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FINITE ELEMENT METHOD WITH ABSORBING BOUNDARY CONDITION (FEM-ABC) APPLIED TO ELECTROMAGNETIC SCATTERING PROBLEMS

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Abstract. This article deals with the application of Finite Element Method (FEM) to a two-dimensional electromagnetic scattering problem. The radar operation for detect submarines and aircraft is just one of many applications of this phenomenon in engineering. The FEM allows the treatment of geometrically complex structures containing non-homogeneous materials. Furthermore, it generates sparse matrices, which allows great computational economy. To be applied to scattering problems, the FEM requires a radiation condition incorporated into the formulation. This work uses an Absorbing Boundary Condition (ABC), which preserves the characteristics of the matrix. This choice gives results in agreement with the analytic solution. The error and sensitivity analysis are evaluated.

1 INTRODUCTION

The electromagnetic scattering occurs when an electromagnetic wave iluminates an object and induces electric current on it. This object, in turn, due to interaction with the incident wave, emits a scattered electromagnetic wave, that overlaps the incident wave. By knowing the characteristics of the object, or scatterer, one can predict the shape of the scattered wave. The radar operation for detect submarines and aircraft (Liu, 2004), the modeling of forests structure by remote sensing (Li, 1998), the study of integrated circuits and printed circuit boards through the analysis of microstrips (Gürel, 1996) are based on that phenomenon.

The electromagnetic scattering problem is an open problem, i.e, the scattered field propagates in all directions without limits. Thus, it is necessary to put an artificial boundary at some distance from scatterer to truncate the geometry of the problem. The region inside this boundary includes the problem domain. On the border, a boundary condition is imposed so that the scattered wave is absorbed by it (Li, 1993). This work uses the Absorbing Boundary Condition (ABC). First order ABC's are widely used because they are simple to implement and have a low computational cost. Moreover, when the boundary should be very close to the scatterer, or when great accuracy is required, the use of a second order ABC is indicated (Lee, 2007). A detailed study of first and second orders ABC's for general problems in two and three dimensions is done in (Stupfel, 1994).

The electromagnetic scattering problem can be solved analytically or by numerical computational methods. The analytical solution, although exact, can be obtained only in very special cases, drastically limiting the number of problems treated. The Finite Element Method (FEM), proposed by R.L. Courant in 1943 (Pelosi, 2007) and first used to solve problems of analysis of structures, is now widely used to solve problems of both electromagnetism and several other areas. For a general review on FEM, see (Coccioli, 1996).

This paper aims to solve the electromagnetic scattering problem due to a plane electromagnetic (EM) wave incident on a scatterer with simple geometry. The Absorbing Boundary Condition of Bayliss-Turkel is used to truncate the computational domain. Initially, the mathematical formulation for the proposed problem is deduced, for both first order and second order ABC. Then, this formulation is validated through a software that solves the problem and compares it with the analytic solution. Then, a sensitivity analysis is done, where some factors that affect the accuracy of results are tested. Finally, in conclusion, is presented a synthesis of the results.

2 MATHEMATICAL FORMULATION

2.1 Problem description

When an EM wave, described by the electric incident field ($\vec{E}^i$) and the magnetic incident field ($\vec{H}^i$), hits on the scatterer, another wave, described by the electric scattered field ($\vec{E}^s$) and the magnetic scattered field ($\vec{H}^s$), is generated. The overlap of these waves changes the total electric and magnetic fields in all the domain for $\vec{E}$ and $\vec{H}$, respectively. So, the electric field is given by:

$$\vec{E} = \vec{E}^i + \vec{E}^s . \tag{1}$$

The same goes for the $\vec{H}$ field. When the incident EM wave is perpendicular to the axis of an infinite scatterer of arbitrary cross section, there is no variation in one direction, say z. In this case, the problem can be solved in two dimensions, through the determination of the fields $\vec{E}$ and $\vec{H}$ in points of a cross section of it. Figure 1 shows an arbitrary two-dimensional scatterer Ω, immersed in the free space Ω_0 and illuminated by a plane and uniform EM wave, which focuses on the scatterer at an angle θ^i.

