, D, T) = (E-S)^+ \ ,$$

boundary condition,

$$P^+(0, D, t) = Ee^{-r_+(T-t)} \ , \quad P^+(S_{max}, D, t) = 0 \ ,$$

$$P^-(0, D, t) = Ee^{-r_-(T-t)} \text{ and } \quad P^-(S_{max}, D, t) = 0.$$

3.2 Variational Formulation

Defining the subset of functions bounded below by $g(S)$

$$\mathcal{K} = \{\mathbf{P} \in \mathcal{W}_\tau(\Omega)\ ;\ \mathbf{P}(S, D, \tau) \ge g(S)\}, \tag{39}$$

where $g(S)$ is the payoff value which, in the case of put option is given by $g(S) = (E - S)^+$ and $g(S) = (S - E)^+$ for call option.

As the system (37) can be rewritten as

$$\frac{\partial \mathbf{p}}{\partial \tau} - \frac{1}{2}\mathbf{Volt}S^2\frac{\partial^2 \mathbf{p}}{\partial S^2} - \mathbf{R}S\frac{\partial \mathbf{p}}{\partial S} - \mathbf{Volt}S\frac{\partial^2 \mathbf{p}}{\partial S \partial D} + \mathbf{\Phi}\frac{\partial \mathbf{p}}{\partial D} - \frac{1}{2}\mathbf{Volt}\frac{\partial^2 \mathbf{p}}{\partial D^2} + \mathbf{\Lambda}\mathbf{p} \geq \mathbf{F} \tag{40}$$

and the system (38) as

$$\frac{\partial \mathbf{p}}{\partial \tau} - \frac{1}{2}\mathbf{Volt}S^2\frac{\partial^2 \mathbf{p}}{\partial S^2} - \mathbf{R}S\frac{\partial \mathbf{p}}{\partial S} - \mathbf{Volt}S\frac{\partial^2 \mathbf{p}}{\partial S \partial D} + \mathbf{\Phi}\frac{\partial \mathbf{p}}{\partial D} - \frac{1}{2}\mathbf{Volt}\frac{\partial^2 \mathbf{p}}{\partial D^2} + \mathbf{\Lambda}\mathbf{p} = \mathbf{F}\ , \tag{41}$$

multiplying (40) and (41) by $\mathbf{v}$, where $\mathbf{v}$ is a test function in $\mathcal{W}_0$, and integrating in Ω we get

$$\begin{aligned}\left(\frac{\partial \mathbf{P}}{\partial \tau}, \mathbf{v}\right) \quad &- \quad \left(\frac{1}{2}\mathbf{Volt}S^2\frac{\partial^2 \mathbf{P}}{\partial S^2}, \mathbf{v}\right) - \left(\mathbf{R}S\frac{\partial \mathbf{P}}{\partial S}, \mathbf{v}\right) - \left(\mathbf{Volt}S\frac{\partial^2 \mathbf{P}}{\partial S \partial D}, \mathbf{v}\right) + \\ &+ \quad \left(\mathbf{\Phi}\frac{\partial \mathbf{P}}{\partial D}, \mathbf{v}\right) - \left(\frac{1}{2}\mathbf{Volt}\frac{\partial^2 \mathbf{P}}{\partial D^2}, \mathbf{v}\right) + \left(\mathbf{\Lambda}\mathbf{P}, \mathbf{v}\right) \geq \left(\mathbf{F}, \mathbf{v}\right).\end{aligned}$$

