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A MODEL FOR IN-PLANE AND OUT-OF-PLANE VIBRATIONS OF NON-HOMOGENEOUS NON-UNIFORM CURVED BEAMS

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Abstract. In this paper the vibration patterns of slender structures with curved axis are addressed. A survey on the literature concerning dynamics of beams with non-homogeneous or functionally graded properties shows that there is a lack of papers devoted to the vibration analysis of curved beams made of graded materials. This problem was tackled by a few authors through two-dimensional approaches where material properties can vary in the radial direction. In-plane bending motions were analyzed with those approaches. However one dimensional models developed to study the out-of-plane motion of FGM curved beams are practically absent in the technical literature. The scope of this article is twofold. The first one is related to the introduction of a linear model of non-homogeneous curved beams containing out-of-plane and in-plane motions. The second topic consists in studying the vibration patterns of different graded materials and geometrical parameters under the presence of initial stresses states. The model is developed by adopting a consistent displacement field which incorporates second order rotational terms. The model also incorporates the shear flexibility due to bending and warping due to twisting effects. A finite element is developed to solve dynamic problems. The model contains several straight beam theories as well as curved beam theories as particular cases. Some comparisons with the available experimental data of the literature are performed in order to illustrate the predictive features of the model.

1 INTRODUCTION

Non-homogeneous materials with properties that can vary gradually along a given direction have emerged as an alluring option to solve the problem of high stress gradients (both normal and tangential) induced in specimens constructed with layers of two or more different materials (e.g. metals and ceramics). It was observed that some layered configurations may lead to stress gradients in the material such that it can undergo into debonding or the presence of cracks or a general failure. The concept of a material with graded properties was explored firstly in the early seventies to design effective thermal barriers in turbine blades. Later, in the middle eighties, in Japan, the name functionally graded materials has been established associated with a particular manufacturing process. Thereafter there has been a remarkable interest in uses and applications of these materials, especially in high technology military crafts, aerospace actuators, special sensors and medical uses as well. In the last ten or twelve years, many researchers focused their attention to study shells and solids constructed with FGM. In the woks of Reddy and Chin (1998), Reddy (2000), Praveen et al. (1999), Kitipornchai et al. (2004), Hosseini Kordkheili and Naghdabadi (2007) among others, one can see interesting studies about non homogeneous shells, plates and solids with graded properties. Relevant and interesting researches about functionally graded straight beams can be found in the recent works of Chakraborty et al. (2003), Goupee and Vel (2006), Ding et al. (2007) and Lü et al. (2008), among others. In these papers different laws that characterize the gradation of the material properties in the cross-sections have been employed. The gradation of properties can be represented in en exponential form or with a power law or any other with "ad-hoc" purposes. In the aforementioned papers three dimensional or a two dimensional models representing a beam were developed. On the other hand there are quite a few papers devoted to study functionally graded curved beams. Dryden (2007) carried out a study on a curved beam by means of an approximation to a two dimensional description based in the hypothesis of plane stresses. Malekzadeh (2009) carried out studies for in-plane vibrations of arches in the context of bi-dimensional formulations. Malekzadeh et al. (2010) developed a model for out-of-plane vibrations of curved beams made of FGM considering thermal effects. Piovan et al. (2008) developed a one dimensional model of curved beams appealing to the variational principle of Hellinger-Reissner. Filipich and Piovan (2010) deduced a theory of thick arches employing a classical strength of material approach. However these last two models were restricted to in-plane motions. Apparently Shafiee et al. (2006) were the first in developing a theory incorporating in-plane and out-of-plane motions in a curved beam made of FGM; however this model was employed to calculate only buckling loads, moreover shear flexibility was not incorporated and dynamic problems were not addressed.

The aim of the present paper is focused to develop a one dimensional model of curved beams with general graded properties. The model is conceived to incorporate in unified fashion in-plane and out-of-plane motions. In order to avoid misunderstandings, these concepts imply the motion in the plane of the beam curvature (the case usually analyzed) and the motions normal to that plane, that is, out-of-plane motions. The shear flexibility is taken into account. The twisting and warping effects are considered as well. It has to be mentioned that very few papers in the literature incorporate warping and twisting neither in the context of functionally graded or non-homogeneous curved beams nor in straight beam. The model is developed by adopting a displacement field which incorporated linear and second order terms. The linearized principle of virtual works is employed in order to obtain the motion equations. The model consists of a set of seven differential equations elastically coupled; however depending on the type of elastic gradation and the features of the cross-section the full system can be decoupled into two subsystems, thus representing in-plane motions and out-of-plane motions. The model is discretized with an isoparametric finite element. Problems of statics, free vibrations with or without presence of initial stresses. Comparisons with the available experimental data are performed as well.

2 MODEL DEVELOPMENT

In Fig. 1 a sketch of the curved beam is shown. The principal reference point **C** is located at the geometric center of the cross-section, where the x-axis is tangent to the circumferential axis of the beam, while y and z are the axes belonging to the cross-section, but not necessarily the principal ones. The present curved beam theory is based on the following assumptions:

1 The cross-section contour is rigid in its own plane.
2 The warping function is defined with respect to point **C**.
3 Material properties can vary with an arbitrary function within the cross-section.
4 The stress tensor, the volume forces and surface forces are composed by initial and incremental terms.
5 The displacement field is described first and second order appealing to semi-tangential rotations.
6 Inertial effects due to higher order displacements are neglected.
7 Structural damping is considered within the context of finite element method through a Rayleigh model.