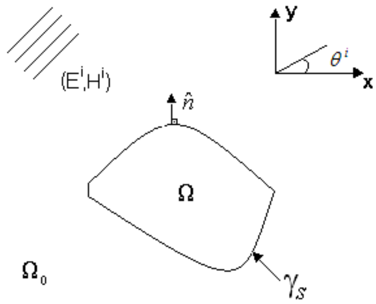

Figure 1: Geometry of an arbitrary two-dimensional scatterer

In this figure, γ_s is the boundary of the scatterer and $\hat{n}$ is the unit exterior normal vector. The above phenomenon is described by the generalized Helmholtz equation (Jin, 1993):

$$\nabla \cdot (\alpha_1 \nabla U) + k_0^2 \alpha_2 U = 0, \qquad \forall \vec{x} \in \Omega. \tag{2}$$

In equation (2), $\alpha_1 = \mu_r^{-1}$ and $\alpha_2 = \varepsilon_r$ when $U = \vec{E}$ and $\alpha_1 = \varepsilon_r^{-1}$ and $\alpha_2 = \mu_r$ when $U = \vec{H}$. Ω is the region inside the scatterer and k_0 is the wave number for free space.

2.2 Weak formulation

The equation (2) defines the so-called strong formulation for the problem, since the *U* field must obey it at all points of the domain. This is a very strong demand for problems to be solved numerically. Instead, is allowed a residual for the strong form (Afonso, 2003):

$$R = \nabla \cdot (\alpha_1 \nabla u) + k_0^2 \alpha_2 u. \tag{3}$$

where the *u* field in equation (3) is an approximation for the exact one in equation (2).

Through the residual weighted method, the so-called weak formulation can be obtained by integration of the residue around the entire domain, Ω, through a weighting function, *w*, and equating the integral to zero (Afonso, 2003):

$$\int_\Omega [\nabla \cdot (\alpha_1 \nabla u)] \cdot w d\Omega + \int_\Omega k_0^2 \alpha_2 w \cdot u d\Omega = 0. \tag{4}$$

Using vector identities and the divergence theorem, the equation (4) can be written as:

$$\int_\Omega \nabla w \cdot (\alpha_1 \nabla u) d\Omega - \int_\Omega k_0^2 \alpha_2 w \cdot u d\Omega - \int_{\gamma_e} \alpha_1 w \frac{\partial u}{\partial n} d\gamma_e = 0. \tag{5}$$

2.3 The Absorbing Boundary Condition

The first two integrals in equation (5) are valid in the domain Ω and are handled directly by the FEM. The third one, nevertheless, requires a special treatment. The normal derivative of the field must satisfy the Sommerfeld's radiation condition (Jin, 1993). However, the implementation of the formulation across an ABC requires a domain extension. Figure 2 illustrates this domain extension, now enclosed by a boundary γ_e, slightly out of the scatterer. All other parameters remain unchanged. The choice of the location of this boundary is influenced by two conflicting interests: accuracy and computational cost. The farther away from the scatterer is placed the ABC boundary, the greater is the precision. On the other hand, the closer it is, the smaller becomes the computational cost.

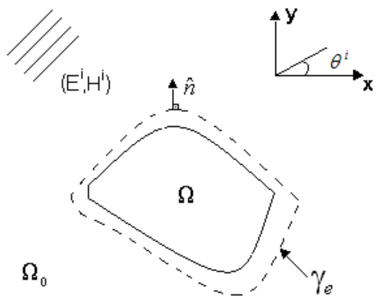

Figure 2: Geometry of the scatterer of Figure 1, after incorporating the boundary γ_e.

To solve this problem computationally, an ABC must be incorporated in equation (5). So, it is assumed that the scattered field can be expressed in the following asymptotic form (Jin, 1993; Peterson, 1989):

$$u^s = \frac{e^{-ik\rho}}{\sqrt{\rho}} \sum_{n=0}^{\infty} \frac{A_n(\varphi)}{\rho^n}. \tag{6}$$

where (ρ,φ) are polar coordinates.

