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A NEW PARAMETER TO ARC-LENGTH METHOD IN NOLINEAR STRUCTURAL ANALYSIS

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Abstract. Path following methods have been widely used in nonlinear analysis due to their greater efficiency in tracing the equilibrium trajectories of a structure with nonlinear behavior. The arc-length method is the most powerful path following method in the solution of equilibrium paths, bifurcation points and limit points by introducing a constraint condition which establishes the nonlinear equations in which the unknown load parameter is determined. Unfortunately, this method presents some difficulties and imperfections in the control of load increment to reach convergence at specific locations along the trajectories of load-deflection. Therefore, this paper proposes to present a modification in the arc-length method introduced by Crisfield (M.A.Crisfield, Compt. Struct., 13:55-62 (1981)) with the objective of improving the performance analysis of equilibrium paths. The mentioned change was firstly proposed by Teng and Luo (J.G.Teng, Y.F.Luo, Comm Num Meth Eng, 14 (1):51-58 (1998)) . It introduces a new parameter with the function to add all previous arc lengths of Crisfield's method up to now and then includes a new current load step, which can achieve convergence to levels of pre-defined values for loads or other parameter (i.e., displacements, strains). With this, the difficulties found in the arc-length method are efficiently overcome.

1 INTRODUCTION

The complete investigation of solution path of the nonlinear systems of equilibrium equations has a strong of practical interest in the study the over critical behaviors of structures, like for example in buckling analysis of shell, such also in the determination to materials instabilities and singular points. Due to this necessity, path-following methods were developed such that allow to follow arbitrary nonlinear solution paths in structural analysis.

Methods which allow general path-following are called arc-length or continuation methods. The essential idea of the path-following method is to add a constraint condition to the set of nonlinear equations from which the unknown load parameter can be determined.

Since path-following methods are well established, a number of different methods (Riks, 1979; Crisfield, 1981; Ramm, 1981; Schweizerhof and Wriggers, 1986) are available and documented in the literature. Among the many existing incremental iterative nonlinear solution methods, the arc-length method as developed by Crisfield (1981) for application in finite element analysis, appears to be the most popular.

The arc-length method is a solution strategy in which the path through a converged solution, at any step, follows a orthogonal direction to the tangent of the solution curve. In this procedure, both the load vector and the displacement field vary. Therefore, the control parameter captures the effects from both load change and displacement change, so both load limit points (snap-through points) and displacement limit points (snap-back points) can be handled effectively. While the arc-length method (or other methods with similar capabilities) can handle complex load-deflection paths effectively, the analyst has no control over the load incrementation scheme to achieve convergence in specific locations along the load-deflection path. There are a number of situations in which such control is required. When nonlinear structural analysis programs are used in structural design, both the ultimate load carrying capacity and the deformations under service loads need to be found.

The present paper aims to present an improved form of the Crisfield's arc-length method so that it can achieve convergence to pre-defined load levels. This improvement was firstly introduced for Teng and Luo (1997), and was developed for bifurcation analysis in which convergence to specified load levels is replaced by convergence to estimated values of a new parameter called accumulated arc-length. This new parameter controls the operation of the arc-length method to converge at the bifurcation point that is being sought. The inclusion of this new parameter modify the conventional arc-length method and is named by Teng and Luo (1998) as the accumulated arc-length method, that leads to a new bifurcation analysis strategy which can efficiently detect the existence of a bifurcation point located anywhere on the load deflection path. The accumulated arc-length method described in this paper can satisfactorily also be applied to analyze post-collapse bifurcation problems, but this will not be approached in this paper.

2 THE CONVENTIONAL ARC-LENGTH METHOD

The solution of nonlinear system equations for a finite element model of a structure using the conventional arc-length method, quote above, is following developed.

Amongst the most varied constraint methods, in this paper we will consider a cylindrical arc-length method developed by Crisfield (1991) adding a simpler constraint condition where the ψ is set to zero in the formulation of the spherical arc-length method defining a cylindrical surface, and therefore the influence of de loads is neglected. The constraint methods are effectively Newton Raphson procedures with an additional constraint to define the direction of iterative path/surface.