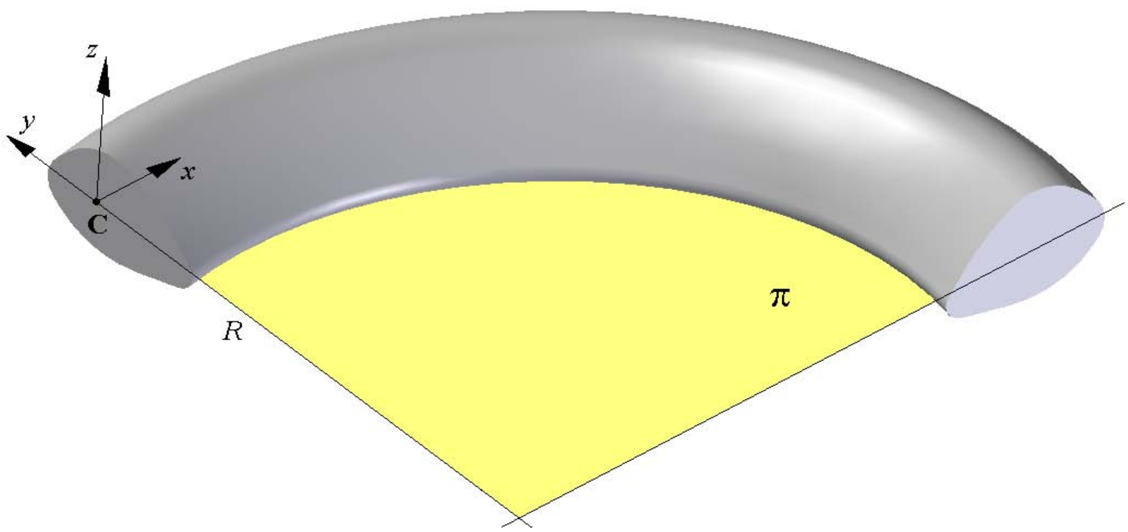

Figure 1: Beam sketch

2.1 Variational formulation

The general expression of the Principle of Virtual Work for a body which presents initial stresses can be written in the following form (Washizu, 1968):

$$\int_V \sigma_{ij}^f \delta e_{ij} dV - \int_V \left(\bar{X}_i^f - \rho \ddot{\bar{u}}_i^f \right) \delta \bar{u}_i^f dV - \int_S \bar{T}_i^f \delta \bar{u}_i^f dS = 0, \quad (1)$$

where $\bar{u}_i^f$ is the displacement vector, e_{ij} is the Green-Lagrange strain tensor, σ_{ij}^f is the second Piola-Kirchhof stress tensor, $\bar{X}_i^f$ is the vector of volume forces, and $\bar{T}_i^f$ is the vector de surface forces. σ_{ij}^f, $\bar{X}_i^f$ and $\bar{T}_i^f$ are defined by the sum of the corresponding incremental (σ_{ij}, $\bar{X}_i$ and $\bar{T}_i$) and initial (σ_{ij}^0, $\bar{X}_i^0$ and $\bar{T}_i^0$) components. The dots over variable mean derivation with respect to the temporal variable (i.e. one or two dots denote first or second derivative). The symbol ρ means the material density. The displacement $\bar{u}_i^f$ is defined by the sum of $\bar{u}_i^L$ and $\bar{u}_i^{NL}$, which denote first and second order (i.e. linear and non-linear) terms of displacements, respectively. V is the volume domain and S is the domain where surface forces are applied.

Now, considering the aforementioned background and employing assumptions 3) to 6) in Eq. (1) it is possible to obtain the general linearized Principle of Virtual Work for a beam with an arbitrary state of initial stresses in the following form:

$$\begin{aligned}\mathcal{W}_T = & \int_V \sigma_{ij}\delta\varepsilon_{ij}^L dV + \int_V \sigma_{ij}^0 \delta\varepsilon_{ij}^{NL} dV - \int_S \bar{T}_i^0 \delta u_i^{NL} dS - \int_V \bar{X}_i^0 \delta u_i^{NL} dV - \\ & - \int_V \left(\bar{X}_i\right)\delta u_i^L dV - \int_S \bar{T}_i \delta u_i^L dS + \int_V \left(\rho \ddot{u}_i^L\right)\delta u_i^L dV = 0,\end{aligned} \tag{2}$$

$$\mathcal{W}_T^0 = \int_V \sigma_{ij}^0 \delta\varepsilon_{ij}^L dV - \int_V \bar{X}_i^0 \delta u_i dV - \int_S \bar{T}_i^0 \delta u_i dS = 0, \tag{3}$$