Equation (6) is the two-dimensional version of the well known Wilcox's expansion to EM fields (Wilcox, 1956). The order of the ABC is related to the number of terms considered in this series, and, in most papers, it is imposed over γ_e a first or second order ABC. In both cases, one takes the derivative of the scattered field with respect to ρ and it is obtained, after mathematical manipulations (Jin, 1993):

$$\alpha_1 \frac{\partial u}{\partial n} = q - \gamma u . \tag{7}$$

The first order ABC is obtained when considering only the first term of equation (6). Thus, the values found for γ and q are:

$$\gamma = \alpha_1 \left[ik_0 + \frac{\kappa}{2} \right], \text{ and} \tag{8}$$

$$q = \alpha_1 \frac{\partial u^i}{\partial n} + \alpha_1 \left[ik_0 + \frac{\kappa}{2} \right] u^i . \tag{9}$$

where $\kappa = 1/\rho$ and $u^i = u - u^s$. Since the incident field is known, $u^i = e^{ik_0(x\cos\theta_i + ysen\theta_i)}$, the terms γ in equation (8) and q in equation (9) are constant for a specific linear element.

To obtain the second order ABC, the first four terms of equation (6) are considered. Thus, the values of γ and q are given by:

$$\gamma = \alpha_1 \left[ik_0 + \frac{\kappa}{2} - \frac{i\kappa^2}{8(i\kappa - k_0)} - \frac{i}{2(i\kappa - k_0)} \frac{\partial^2}{\partial \gamma_e^2} \right], \text{ and} \tag{10}$$

$$q = \alpha_1 \frac{\partial u^i}{\partial n} + \alpha_1 \left[ik_0 + \frac{\kappa}{2} - \frac{i\kappa^2}{8(i\kappa - k_0)} \right] u^i - \frac{i\alpha_1}{2(i\kappa - k_0)} \frac{\partial^2 u^i}{\partial \gamma_e^2} . \tag{11}$$

It is noteworthy that the term γ in equation (10) is not constant, but a differential operator. The term q in equation (11), however, remains constant. Equations (7) to (11) define the Bayliss-Turkel ABC (Bayliss, 1982).

2.4 Incorporation of the ABC

The FEM formulation with first order ABC is obtained by direct substitution of the values of γ and q given by equations (8) and (9), respectively:

$$\int_\Omega \nabla w \cdot (\alpha_1 \nabla u) d\Omega - \int_\Omega k_0^2 \alpha_2 w \cdot u d\Omega - \int_{\gamma_e} q \cdot w d\gamma_e + \int_{\gamma_e} \gamma \cdot w \cdot u d\gamma_e = 0 . \tag{12}$$

where γ_e is the ABC boundary.

However, when the values of γ and q are given by equations (10) and (11), respectively, the obtainment of the FEM formulation with second order ABC is a bit more elaborate. Using integration by parts, it is obtained, after mathematical manipulations, the following expression:

$$\int_\Omega \nabla w \cdot (\alpha_1 \nabla u) d\Omega - \int_\Omega k_0^2 \alpha_2 w \cdot u d\Omega - \int_{\gamma_e} q \cdot w d\gamma_e + \int_{\gamma_e} \gamma_1 \cdot w \cdot u d\gamma_e$$

$$- \int_{\gamma_e} \gamma_2 \cdot \frac{\partial u}{\partial \gamma_e} \cdot \frac{\partial w}{\partial \gamma_e} d\gamma_e + \gamma_2 \cdot w \cdot \frac{\partial u}{\partial \gamma_e} \Bigg]_{(x_1, y_1)}^{(x_2, y_2)} = 0 . \tag{13}$$

where (x_1, y_1) and (x_2, y_2) are the coordinates of the boundary element in question. The constants γ_1 and γ_2 in equation (13) are related with γ in equation (10) by:

$$\gamma = \gamma_1 + \gamma_2 \frac{\partial^2}{\partial \gamma_e^2} \text{, where} \tag{14}$$

$$\gamma_1 = \alpha_1 \left[ik_0 + \frac{\kappa}{2} - \frac{i\kappa^2}{8(i\kappa - k_0)} \right] \text{, and} \tag{15}$$

$$\gamma_2 = -\frac{i\alpha_1}{2(i\kappa - k_0)} . \tag{16}$$

It is noticed that the second order FEM-ABC formulation, given by equation (13), reduces to first order one, given by equation (12), when $\gamma_2 = 0$ and γ_1 is calculated by equation (8).