Taking into account Eq. (31), Eq. (32) and Eq. (33), we establish

$$\begin{aligned}\left(\frac{\partial \mathbf{P}}{\partial \tau}, \mathbf{v}\right) \quad &- \quad \left(\frac{1}{2}\mathbf{Volt}\frac{\partial}{\partial S}\left\{S^2\frac{\partial \mathbf{P}}{\partial S}\right\}, \mathbf{v}\right) - \left(\mathbf{Volt}\frac{\partial}{\partial S}\left\{S\frac{\partial \mathbf{P}}{\partial D}\right\}, \mathbf{v}\right) - \\ &- \quad \left(\frac{1}{2}\frac{\partial}{\partial D}\left\{\mathbf{Volt}\frac{\partial \mathbf{P}}{\partial D}\right\}, \mathbf{v}\right) - \left(\Big[\mathbf{R} - \mathbf{Volt}\Big]S\frac{\partial \mathbf{P}}{\partial S}, \mathbf{v}\right) + \left(\mathbf{\Lambda}\ \mathbf{P}, \mathbf{v}\right) + \\ &+ \quad \left(\left[\mathbf{\Phi} + \mathbf{Volt} + \frac{\partial}{\partial D}\Big\{\mathbf{Volt}\Big\}\right]\frac{\partial \mathbf{P}}{\partial D}, \mathbf{v}\right) \geq \left(\mathbf{F}, \mathbf{v}\right),\end{aligned} \tag{42}$$

which leads

$$\begin{aligned}\left(\frac{\partial \mathbf{p}}{\partial \tau}, \mathbf{v}\right) \quad &+ \quad \left(\frac{1}{2}\mathbf{Volt}S\frac{\partial \mathbf{P}}{\partial S}, S\frac{\partial \mathbf{v}}{\partial S}\right) + \left(\mathbf{Volt}S\frac{\partial \mathbf{P}}{\partial D}, \frac{\partial \mathbf{v}}{\partial S}\right) + \\ &+ \quad \left(\frac{1}{2}\mathbf{Volt}\frac{\partial \mathbf{P}}{\partial D}, \frac{\partial \mathbf{v}}{\partial D}\right) - \left(\Big[\mathbf{R} - \mathbf{Volt}\Big]S\frac{\partial \mathbf{P}}{\partial S}, \mathbf{v}\right) + \\ &+ \quad \left(\left[\mathbf{\Phi} + \mathbf{Volt} + \frac{\partial}{\partial D}\Big\{\mathbf{Volt}\Big\}\right]\frac{\partial \mathbf{P}}{\partial D}, \mathbf{v}\right) + \left(\mathbf{\Lambda}\ \mathbf{P}, \mathbf{v}\right) \geq \left(\mathbf{F}, \mathbf{v}\right).\end{aligned}$$

Thus, the problem of American put options pricing can be formulated as follows

PROBLEM IAPHRV: For $\tau \in [0, T]$ we find $P \in \mathcal{K}$, $P(S, D, \tau) = \Big[P^+(S, D, \tau) + P^-(S, D, \tau)\Big]/2$, satisfying the variational inequality

$$\left(\frac{\partial \mathbf{P}}{\partial \tau}, \mathbf{v}\right) + a(\mathbf{P}, \mathbf{v}) \geq \left(\mathbf{F}, \mathbf{v}\right), \quad \forall\ \mathbf{v} \in \mathcal{W}_0,$$

and the initial condition

$$\left(\mathbf{P}(0), \mathbf{v}\right) = \begin{bmatrix} \left(P^+(0), v^+\right) \\ \\ \left(P^-(0), v^-\right) \end{bmatrix}, \quad \forall\ \mathbf{v} \in \mathcal{W}_0,$$

with the bilinear form given by

$$
\begin{aligned}
a(\mathbf{P},\mathbf{v}) &= \left(\frac{1}{2}\mathbf{Volt}S\frac{\partial \mathbf{P}}{\partial S}, S\frac{\partial \mathbf{v}}{\partial S}\right) + \left(\mathbf{Volt}S\frac{\partial \mathbf{P}}{\partial D}, \frac{\partial \mathbf{v}}{\partial S}\right) + \left(\frac{1}{2}\mathbf{Volt}\frac{\partial \mathbf{P}}{\partial D}, \frac{\partial \mathbf{v}}{\partial D}\right) + \Big(\mathbf{\Lambda}\ \mathbf{P},\mathbf{v}\Big) - \\
&- \left(\Big[\mathbf{R}-\mathbf{Volt}\Big]S\frac{\partial \mathbf{P}}{\partial S},\mathbf{v}\right) + \left(\Big[\mathbf{\Phi}+\mathbf{Volt}+\frac{\partial}{\partial D}\Big\{\mathbf{Volt}\Big\}\Big]\frac{\partial \mathbf{P}}{\partial D},\mathbf{v}\right) \qquad (43)
\end{aligned}
$$

On existence and uniqueness of this kind of parabolic variational inequality see, for example, Brézis (1984).