Eq. (2) is subjected to the constraint Eq. (3), which implies the condition of self-equilibrium of initial stresses and initial volume and surface forces. The first term of Eq. (2) denotes the virtual work due to internal forces, the second term gives the virtual work due to initial stresses, the third and fourth terms are the virtual work of initial volume and surface forces due to non linear components of displacement field, the fifth and sixth terms are the virtual work of incremental volume and surface forces due to linear components of displacement field. The seventh term of Eq. (2) is the virtual work of inertial forces, where ρ is the material density and $\ddot{u}_i^L$ are the acceleration components of a point. Clearly dots over variables should be interpreted as derivation with respect to the time. ε_{ij}^L are the typical linear strain components, whereas ε_{ij}^{NL} are non linear strains components, given by the following expressions:

$$\varepsilon_{ij}^L = \frac{1}{2}\left(\frac{\partial u_j^L}{\partial x_i} + \frac{\partial u_i^L}{\partial x_j}\right), \varepsilon_{ij}^{NL} \cong \frac{1}{2}\left(\frac{\partial u_j^{NL}}{\partial x_i} + \frac{\partial u_i^{NL}}{\partial x_j}\right) + \frac{1}{2}\left(\frac{\partial u_h^L}{\partial x_i}\frac{\partial u_h^L}{\partial x_j}\right). \tag{4}$$

The higher-order strain components due to second-order displacements are neglected in the Green-Lagrange strain tensor (Piovan and Cortínez, 2007; Kim et al., 2005).

2.2 Kinematic relationships

Taking into account assumptions 1), 2) and 5) it is possible to develop the general displacement field, for an arbitrary point of a curved beam (including first-order and second-order terms of rotation parameters), in the following form:

$$\begin{Bmatrix} u_x^L \\ u_y^L \\ u_z^L \end{Bmatrix} = \begin{Bmatrix} u_{xc} - \omega\Phi_W \\ u_{yc} \\ u_{zc} \end{Bmatrix} + \begin{bmatrix} 0 & -\Phi_3 & \Phi_2 \\ \Phi_3 & 0 & -\Phi_1 \\ -\Phi_2 & \Phi_1 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ y \\ z \end{Bmatrix}, \tag{5}$$

$$\begin{Bmatrix} u_x^{NL} \\ u_y^{NL} \\ u_z^{NL} \end{Bmatrix} = \begin{bmatrix} -\Phi_3^2 - \Phi_2^2 & \Phi_1\Phi_2 & \Phi_1\Phi_3 \\ \Phi_1\Phi_2 & -\Phi_1^2 - \Phi_3^2 & \Phi_2\Phi_3 \\ \Phi_1\Phi_3 & \Phi_2\Phi_3 & -\Phi_1^2 - \Phi_2^2 \end{bmatrix} \begin{Bmatrix} 0 \\ y \\ z \end{Bmatrix}, \tag{6}$$

where, ω is the warping function of a beam , Φ_W, Φ_1, Φ_2 and Φ_3 are defined in terms of rotational and warping parameters as follows:

$$\Phi_1 = \phi_x, \quad \Phi_2 = \theta_y, \quad \Phi_3 = \theta_z - \frac{u_{xc}}{R}, \quad \Phi_W = \theta_x + \frac{\theta_y}{R} \tag{7}$$

In the previous equations, u_{xc}, u_{yc}, u_{zc} are the displacements of the reference center; ϕ_x is the twisting angle; θ_y and θ_z are bending rotational parameters, and finally θ_x is a measure of the warping intensity.

The warping function of a beam with curved axis can be approximated (Yang and Kuo, 1987) in the following form:

$$\omega = \bar{\omega}\mathcal{F}, \qquad with \qquad \mathcal{F} = \frac{R}{R+y}, \tag{8}$$

where, $\bar{\omega}$ is the warping function of the straight beam, which is case dependent of the material gradation function (i.e. the variation of the shear modulus of elasticity) and can be calculated solving the following differential equations (Lekhnitskii, 1981):

$$\begin{aligned} &\frac{\partial}{\partial z}\left(\frac{1}{G_{xy}}\frac{\partial \psi}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{1}{G_{xz}}\frac{\partial \psi}{\partial y}\right) = -2\Theta, \\ &\frac{\partial}{\partial z}\left(G_{xz}\frac{\partial \bar{\omega}}{\partial z}\right) + \frac{\partial}{\partial y}\left(G_{xy}\frac{\partial \bar{\omega}}{\partial y}\right) - \Theta\left(z\frac{\partial G_{xy}}{\partial y} - y\frac{\partial G_{xz}}{\partial z}\right) = 0, \end{aligned} \tag{9}$$

where, G_{xy} and G_{xz} are the shear moduli that can vary according to the given rule; Θ is a prescribed twisting angle per unit length employed to normalize the warping function and ψ is the so called Prandtl stress function employed to define the shear stresses τ_{xy} and τ_{xz} in terms of its spatial derivatives. These equations are subjected to the following boundary conditions:

$$\begin{aligned} &\psi = 0, \quad \text{on} \quad \Lambda_0, \\ &\frac{\partial \bar{\omega}}{\partial n} = \left(\frac{\partial \psi}{\partial z} + \Theta G_{xy} z, -\frac{\partial \psi}{\partial y} - \Theta G_{xz} y\right) \times \hat{n}_u, \quad \text{on} \quad \Lambda_0. \end{aligned} \tag{10}$$

In Eq. (10), Λ_0 is the contour of the cross-section, $\hat{n}_u$ is the unit vector normal to the cross-sectional contour and $\partial(\bullet)/\partial n$ is the normal derivative operator.

Normally if shear moduli are graded with a general expression, $\bar{\omega}$ has to be calculated with numerical or computational approaches; however under certain particular conditions and type of gradations, $\bar{\omega}$ may be deduced with a closed form solution (Lekhnitskii, 1981).