3 RESULTS

The surface of many practical scatterers, such as fuselage of aircraft, missiles, etc. can be often represented by cylindrical structures (Li, 1998). Therefore, the cylinders are one of the most important classes of geometric surfaces (Balanis, 1989). The circular cylinder, due to its simplicity and the fact that the solution can be represented in terms of well known and tabulated functions (such as Bessel functions and Hankel functions) is one of the geometry most widely used to represent practical scatterers.

The formulation developed in the previous section – equations (12) and (13) – allows solving 2D electromagnetic scattering problems of arbitrary cross section. The problem proposed in this work consists of a dielectric circular cylinder of radius *a* and infinite in the *z* direction, illuminated at normal incidence by a monochromatic and uniform plane wave, as shown in figure 3.

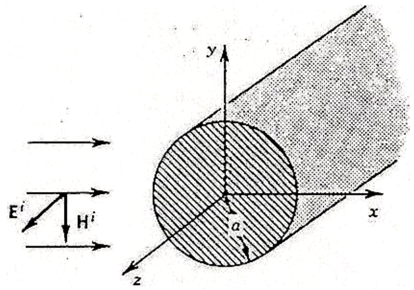

Figure 3: Description of the scatterer and the incident wave. Figure extracted and adapted from [14].

Figure 4 shows the geometry of this problem, a cross section of figure 3. This figure is essentially a simplification of the geometry shown in figure 2.

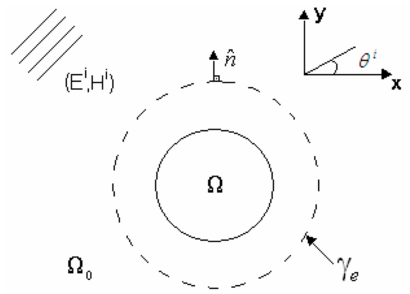

Figure 4: The geometry of the proposed problem.

After these simplifications, the validation of the formulation is made based on the calculation of the electric scattered field by the cylinder described in Figure 3. This is a classical problem of electromagnetic theory and it has analytical solution (Harrington, 1961). For comparison, the following parameters are chosen: angle of incidence of the wave, $\theta^i = 180^0$; frequency of the incident wave, f = 0.3 GHz (λ = 1 m); radius of the cylinder, a = 0.3 λ; relative electrical permittivity of the material that forms the cylinder, ε_r = 3.0; relative magnetic permeability of the same material, μ_r = 1.0.

In this paper, the domain is discretized by the software *Triangle* (Shewchuk, 1996), a two-dimensional mesh generator of high quality that makes the Delaunay triangulation of a region from the description of the boundary of the region.

3.1 The FEM formulation with first order ABC

The results presented below are intended to continue the studies done in (Barbosa, 2009; Barbosa, 2010). Initially, the ABC boundary is set at 0.6λ from the surface of the cylinder. When the maximum area of any one element is fixed by the Triangle in, say, 0.002λ^2, is generated a mesh, called mesh 1. This mesh has 2,384 elements, 296 of which are inside the scatterer, and 1,328 nodes, 90 of them on the surface of the cylinder and other 270 on the ABC boundary.

The first order FEM-ABC formulation, obtained in equation (12), when implemented from mesh 1, provides the solution presented in figure 5.