Next we present a finite element approximation to *PROBLEM IAPHRV*. in the space domain combined with an implicit finite difference approximation in the time domain.

3.3 Finite Element Approximation

For the construction of a fully discrete approximation we introduce the set

$$\mathcal{W}^A_{\Delta t,h} = \{\mathbf{P}^n_h \in \mathcal{W}_{\Delta t,h}\ ;\ \mathbf{P}^n_h(S,D,\tau_n) \geq g(S)\}.$$

We should point out that the inequality arising in the previous definition will be considered only in nodal points of the triangular mesh.

Thus the fully discrete problem for *put american options* is given by

PROBLEM APHRhm: For $n = 0,1,2,\cdots$, we find $P^n_h = \Big[P^{n,+}_h + P^{n,-}_h\Big]/2 \in \mathcal{W}^A_{\Delta t,h}$ such that

$$\left(\frac{1}{\Delta\tau}\Big\{\mathbf{P}^{n+1}_h - \mathbf{P}^n_h\Big\}, \mathbf{v}_h - \mathbf{P}^{n+1}_h\right) + a(\mathbf{P}^{n+1}_h, \mathbf{v}_h - \mathbf{P}^{n+1}_h) \geq \Big(\mathbf{F}_h, \mathbf{v}_h\Big),\ \mathbf{v}_h \in \mathcal{W}_h \cap H^1_0(\Omega) \qquad (44)$$

with initial condition

$$\Big(\mathbf{p}^0_h, \mathbf{v}_h\Big) = \Big(\mathbf{p}(0), \mathbf{v}_h\Big)\ , \quad \forall \mathbf{v}_h \in \mathcal{W}_h \cap H^1_0(\Omega)\ .$$

4 SOLUTION OF THE ALGEBRAIC PROBLEMS

The numerical results were obtained with the Picard's Algorithm with the method of Successive Over-Relaxation - SOR(ω) for European option pricing. This last method was replaced by method of Successive Over-Relaxation SOR(ω) with projection on the convex set when the American option pricing problem is solved.

4.1 Iterative Method of Successive Over-Relaxation- SOR(ω)

In iterative methods for solution of linear systems $Ax = b$ is generally evaluated the norm of the residue of "$r = b - Ax$" of the approximated solution. The process stop when the residue satisfies a stopping criterion.

In the numerical approximation of the European option pricing we use the method of Successive Over-Relaxation which leads, in each time step, to a sequence of approximations to $p^n_{h,i}$, $p^{n,(k)}_{h,i}$, which uses a combination of $p^{n,(k+1)}_{h,i}$ and $p^{n,(k)}_{h,i}$, i.e,

$$p^{n,(k+1)}_{h,i} = (1-\omega)p^{n,(k)}_{h,i} + \omega p^{n,(k+1)}_{h,i}$$

leading to

$$p^{n,(k+1)}_{h,i} = (1-\omega)p^{n,(k)}_{h,i} + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}p^{n,(k+1)}_{h,j} - \sum_{j=i+1}^{n} a_{ij}p^{n,(k)}_{h,j}\right).$$

For the American option pricing, in the numerical computation of an approximation for the solution of the system (44), we use the iterative method of Successive Over-Relaxation - SOR(ω) with projection on convex set at each instant of time, i.e.,

$$\widetilde{P}_{h,j}^{n,(k+1)} = (1-\omega)P_{h,i}^{n,(k)} + \frac{\omega}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}P_{h,j}^{n,(k+1)} - \sum_{j=i+1}^{n} a_{ij}P_{h,j}^{n,(k)}\right]$$

$$P_{h,j}^{n,(k+1)} = \min\left[\widetilde{P}_{h,j}^{n,(k+1)}, g(S)\right]$$

where a_{ij} are the coefficients of the matrix of the system of inequations (44) resulting from the piecewise linear approximation. In the case of put options $g(S) = (E-S)^+$ and $g(S) = (S-E)^+$ for call option.