The most representative strain components are given by:

$$\varepsilon_{xx}^L = \left(\frac{\partial u_x^L}{\partial x} + \frac{u_y^L}{R}\right)\mathcal{F}, \quad 2\varepsilon_{xy}^L = \left(\frac{\partial u_y^L}{\partial x} - \frac{u_x^L}{R}\right)\mathcal{F} + \frac{\partial u_x^L}{\partial y}, \quad 2\varepsilon_{xz}^L = \frac{\partial u_z^L}{\partial x}\mathcal{F} + \frac{\partial u_x^L}{\partial z}, \tag{11}$$

$$\varepsilon_{xx}^{NL} = \left(\frac{\partial u_x^{NL}}{\partial x} + \frac{u_y^{NL}}{R}\right)\mathcal{F} + \frac{1}{2}\left[\left(\frac{\partial u_x^L}{\partial x} + \frac{u_y^L}{R}\right)^2 + \left(\frac{\partial u_y^L}{\partial x} - \frac{u_x^L}{R}\right)^2 + \left(\frac{\partial u_z^L}{\partial x}\right)^2\right]\mathcal{F}^2, \tag{12}$$

$$2\varepsilon_{xy}^{NL} = \left(\frac{\partial u_y^{NL}}{\partial x} - \frac{u_x^{NL}}{R}\right)\mathcal{F} + \frac{\partial u_x^{NL}}{\partial y} + \left[\frac{\partial u_x^L}{\partial y}\left(\frac{\partial u_x^L}{\partial x} + \frac{u_y^L}{R}\right) + \frac{\partial u_y^L}{\partial y}\left(\frac{\partial u_y^L}{\partial x} - \frac{u_x^L}{R}\right) + \left(\frac{\partial u_z^L}{\partial y}\frac{\partial u_z^L}{\partial x}\right)\right]\mathcal{F}, \tag{13}$$

$$2\varepsilon_{xz}^{NL} = \left(\frac{\partial u_z^{NL}}{\partial x}\mathcal{F} + \frac{\partial u_x^{NL}}{\partial z}\right) + \left[\frac{\partial u_x^L}{\partial z}\left(\frac{\partial u_x^L}{\partial x} + \frac{u_y^L}{R}\right) + \frac{\partial u_y^L}{\partial z}\left(\frac{\partial u_y^L}{\partial x} - \frac{u_x^L}{R}\right) + \left(\frac{\partial u_z^L}{\partial z}\frac{\partial u_z^L}{\partial x}\right)\right]\mathcal{F}, \tag{14}$$

Now, substituting Eqs. (5)-(7) into Eqs (11)-(14) and neglecting the higher order terms, the linear and non linear components of strain may be written in the following form:

$$\begin{aligned} \varepsilon_{xx}^L &= \left[\varepsilon_{D1} + z\varepsilon_{D2} - y\varepsilon_{D3} - \omega\varepsilon_{D4}\right]\mathcal{F}, \\ 2\varepsilon_{xy}^L &= \left[\varepsilon_{D5} + \frac{\partial\bar{\omega}}{\partial y}\varepsilon_{D7} - \left(z + \frac{\partial\bar{\omega}}{\partial y}\right)\varepsilon_{D8}\right]\mathcal{F}, \\ 2\varepsilon_{xz}^L &= \left[\varepsilon_{D6} + \frac{\partial\bar{\omega}}{\partial z}\varepsilon_{D7} + \left(y - \frac{\partial\bar{\omega}}{\partial z}\right)\varepsilon_{D8}\right]\mathcal{F}, \end{aligned} \tag{15}$$

$$\begin{aligned} \varepsilon_{xx}^{NL} = \; & \frac{\mathcal{F}}{2}\left[y\left(\Phi_1'\Phi_2 + \Phi_1\Phi_2'\right) + z\left(\Phi_1'\Phi_3 + \Phi_1\Phi_3'\right) - \frac{y\left(\Phi_1^2 + \Phi_3^2\right)}{R} + \frac{z\Phi_2\Phi_3}{R}\right] + \\ & \frac{\mathcal{F}^2}{2}\left[\left(\varepsilon_{D1} + z\varepsilon_{D2} - y\varepsilon_{D3} - \omega\varepsilon_{D4}\right)^2 + \left(\varepsilon_{D6} + y\varepsilon_{D8} - \frac{y+R}{R}\Phi_2\right)^2\right] \\ & \frac{\mathcal{F}^2}{2}\left[\left(\varepsilon_{D5} - z\varepsilon_{D8} + \frac{y+R}{R}\Phi_3 + \frac{\omega\Phi_W}{R}\right)^2\right], \\ 2\varepsilon_{xy}^{NL} = \; & \mathcal{F}\left[\left(-\Phi_2'\Phi_3 + \Phi_2\Phi_3'\right)\frac{z}{2} - \frac{\Phi_1\Phi_2}{2} + \frac{\Phi_1\Phi_3 z}{2R} - \Phi_3\varepsilon_{D1} + \Phi_3\Phi_W'\omega\right] + \\ & +\mathcal{F}\left[\Phi_1\varepsilon_{D6} - \Phi_W\frac{\partial\omega}{\partial y}\left(\varepsilon_{D1} + z\varepsilon_{D2} - y\varepsilon_{D3} - \omega\varepsilon_{D4}\right)\right], \\ 2\varepsilon_{xz}^{NL} = \; & \mathcal{F}\left[\left(\Phi_2'\Phi_3 - \Phi_2\Phi_3'\right)\frac{y}{2} - \frac{\Phi_1\Phi_3}{2} - \frac{\Phi_1\Phi_3 y}{2R} + \Phi_2\left(\varepsilon_{D1} - \Phi_W'\omega\right)\right] + \\ & +\mathcal{F}\left[-\Phi_1\varepsilon_{D5} - \Phi_W\frac{\partial\omega}{\partial z}\left(\varepsilon_{D1} + z\varepsilon_{D2} - y\varepsilon_{D3} - \omega\varepsilon_{D4}\right) - \Phi_1\Phi_W\frac{\omega}{R}\right], \end{aligned} \tag{16}$$