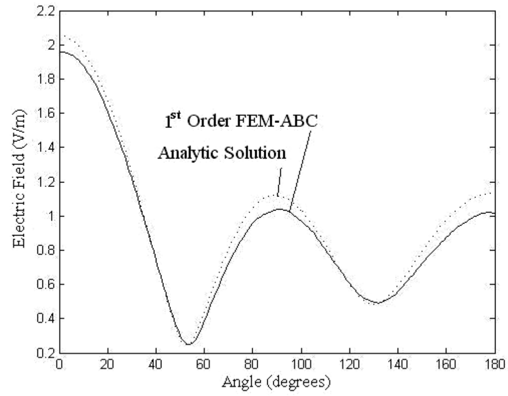

Figure 5: Total electric field on the surface of dielectric cylinder:
comparison between the first order FEM-ABC solution and the analytical one.

In this figure, the total electric field is calculated on the surface of the cylinder. The numerical solution is shown in continuous line, while the analytical one is shown in dotted line. There is a reasonable agreement between the two solutions.

To evaluate the distance between these two solutions, the percentage error at each point on the surface of the cylinder is determined by:

$$\Delta E(\%) = \frac{\left| E_{exact} - E_{calculated} \right|}{E_{exact}} \times 100 \,. \tag{17}$$

This result and the average percentage error (6.0%) are presented in figure 6.

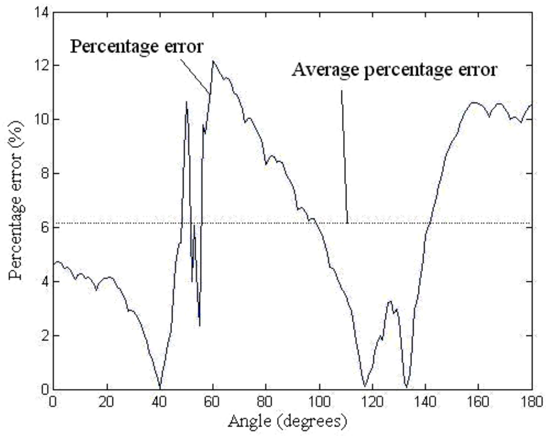

Figure 6: Percentage error and average percentage error when using a first order ABC.

Then, the ABC boundary is placed 1.2λ of the scatterer's surface. The maximum area of an element, which determines the mesh density, remains the same. For this farther away boundary, the mesh generated by Triangle, called mesh 2, has 6,024 elements and 3,238 nodes, with 450 of these on the ABC boundary. The number of elements inside the scatterer and the number of nodes on the surface of the cylinder remains the same.

The information in this mesh, compared with mesh 1, indicates that when the ABC boundary is placed more distant from the scatterer, the number of nodes and elements generated increases significantly, and this increases the computational cost. For mesh 2, the first order FEM-ABC formulation gives the solution shown in figure 7.

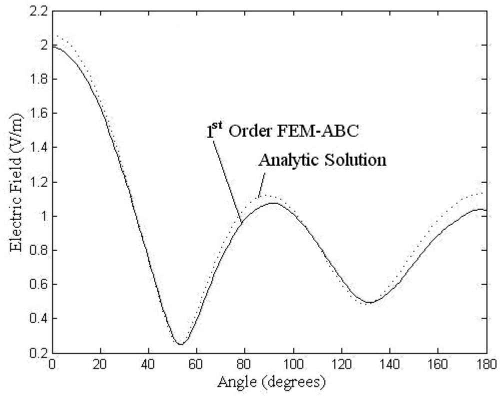

Figure 7: Solution with first order FEM-ABC for mesh 2.

The average percentage error in this case is 5.2%. It is noticed that the removal of the ABC boundary contributes to increase the accuracy of results. This is because the equation (6) is an approximate expression of the *u* field in the near-field region, which is more accurate the farther away the ABC boundary is placed the scatterer.

3.2 The FEM formulation with second order ABC

The same problem is, then, solved from the second order FEM-ABC formulation, presented in equation (13). The validation of the formulation presented in this equation is the more important objective of this paper. When the implementation is made from mesh 1, is obtained the solution shown in figure 8, with an average percentage error of 5.1%.