The iterative method converges to $0 < \omega < 2$. For $\omega = 1$ would be the equivalent to the Gauss-Seidel method, $\omega < 1$ has been under-relaxation and $\omega > 1$ has been over-relaxation.

4.2 Picard's Algorithm

In all time steps is performed such procedures.

1. For $n = 1, ..., N$:

   i - Resets the force vectors, $F_- = \mathbf{0}$ and $F_+ = \mathbf{0}$;

   ii - Loop to adjust the force vector. For $const = 1, ..., 10$:

   A . Solve the equation for positive past information, $V^+(S, D, \tau_n)$.

   B . Solve the equation for negative past information, $V^-(S, D, \tau_n)$.

   C . Adjust the force vector positive and negative, $F_+ = \Delta t\ \lambda_-\ V^{+,i}_{j(1+h_-)}$ and $F_- = \Delta\tau\ \lambda_+\ V^{-,i}_{j(1+h_+)}$.

5 NUMERICAL RESULTS

The aim of the this section is to illustrate some important aspects concerning the application of numerical methods in the financial market, more precisely, the use of finite element methods for numerical solution of equations and inequations related to European and American pricing of options with memory in assets and volatility.

The code of implementation for the finite element method is written in Matlab in the spirit of the series Alberty et al. (1999, 2002); Cartensen and Klose (2002) and more Becker et al. (1981)

We remark that the purpose of the simulations is to provide an easier overview of the theoretical results presented in this paper.

5.1 Example: Recovering the Standard Black and Scholes Model

The economics parameters of modeling are: $\$36.00$ is o initial price of underlying asset, exercise price is $\$40$, interest rate is $6\%/year$, Here in this example the volatility is constant (σ) de $30\%/year$ and the time of maturity of the option is one year. The memory parameters are zeros.

In the implementation of the Finite Element Method we have the addition of the following parameters: 51 is the number of step of time and 51 is the number of é número de underlying asset (space).

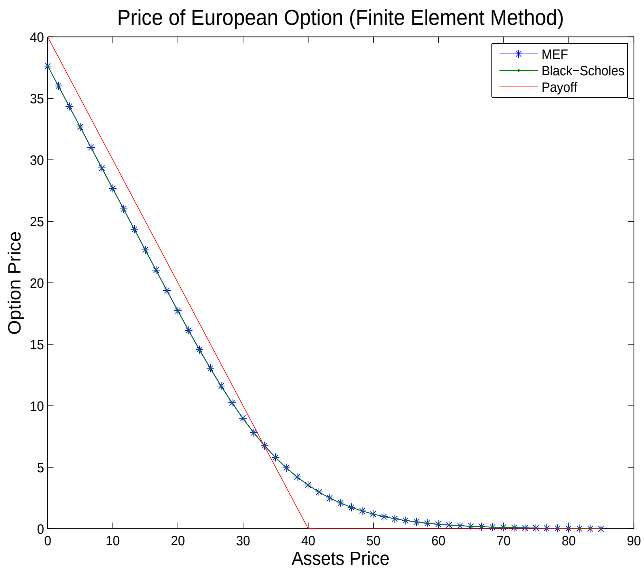

Figure 1: European option without memory, assumptions of Black and Scholes

Verified by the figure, Fig. 1, that we recover the Black and Scholes model, ie has the numerical value found by the Finite Element Method is equal to the Black and Scholes formula. Detailed study of the convergence of Numerical Solution for Black-Scholes formula is presented in the master's thesis Thomaz (2005). In the next figure, Fig. 2, we see the option value over time of maturation.