In Eqs. (15)-(16) the following definitions are introduced:

$$\begin{aligned} &\varepsilon_{D1} = u_{xc}' + \frac{u_{yc}}{R}, \varepsilon_{D2} = \theta_y' - \frac{\phi_x}{R}, \varepsilon_{D3} = \theta_z' - \frac{u_{xc}'}{R}, \varepsilon_{D4} = \theta_x' + \frac{\theta_y'}{R}, \\ &\varepsilon_{D5} = u_{yc}' - \theta_z, \varepsilon_{D6} = u_{zc}' + \theta_y, \varepsilon_{D7} = \phi_x' - \theta_x, \varepsilon_{D8} = \phi_x' + \frac{\theta_y}{R}, \end{aligned} \tag{17}$$

The apostrophes in Eqs. (15)-(17) mean derivation with respect to the x-variable.

2.3 Motion equations

In order to obtain the motion equations the principle of virtual works given in Eq. (2) is shrunk into the following form:

$$\delta\mathcal{T}_U + \delta\mathcal{T}_{G1} + \delta\mathcal{T}_{G2} + \delta\mathcal{T}_K + \delta\mathcal{T}_{EF} = 0 \tag{18}$$

where, $\delta\mathcal{T}_U$, $\delta\mathcal{T}_{G1}$, $\delta\mathcal{T}_{G2}$, $\delta\mathcal{T}_K$ and $\delta\mathcal{T}_{EF}$ are the virtual work of internal forces (due to strain), the virtual work due to initial initial stresses, the virtual work due to initial forces, the virtual work of inertial forces, and the virtual work done by external forces, respectively. These terms may be written as follows:

$$\begin{aligned}\delta\mathcal{T}_U = & \int_L \left[Q_x\delta\varepsilon_{D1} + M_y\delta\varepsilon_{D2} + M_z\delta\varepsilon_{D3} + B\delta\varepsilon_{D4}\right] dx + \\ & \int_L \left[Q_y\delta\varepsilon_{D5} + Q_z\delta\varepsilon_{D6} + T_w\delta\varepsilon_{D7} + T_{sv}\delta\varepsilon_{D8}\right] dx,\end{aligned} \tag{19}$$

$$\begin{aligned}\delta\mathcal{T}_{G1} = \; & \delta\int_L \left[\frac{M_z^0}{2}\left(\frac{\Phi_1^2+\Phi_3^2}{R} - \Phi_1'\Phi_2 - \Phi_1\Phi_2'\right)\right] dx + \\ & \delta\int_L \left[\frac{M_y^0}{2}\left(\frac{\Phi_2\Phi_3}{R} + \Phi_1'\Phi_3 + \Phi_1\Phi_3'\right)\right] dx + \\ & \delta\int_L \left[(\bar{\mathbf{d}}_a)^T\mathbf{B}_a^0\bar{\mathbf{d}}_a + (\bar{\mathbf{d}}_b)^T\mathbf{B}_b^0\bar{\mathbf{d}}_b + (\bar{\mathbf{d}}_c)^T\mathbf{B}_c^0\bar{\mathbf{d}}_c\right] dx + \\ & \delta\int_L \left[\frac{M_x^0}{2}\left(\Phi_2'\Phi_3 - \Phi_2\Phi_3' - \frac{\Phi_1\Phi_3}{R}\right)\right] dx + \\ & \delta\int_L \left[Q_y^0\left(\Phi_1\varepsilon_{D6} - \Phi_3\varepsilon_{D1} - \frac{\Phi_1\Phi_2}{2}\right)\right] dx + \\ & \delta\int_L \left[Q_z^0\left(\Phi_2\varepsilon_{D1} - \Phi_1\varepsilon_{D5} - \frac{\Phi_1\Phi_3}{2}\right)\right] dx + \\ & \delta\int_L \left[Q_{z\omega}^0\left(\Phi_2\Phi_W' + \frac{\Phi_1\Phi_W}{R}\right) - Q_{y\omega}^0\left(\Phi_3\Phi_W'\right)\right] dx + \\ & \delta\int_L \left[T_w^0\Phi_W\varepsilon_{D1} + T_{wz}^0\Phi_W\varepsilon_{D2} + T_{wy}^0\Phi_W\varepsilon_{D3} + T_{w\omega}^0\Phi_W\varepsilon_{D4}\right] dx,\end{aligned} \tag{20}$$

$$\delta\mathcal{T}_{G2} = -\int_L \left[\bar{\mathbf{X}}_1^0\delta u_{xc} + \bar{\mathbf{X}}_3^0\delta\theta_z + \bar{\mathbf{X}}_5^0\delta\theta_y + \bar{\mathbf{X}}_6^0\delta\phi_x\right] dx \tag{21}$$