In the standard method, the iterative displacement which account for a flexible load level are given by:

$$\delta\mathbf{p}_i^n = \delta\bar{\mathbf{p}}_i^n + \delta\lambda_i^n \delta\mathbf{p_t}_i^n \tag{1}$$

where $\delta\bar{\mathbf{p}}_i^n$ are the iterative displacements for the *n*th iteration within the *i*th loading step, related to the residual force vector as in the standard Newton Raphson method.

For an arbitrary load level $\mathbf{f}$ in (generally the initial or reference load level), the tangent displacement vector may be calculated as

$$\delta\mathbf{p_t}_i^n = \mathbf{K_t}_i^{-1n}\mathbf{f} \tag{2}$$

where $\mathbf{K_t}_i^{-1n}$ is the tangent stiffness matrix formed at the beginning of the *i*th loading step and it is kept constant during subsequent iterations. At the *n*th iteration within the *i*th loading step, the displacement increments due to residual forces $\mathbf{g}_i^j$ are given by,

$$\delta\bar{\mathbf{p}}_i^n = -\mathbf{K_t}_n^{-1n}\mathbf{g}_i^j \tag{3}$$

The incremental displacement can then be updated as

$$\Delta\mathbf{p}_i^n = \Delta\mathbf{p}_{i-1}^n + \delta\bar{\mathbf{p}}_i^n + \delta\lambda_i^n \delta\mathbf{p_t}_i^n \tag{4}$$

where $\delta\lambda_i^n$ is the change of the load factor for the *n*th iteration within the *i*th loading step.

The change of load factor $\delta\lambda$ is constrained by the arc-length increment Δl_i for the *i*th loading step through the following equation:

$$\Delta\mathbf{p}_i^{nt}\Delta\mathbf{p}_i^n = \Delta l_i^2 \tag{5}$$

For the iterations, Eq. 4 and Eq. 5 give the following quadratic equation to define $\delta\lambda_i^n$:

$$a_1\delta\lambda_i^{n^2} + a_2\delta\lambda_i^n + a_3 = 0 \tag{6}$$

$$a_1 = \delta\mathbf{p_t}_i^{nt}\delta\mathbf{p_t}_i^n \tag{7}$$

$$a_2 = 2\delta\mathbf{p_t}_i^n(\Delta\mathbf{p}_{i-1}^n + \delta\bar{\mathbf{p}}_i^n) \tag{8}$$

$$a_3 = (\Delta\mathbf{p}_{i-1}^n + \delta\bar{\mathbf{p}}_i^n)^t(\Delta\mathbf{p}_{i-1}^n + \delta\bar{\mathbf{p}}_i^n) - \Delta l_i^2 \tag{9}$$

The appropriate root is one which maintains a positive angle between the original and updated displacements within the *i*th loading step, or the one closer to the linear solution if both angles are positive. The arc-length increment Δl_i for the *i*th loading step is determined by the following procedure:

$$\Delta l_i = \Delta l_{i-1}\left(\frac{I_d}{I_{i-1}}\right)^{1/2} \tag{10}$$

where Δl_{i-1} is the arc-length increment of the previous loading step, I_d is the desired number of iterations for the *i*th loading step before convergence and I_{i-1} is the number of iterations required to converge in the $(i-1)$th loading step.

3 THE ACCUMULATED ARC-LENGTH METHOD

According to the procedure, first, conventional arc-length method presented above is used to trace the load-displacement path. When the path approaches to the pre-defined state, the accumulated arc-length process (Teng and Luo, 1998) is started to achieve the desired level of convergence. For this propose the standard arc-length is modified.

Firstly a new parameter that define the sum of all arc-lengths up to and including the current load step is introduced. This new parameter is express by L_i, and as cited above, is called accumulated arc-length at the *i*th loading step L_i as following bellow

$$L_i = \sum_{k=1}^{i} \Delta l_k \tag{11}$$

where Δl_k is the arc-length increment of the *k*th loading step. The L parameter represents the current state of the structure and depends on the characteristics of the structure and its loading, and also the process of load incrementation during the analysis.

During the conventional arc-length method of Crisfield as described above is employed to trace the load deflection path until the load factor reaches a desired load factor, is necessary a continuous monitoring to see if the converged load level λ_i is near the pre-defined load level λ_d. Then, the accumulated arc-length process proposed below is started once the load factor li at the convergence of the *i*th loading step is close to λ_d.