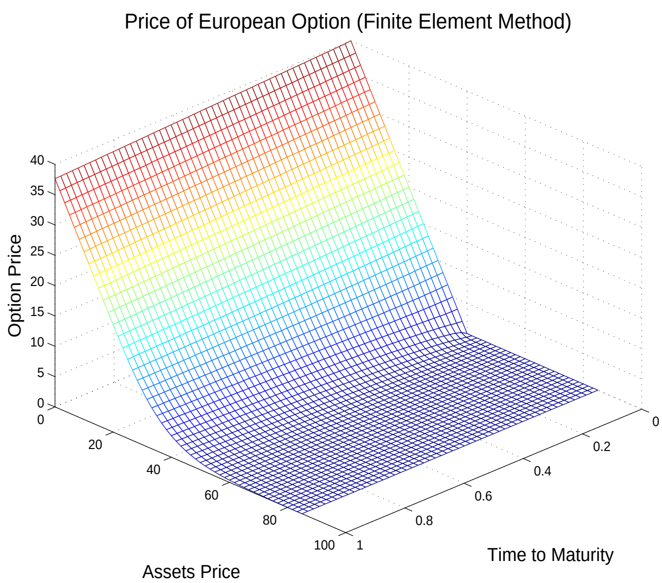

Figure 2: Over time

5.2 Example: Memory in Assets and Stochastic Volatility of Hobson and Rogers

In this example, we present an European put option. The Economics parameters are: $\$36.00$ is o initial price of underlying asset, exercise price is $\$50$, interest rate is $20\%/year$. Here in this example the volatility ($\sigma(D)$) is not constant being a function of D, i.e. volatility function of Hobson and Rogers. The time of maturity of the option is one year.

The parameter η is said to be the minimal level of implied volatility and is found in the volatility function $\sigma(D)$. We shall use $\eta = 0.4$. Furthermore we have $\varepsilon = 5$, a scaling parameter for the influence of the initial offset in the volatility function $\sigma(D)$, $\theta = 1$, the rate that past information gets discounted in the offset function. $\lambda = 0.1$ is discontinuity parameter of the underlying asset.

The method numerical parameters are: 31 is the number of step of time, 51 is the number of é número de underlying asset (space) and 41 is the number of step of Deviation. To generate numerical solution we have plane over $S \times D$. This example we use $D \in [-2; 2]$ and $S \in [0; 122.14]$.

We found that over time the behavior of the solution curve has a slope greater due to the parameters of the memory, both in the price as in the volatility.

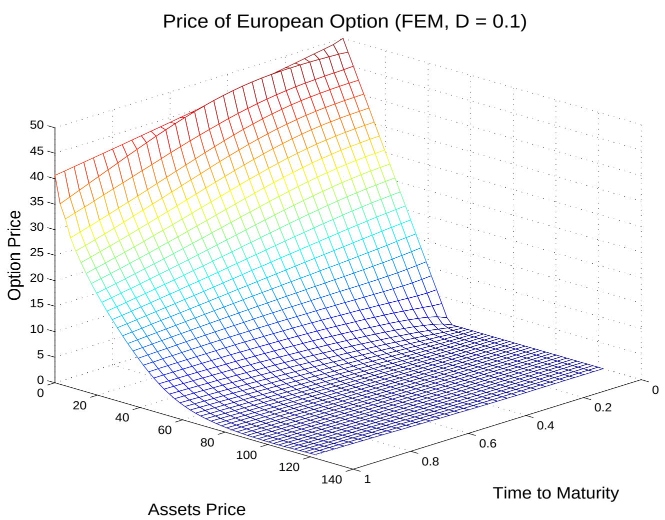

Figure 3: Option pricing in over time

American put option with the same parameters.