$$\delta\mathcal{T}_K = \int_L \left[\mathcal{M}_1\delta u_{xc} + \mathcal{M}_2\delta u_{yc} + \mathcal{M}_3\delta\theta_z + \mathcal{M}_4\delta u_{zc} + \mathcal{M}_5\delta\theta_y + \mathcal{M}_6\delta\phi_x + \mathcal{M}_7\delta\theta_x\right] dx \tag{22}$$

$$\delta\mathcal{T}_{EF} = -\int_L \left[\mathcal{P}_1\delta u_{xc} + \mathcal{P}_2\delta u_{yc} + \mathcal{P}_3\delta\theta_z + \mathcal{P}_4\delta u_{zc} + \mathcal{P}_5\delta\theta_y + \mathcal{P}_6\delta\phi_x + \mathcal{P}_7\delta\theta_x\right] dx \tag{23}$$

In Eq. (19) the internal forces Q_x, M_y, M_z, and B are axial force, bending moment in y-direction, bending moment in z-direction, and bi-moment, respectively; whereas the internal forces Q_y, Q_z, T_w, and T_{sv} are shear force in y-direction, shear force in z-direction, twisting moment due to warping and twisting moment due to pure torsion, respectively. These internal forces are defined as stress-resultant in the following form:

$$\begin{aligned} \{Q_x, M_y, M_z, B\} &= \int_A \sigma_{xx}\{1, z, -y, -\omega\}\, dydz, \\ \{Q_y, Q_z\} &= \int_A \{\sigma_{xy}, \sigma_{xz}\}\, dydz, \\ T_w &= \int_A \left(\sigma_{xy}\frac{\partial\bar{\omega}}{\partial y} + \sigma_{xz}\frac{\partial\bar{\omega}}{\partial z}\right) dydz, \\ T_{sv} &= \int_A \left[-\sigma_{xy}\left(z + \frac{\partial\bar{\omega}}{\partial y}\right) + \sigma_{xz}\left(y - \frac{\partial\bar{\omega}}{\partial z}\right)\right] dydz. \end{aligned} \tag{24}$$

In Eq. (20), Q_x^0, M_y^0, M_z^0, B^0, Q_y^0, Q_z, T_w^0, T_{sv}^0 and M_x^0 (=$T_{sv}^0 + T_w^0$) are the initial forces and moments defined in the same form of Eq. (24) but respect to the initial stress state, i.e. substituting σ_{ij} by σ_{ij}^0. On the other hand $Q_{y\omega}^0$, $Q_{z\omega}^0$, T_{wy}^0, T_{wz}^0, $T_{w\omega}^0$ are generalized initial shear stress resultants defined according to the following expressions:

$$\begin{aligned} \{T_w^0, T_{wz}^0, T_{wy}^0, T_{w\omega}^0\} &\int_A \left(\sigma_{xy}^0\frac{\partial\bar{\omega}}{\partial y} + \sigma_{xz}^0\frac{\partial\bar{\omega}}{\partial z}\right)\{1, z, -y, -\omega\}\, dydz \\ \{Q_{y\omega}^0, Q_{z\omega}^0\} &\int_A \omega\,\{\sigma_{xy}^0, \sigma_{xz}^0\}\, dydz \end{aligned} \tag{25}$$

On the other hand $\mathbf{B}_a^0$, $\mathbf{B}_b^0$ and $\mathbf{B}_c^0$ are matrices containing generalized initial normal stress resultants. These matrices and the vectors $\bar{\mathbf{d}}_a$, $\bar{\mathbf{d}}_b$, and $\bar{\mathbf{d}}_c$ are defined as follows:

$$\mathbf{B}_a^0 = \int_A \sigma_{xx}^0 \mathcal{F}(\bar{\mathbf{g}}^a)^T \bar{\mathbf{g}}^a dydz, \quad \bar{\mathbf{g}}^a = \{1, z, -y, -\omega\} \tag{26}$$

$$\mathbf{B}_b^0 = \int_A \sigma_{xx}^0 \mathcal{F}(\bar{\mathbf{g}}^b)^T \bar{\mathbf{g}}^b dydz, \quad \bar{\mathbf{g}}^b = \left\{1, y, -\frac{R+y}{R}\right\} \tag{27}$$

$$\mathbf{B}_c^0 = \int_A \sigma_{xx}^0 \mathcal{F}(\bar{\mathbf{g}}^c)^T \bar{\mathbf{g}}^c dydz, \quad \bar{\mathbf{g}}^c = \left\{1, -z, \frac{R+y}{R}, \frac{\omega}{R}\right\} \tag{28}$$

$$\bar{\mathbf{d}}_a = \{\varepsilon_{D1}, \varepsilon_{D2}, \varepsilon_{D3}, \varepsilon_{D4}\}^T, \quad \bar{\mathbf{d}}_b = \{\varepsilon_{D5}, \varepsilon_{D8}, \Phi_2\}^T, \quad \bar{\mathbf{d}}_c = \{\varepsilon_{D6}, \varepsilon_{D8}, \Phi_3, \Phi_W\}^T \tag{29}$$