In the formulation of the accumulated arc-length , is necessary to define a new parameter ψ just to simplify the accumulated arc-length formulation:

$$\psi(L_i) = \lambda_i - \lambda_d \tag{12}$$

The desired arc-length increment for the $(i+1)$th loading step Δl_d is to make the accumulated arc-length L_d satisfy the following equation:

$$\psi(L_d) = 0 \tag{13}$$

where

$$L_d = L_{i+1} = L_i + \Delta l_{i+1} = L_i + \Delta l_d \tag{14}$$

Taylor's expansion of equation (11) leads to

$$\psi(L_d) = \psi(L_i + \Delta l_d) = \psi(L_i) + \frac{d\psi(L_i)}{dL}\Delta l_d + \frac{1}{2}\frac{d^2\psi(L_i)}{dL^2}\Delta {l_d}^2 + ... = 0 \tag{15}$$

3.1 Linear approximation to the desired arc-length increment

In this present work, it will be omitted the second and higher order terms, resulting in a linear approximation to the desired arc-length increment Δl_d,

$$\psi(L_i) + \frac{d\psi(L_i)}{dL}\Delta l_d = 0 \tag{16}$$

With this consideration, it will reach the following equation:

$$\Delta l_d = -\frac{\psi(L_i)}{d\psi(L_i)/dL} \tag{17}$$

Where the first derivative of ψ with regard to L at loading steps i and i-1 are approximated by the following backward finite difference is,

$$\frac{d\psi(L_i)}{dL} = \frac{\lambda_i - \lambda_{i-1}}{L_i - L_{i-1}} \tag{18}$$

Introducing Eq. 17 into Eq. 18, the new arc-length increment Δl_d for the $(i-1)$th loading step in order to reach the desired load level is found as

$$\Delta l_d = \frac{\lambda_d - \lambda_i}{\lambda_i - \lambda_{i-1}} (L_i - L_{i-1}) \tag{19}$$

The resulting formula developed above is the formula for the desired arc-length increment.

In the solution process the magnitude of Δl achieved, should be compared with each other and the smaller one is used as the arc-length.

4 NUMERICAL EXAMPLES

The method proposed here has been coded into the latest version of the MATLAB program. Two numerical examples are presented below to demonstrate the validity and capability of the proposed method.

The first example is a simple truss structure with two bar element and the truss has geometric symmetry, see Figure 1. But a pertubation was introduced in material properties setting a Young's Modulus different in each bar, in the first bar the Young's Modulus adopted was 1.07×10^5 MPa. The pre-defined load level for convergence considered was 300 kN.

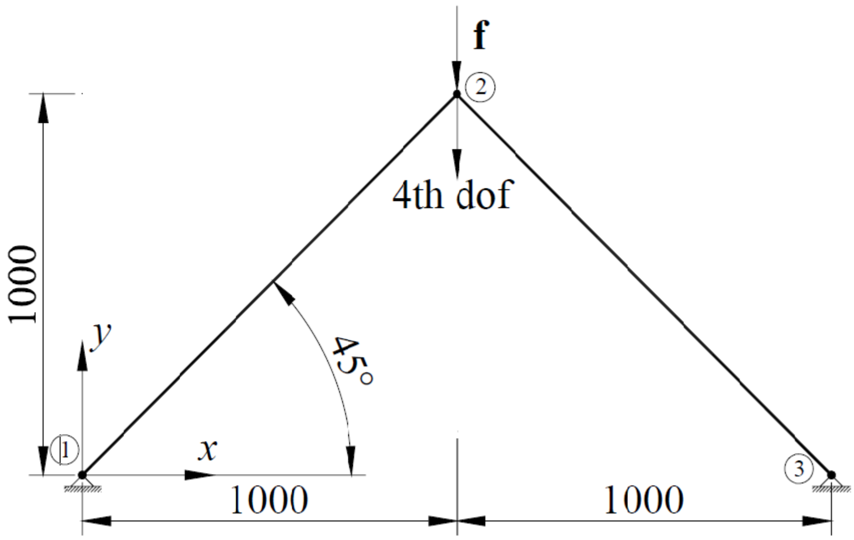

Figure 1: Two bar symmetric structure

In this Figure 2, it is possible to observe a snap-through behavior. With accumulated arc-length method the convergence level was most efficient, and was very effective for convergence to a pre-defined value.