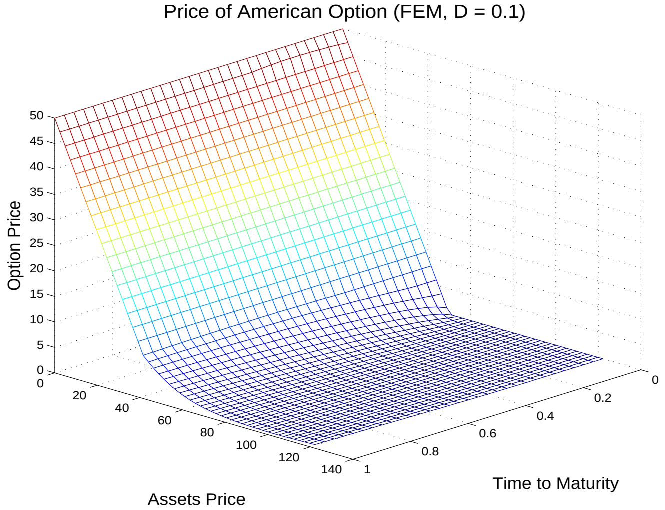

Figure 4: Option pricing in over time

If we use $\eta = 0.35$ with $\varepsilon = 5$, almost no influence of the initial offset in the volatility function $\sigma(D)$, $\theta = 1$, the rate that past information gets discounted in the offset function. $\lambda = 0.1$ is discontinuity parameter of the underlying asset. The Economics parameters are, $\$36.00$ is o initial price of underlying asset, exercise price is $\$80$, interest rate is $10\%/year$. The time of maturity of the option is one year. We find European put option pricing in $D = 0.1$

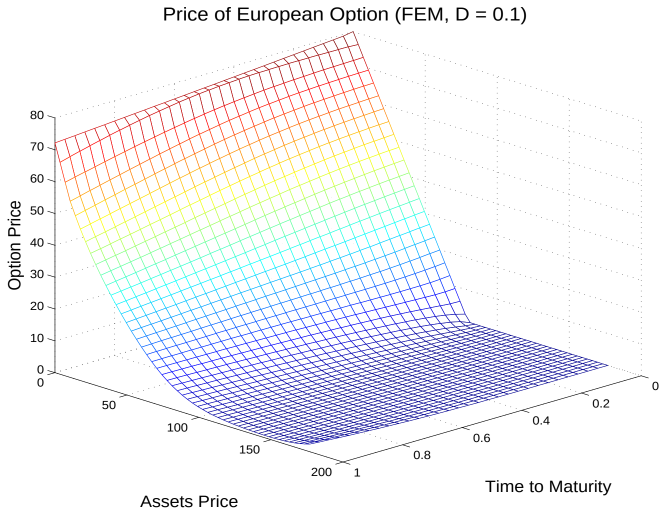

Figure 5: Option pricing in over time

European put option pricing with the same parameters in $D = 0.5$ and $D = 1.0$,

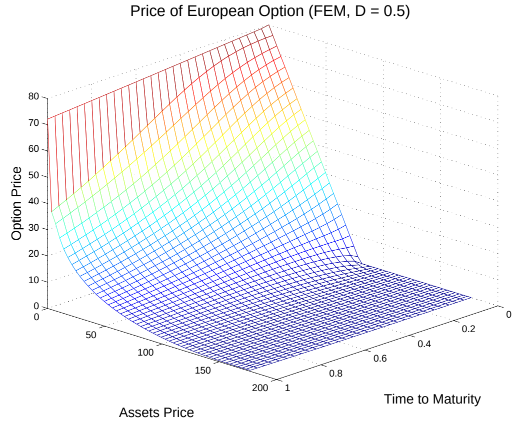

Figure 6: Option pricing in over time ($D = 0.5$)

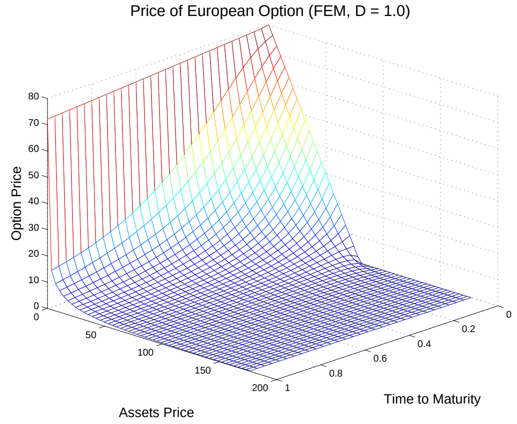

Figure 7: Option pricing in over time ($D = 1.0$)

We observed that for higher values of D we have a increase in the value of the volatility function which results in one greater displacement of the option price curve over time.