In Eq. (21) the functions $\bar{\mathbf{X}}_j^0$, $j = 1, 3, 5, 6$ associated with the initial volume forces are calculated with the following expressions:

$$\begin{aligned} \bar{\mathbf{X}}_1^0 &= -\frac{N_1^0}{R^2}u_{xc} + \frac{N_1^0}{R}\theta_z + \frac{N_3^0 + N_4^0}{2R}\theta_y - \frac{N_6^0}{2R}\phi_x \\ \bar{\mathbf{X}}_3^0 &= \frac{N_1^0}{R}u_{xc} - N_1^0\theta_z - \frac{N_3^0 + N_4^0}{2}\theta_y + \frac{N_6^0}{2}\phi_x \\ \bar{\mathbf{X}}_5^0 &= \frac{N_3^0 + N_4^0}{2R}u_{xc} - \frac{N_3^0 + N_4^0}{2}\theta_z - N_2^0\theta_y - \frac{N_5^0}{2}\phi_x \\ \bar{\mathbf{X}}_6^0 &= -\frac{N_6^0}{2R}u_{xc} + \frac{N_6^0}{2}\theta_z - \frac{N_5^0}{2}\theta_y - \left(N_1^0 + N_2^0\right)\phi_x \end{aligned} \tag{30}$$

where

$$\left\{N_1^0, N_2^0, N_3^0\right\} = \int_A \left\{y\bar{X}_y^0, z\bar{X}_z^0, y\bar{X}_z^0\right\} \frac{dydz}{\mathcal{F}}$$
$$\left\{N_4^0, N_5^0, N_6^0\right\} = \int_A \left\{z\bar{X}_y^0, y\bar{X}_x^0, z\bar{X}_x^0\right\} \frac{dydz}{\mathcal{F}} \tag{31}$$

The inertia forces $\mathcal{M}_j$, $j = 1, ..., 7$, introduced in Eq. (22) are defined in terms of the accelerations in the following form:

$$\left\{\begin{array}{c} \mathcal{M}_1 \\ \mathcal{M}_2 \\ \mathcal{M}_3 \\ \mathcal{M}_4 \\ \mathcal{M}_5 \\ \mathcal{M}_6 \\ \mathcal{M}_7 \end{array}\right\} = \left[\begin{array}{ccccccc} I_{11} & 0 & I_{13} & 0 & I_{15} & 0 & I_{17} \\ & I_{22} & 0 & 0 & 0 & I_{26} & 0 \\ & & I_{33} & 0 & I_{35} & 0 & I_{37} \\ & & & I_{44} & 0 & I_{46} & 0 \\ & & & & I_{55} & 0 & I_{57} \\ & sym & & & & I_{66} & 0 \\ & & & & & & I_{77} \end{array}\right] \left\{\begin{array}{c} \ddot{u}_{xc} \\ \ddot{u}_{yc} \\ \ddot{\theta}_z \\ \ddot{u}_{zc} \\ \ddot{\theta}_y \\ \ddot{\phi}_x \\ \ddot{\theta}_x \end{array}\right\} \tag{32}$$

where:

$$\begin{gathered} I_{11} = J_{11}^\rho - \frac{2J_{13}^\rho}{R} + \frac{J_{33}^\rho}{R^2}, \quad I_{13} = J_{13}^\rho - \frac{J_{33}^\rho}{R}, \quad I_{15} = J_{12}^\rho + \frac{J_{14}^\rho}{R} - \frac{J_{23}^\rho}{R} - \frac{J_{34}^\rho}{R^2}, \\ I_{17} = J_{14}^\rho - \frac{J_{34}^\rho}{R}, \quad I_{22} = I_{44} = J_{11}^\rho, \quad I_{26} = -J_{12}^\rho, \quad I_{33} = J_{33}^\rho, \\ I_{35} = J_{23}^\rho + \frac{J_{34}^\rho}{R}, \quad I_{37} = J_{34}^\rho, \quad I_{46} = -J_{13}^\rho, \quad I_{55} = J_{22}^\rho + \frac{2J_{24}^\rho}{R} + \frac{J_{44}^\rho}{R^2}, \\ I_{57} = J_{24}^\rho + \frac{J_{44}^\rho}{R}, \quad I_{66} = J_{22}^\rho + J_{33}^\rho, \quad I_{77} = J_{44}^\rho. \end{gathered} \tag{33}$$

The coefficients J_{ik}^ρ, $i, k = 1, ..., 7$ are defined according to:

$$J_{ik}^\rho = \int_A \rho \bar{\mathbf{g}}_i^a \bar{\mathbf{g}}_k^a \frac{dydz}{\mathcal{F}}, \quad \bar{\mathbf{g}}^a = \{1, z, -y, -\omega\}. \tag{34}$$

The external distributed forces $\mathcal{P}_j$, $j = 1, ..., 7$ introduced in Eq (23) can be written, in terms of general volume forces (i.e. $\bar{X}_x$, $\bar{X}_y$ and $\bar{X}_z$), in the following form:

$$\left\{\mathcal{P}_1, \mathcal{P}_3, \mathcal{P}_5, \mathcal{P}_7\right\} = \int_A \bar{X}_x \left\{\frac{y+R}{R}, z - \frac{\omega}{R}, -y, -\omega\right\} \frac{dydz}{\mathcal{F}},$$
$$\left\{\mathcal{P}_2, \mathcal{P}_4, \mathcal{P}_6\right\} = \int_A \left\{\bar{X}_y, \bar{X}_z, \bar{X}_z y - \bar{X}_y z\right\} \frac{dydz}{\mathcal{F}}. \tag{35}$$