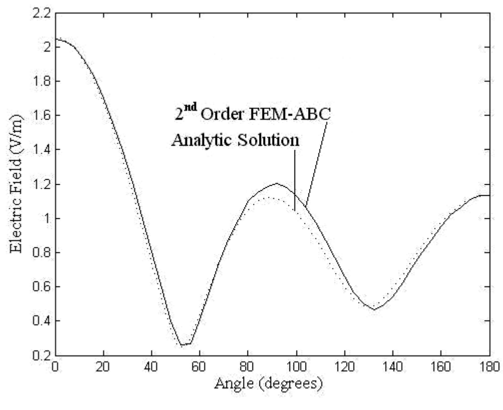

Figure 8: Solution with second order ABC for mesh 1.

When, however, the implementation is made from mesh 2, the average percentage error is reduced to 2.1%. Figure 9 shows this solution. In this case, the fit of the curves is significantly larger.

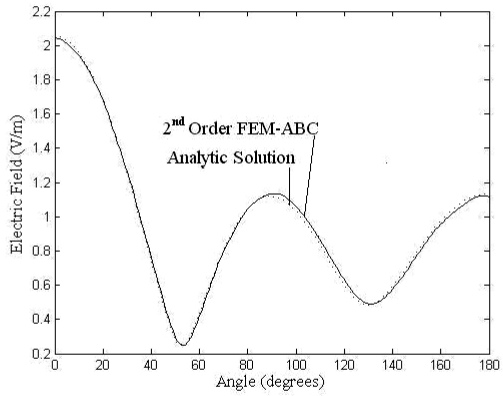

Figure 9: Solution with second order ABC for mesh 2.

Note that, for both meshes, the results are more accurate with a second order ABC. In particular, the increase in accuracy is highest when there is a combination of factors: use of higher order ABC in a boundary further away.

To measure the relationship between these factors and the computational cost, the total processing time is calculated for each implementation obtained. These times are shown in Table 1.

	1^{st} order ABC	2^{nd} order ABC
Mesh 1	1.8	1.7
Mesh 2	8.0	8.2

Table 1: Processing times, in seconds

The hardware used is a computer Intel Core 2 Duo processor, 3GHz, 2GB DDR2. An analysis of this table shows that the use of a higher order ABC practically does not affect the computational cost. But the remoteness of the ABC boundary increases significantly this time. This is due to the increase in the size of the array to be reversed, where the number of rows or columns is given by the number of nodes in the mesh.

With this paper one can emphasize several advantages of the FEM-ABC. A first one is the generation of sparse matrices. For example, for the array generated from the mesh 2, which has 3,238x3,238 elements, only 21,760 of them (0.21%) are not null. The location of these terms is represented by dark spots in figure 10. It is also observed in this figure that this matrix is symmetric.

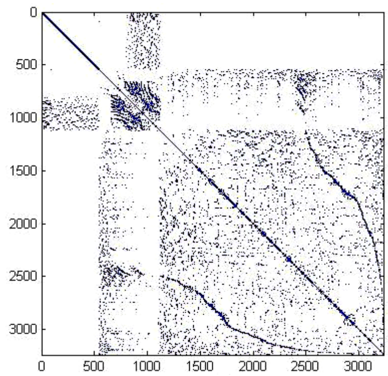

Figure 10: Location of non-zero terms in the matrix generated by the FEM-ABC for the mesh 2.

Thus, the use of efficient methods for solving sparse matrix systems allows that more complex problems are solved in reduced times, one of the main advantages of FEM-ABC. The use of such methods is proposed as future work. Another major advantage of FEM-ABC is the easiness to treats not homogeneous domains, since the value of ε_r is reported for each element.

4 CONCLUSION

This paper presents a study of two-dimensional electromagnetic scattering. It is shown that this problem has many practical applications in engineering and can be successfully solved by the Finite Element Method. An absorbing boundary condition of Bayliss-Turkel is incorporated in the formulation, which gives the solution of the problem. The results agree with the analytical solution. Both error and sensitivity analysis show that the accuracy of results increases when the boundary moves away from the scatterer or when a second order ABC is used. The determination of the density of matrix obtained and the processing time in each case show the importance of using efficient methods of solution of sparse matrix systems.

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