In the second example is used a star shaped dome according to Crisfield (1991) and Wriggers (2008), that consists of nonlinear truss elements, the top view and front view of the dome are depicted in Figure 3. This structure was modeled by the St. Venant elastic constitutive equation with a Young modulus of E = 1079.6. The outer nodes of the finite element mesh are located on a circle with radius R = 50 while the inner nodes lye on a circle with radius R = 25. The inner nodes are located at a height of H = 6. 216 and the mid node is located at a height of H = 8. 216. The structure is simply supported at all outer nodes. A point load is applied at the apex of the dome.

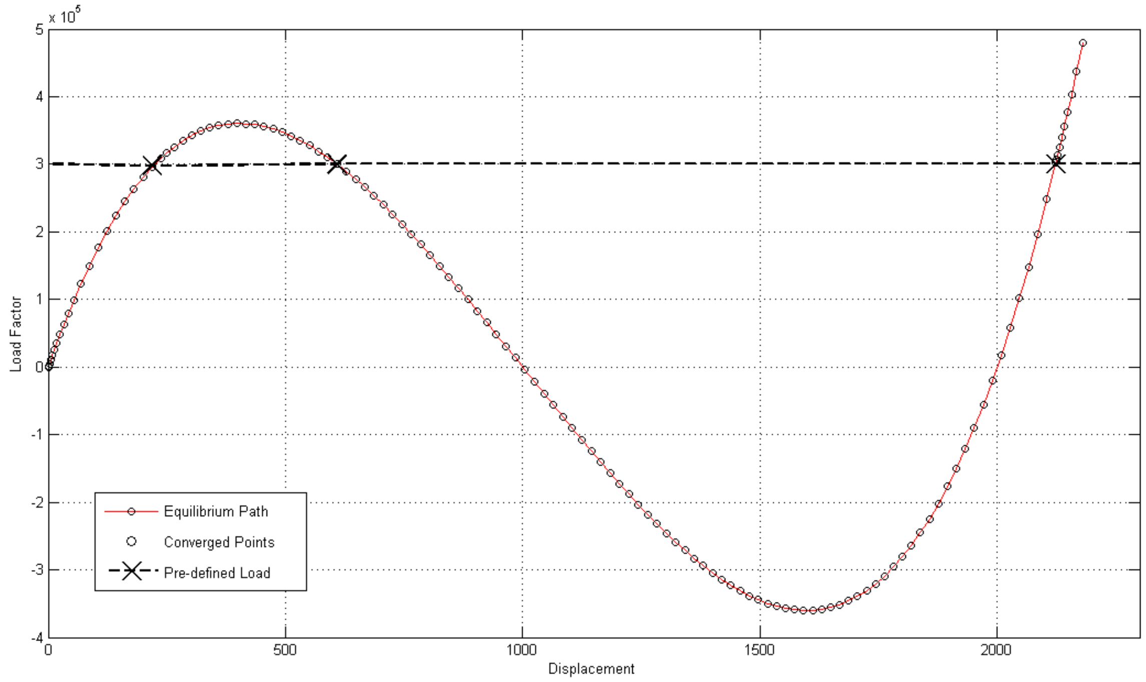

Figure 2: Load-deflection path tracing with convergence to a pre-defined load level