6 CONCLUSIONS

In this paper a new model for the evolution of option pricing when memory price and volatility underlying asset are considered. We combine models where certain memory effect is presented in the price of the underlying asset due to Crescenzo and Pellerey (2002) and developed later by Ratanov (2007), with the original class of models of non-constant volatility developed Hobson and Rogers (1998).

The use of Jump-Telegraph-Diffusion-Drift processes (JTDD process) for underlying asset models leads to an increasing of the difficulties in the modeling process. In fact, such JTDD process induces a system of equations involving the options pricing into different assets pricing: system (25) for put European options; system (26) for call European options; system of differential inequalities (37) for put American options.

The solution of the previous systems of PDEs were approximated using a combination between Galerkin method and Implicit Euler's method. The fully discrete problem was numerically solved using the Picard's algorithm.

A direct extension for this work would be to consider the stochastic interest rate where a three-factor model will obtained. Empirical studies in this case have crucial importance, mainly for the calibration of volatility function, similar to Foschi and Pascucci (2009) which was done for models with memory in asset pricing.

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REFERENCES

Alberty J., Cartensen C., and Funken S.A. Remarks around 50 lines of matlab: short finite element implementation. *Numerical Algorithms*, 20:117–137, 1999.

Alberty J., Cartensen C., Funken S.A., and Klose R. Matlab implementation of the finite element method in elasticity. *Cumputing*, 69:239–263, 2002.

Becker E.B., Carey G.F., and Oden J.T. *Finite Elements An Introduction*, volume 1. Prentice-Hall, 1981.

Black F. and Scholes M. The pricing of options and corporate liabilities. *Journal of Political Economy*, 4:323–338, 1973.

Brézis H. *Análisis Funcional Teoria y Aplicaciones*. Alianza editorial, Madrid, 1984.

Cartensen C. and Klose R. Elastoviscoplastic finite element analysis in 100 lines of matlab. *J. Numer. Math.*, 10:157–192, 2002.

Crescenzo A.D. and Pellerey F. On price's evolutions based on geometric telegrapher's process. *Applied Stochastic Models in Business and Industry*, 18(2):171–184, 2002.

Foschi P. and Pascucci A. Calibration of a path-dependent volatility model: Empirical tests. *Computational Statistics and Data Analysis*, 53:2219–2235, 2009.

Francesco M.D. and Pascucci A. On the complete model with stochastic volatility by hobson and rogers. *Proc. R. Soc. Lond. A*, 8(460):3327–3338, 2004.

Goldstein S. On diffusion by discontinuous movements and the telegraph equation. *The Quarterly Journal of Mechanics and Applied Mathematics*, 4:129–156, 1951.

Hobson D. and Rogers L. Complete models with stochastic volatility. *Math. Finance*, 8:27 – 48, 1998.

Kac M. A stochastic model related to the telegrapher's equation. *Rocky Mountain Journal of Mathematics*, 4:497–509, 1974.

Platania A. and Rogers L. Putting the hobson-rogers model to the test. *Preprint*, 2003.

Ratanov N. Jump telegraph processes and financial markets with memory. *Journal of Applied Mathematics and Stochastic Analysis*, 2007. Article ID 72326.

Ratanov N. Jump telegraph-diffusion option pricing. UNIMI - Research Papers in Economics, Business, and Statistics Working Paper 33, Universitá degli Studi di Milano, 2008.

Thomaz J.C.A. *Solução Numérica de Equações Diferenciais para Precificação de Opções*. Master's Thesis, Laboratório Nacional de Computação Científica, 2005.

Wilmott P., Dewynne J., and Howison S. *Option Pricing, Mathematical Models and Computation*. Oxford Financial Press., 2000.