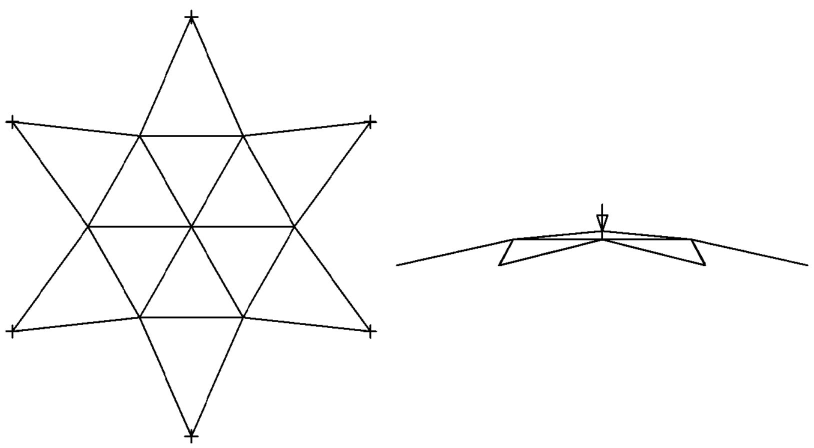

Figure 3: Top view and front view of the star shaped dome

The Figure 4 shows the load-deflexion path of the star dome considering a pre-defined load level of 50.

Also in this structure is easily to note that the accumulated arc length method had very effective convergence in the pre-defined load level adopted. This figure depicts the complex nonlinear behavior of this simple structure.

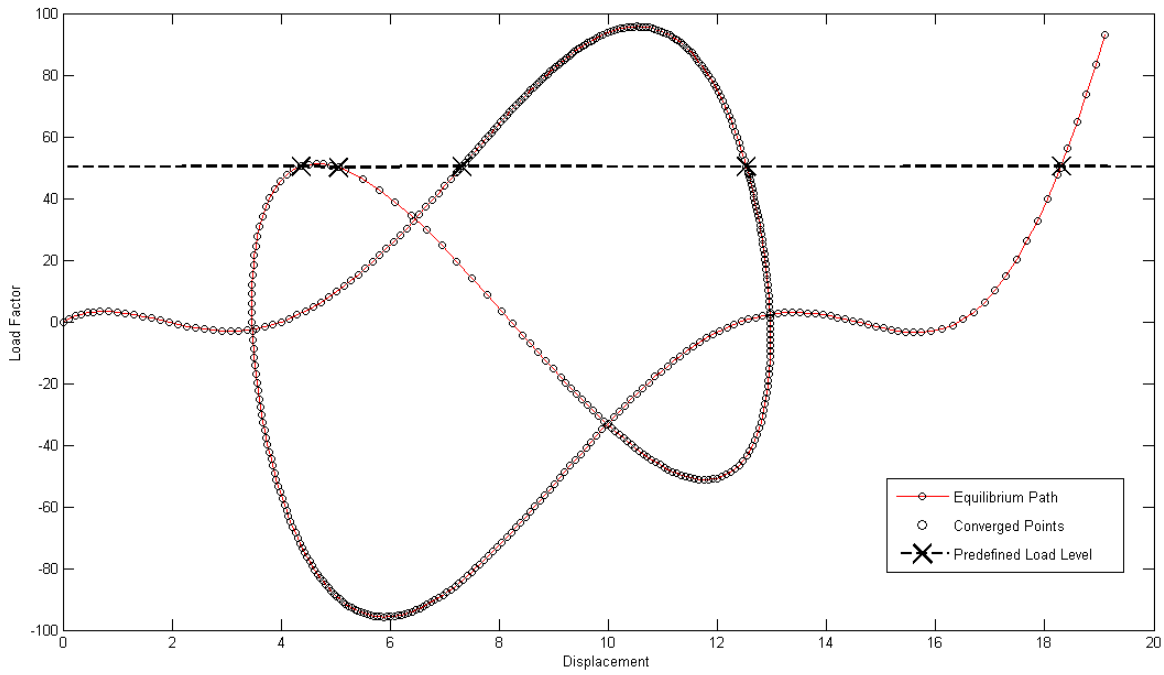

Figure 4: Load-deflection path tracing with convergence to a pre-defined load level

5 CONCLUSIONS

This paper has presented a modification or an improvement in the conventional arc-length method, in which can be previous defined load level to reach the convergence.

As can be readily observed the numerical formulation, this process is easy to implement in nonlinear finite element and arc-length method programs.

The examples depicted above prove that this new method can improve the conventional arc-length method by achieving the convergence level easier and faster than one, i.e., with a smaller number of iterative steps in each increment and with also a smaller number of increments, thereby gaining a greater computational efficiency.

Finally, the numerical examples mentioned here have demonstrated the effectiveness of the proposed method as a important tool to examine and predefine bifurcation and limits points, converging efficiently to pre-defined load level.

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