Now performing the conventional steps of variational calculus in Eq. (18), it is possible to arrive to the following differential system of seven equations:

$$
\begin{aligned}
-\frac{\partial}{\partial x}\left[Q_x-\frac{M_z}{R}+\mathcal{G}_{11}^0\right]+\mathcal{G}_{10}^0-\bar{\mathbf{X}}_1^0+\mathcal{M}_1-\mathcal{P}_1=0,\\
-\frac{\partial}{\partial x}\left[Q_y+\mathcal{G}_{21}^0\right]+\frac{Q_x}{R}+\mathcal{G}_{20}^0+\mathcal{M}_2-\mathcal{P}_2=0,\\
-\frac{\partial}{\partial x}\left[M_z+\mathcal{G}_{31}^0\right]-Q_y+\mathcal{G}_{30}^0-\bar{\mathbf{X}}_3^0+\mathcal{M}_3-\mathcal{P}_3=0,\\
-\frac{\partial}{\partial x}\left[Q_z+\mathcal{G}_{41}^0\right]+\mathcal{G}_{40}^{(0)}+\mathcal{M}_4-\mathcal{P}_4=0,\\
-\frac{\partial}{\partial x}\left[M_y+\mathcal{G}_{51}^0+\frac{B}{R}\right]+Q_z+\frac{T_{sv}}{R}+\mathcal{G}_{50}^0-\bar{\mathbf{X}}_5^0+\mathcal{M}_5-\mathcal{P}_5=0,\\
-\frac{\partial}{\partial x}\left[T_{sv}+T_w+\mathcal{G}_{61}^0\right]-\frac{M_y}{R}+\mathcal{G}_{60}^0-\bar{\mathbf{X}}_6^0+\mathcal{M}_6-\mathcal{P}_6=0,\\
-\frac{\partial}{\partial x}\left[B+\mathcal{G}_{71}^0\right]-T_w+\mathcal{G}_{70}^0+\mathcal{M}_7-\mathcal{P}_7=0,
\end{aligned}
\tag{36}
$$

subjected to the following boundary equations:

$$
\begin{aligned}
-\left(\bar{Q}_x-\frac{\bar{M}_z}{R}\right)+\left(Q_x-\frac{M_z}{R}\right)+\mathcal{G}_{11}^0-\bar{\mathbf{S}}_1^0=0, \quad or \quad \delta u_{xc}=0,\\
-\bar{Q}_y+Q_y+\mathcal{G}_{21}^0=0, \quad or \quad \delta u_{yc}=0,\\
-\bar{M}_z+M_z+\mathcal{G}_{31}^{(0)}-\bar{\mathbf{S}}_3^0=0, \quad or \quad \delta\theta_z=0,\\
-\bar{Q}_z+Q_z+\mathcal{G}_{41}^{(0)}=0, \quad or \quad \delta u_{zc}=0,\\
-\left(\bar{M}_y-\frac{\bar{B}}{R}\right)+\left(M_y-\frac{B}{R}\right)+\mathcal{G}_{51}^{(0)}-\bar{\mathbf{S}}_5^0=0, \quad or \quad \delta\theta_y=0,\\
-\bar{M}_x+M_x+\mathcal{G}_{61}^{(0)}-\bar{\mathbf{S}}_6^0=0, \quad or \quad \delta\phi_x=0,\\
-\bar{B}+B+\mathcal{G}_{71}^{(0)}=0, \quad or \quad \delta\theta_x=0
\end{aligned}
\tag{37}
$$

where: $\bar{Q}_x$, $\bar{Q}_y$, $\bar{Q}_z$, $\bar{M}_z$, $\bar{M}_y$, $\bar{M}_x$ and $\bar{B}$ are prescribed forces acting on the boundaries. $\bar{\mathbf{S}}_j^0$, $j=1,3,5,6$, are initial surface forces. $\mathcal{G}_{j1}^0$ and $\mathcal{G}_{j0}^0$, $j=1,...,7$ are forces that collect all the initial stress resultants associated with the corresponding variational variable. Due to space reasons the expressions of $\mathcal{G}_{j1}^0$ and $\mathcal{G}_{j0}^0$ are not provided.

2.4 Constitutive equations in terms of strain components

The stress-strain relations are connected with the distribution of the material constituents in the graded configuration of the cross section. Generally for functionally graded materials, the constitutive stress-strain relations can be represented in the following form (Malekzadeh et al., 2010):

$$
\sigma_{xx}=E_{xx}(y,z)\,\varepsilon_{xx}^L, \quad \sigma_{xy}=G_{xy}(y,z)\,2\varepsilon_{xy}^L, \quad \sigma_{xz}=G_{xz}(y,z)\,2\varepsilon_{xz}^L, \tag{38}
$$

where, $E_{xx}(y,z)$ is the longitudinal elasticity modulus, whereas $G_{xy}(y,z)$ and $G_{xz}(y,z)$ are the transversal elasticity moduli. It should be mentioned that, $G_{xy}(y,z)$ and $G_{xz}(y,z)$ may be affected by given coefficients (κ_{xy}, κ_{xz}) in order to enhance the characterization of shear stresses, as it is done in the classic Timoshenko beam theory or in other first order shear theories (Malekzadeh et al., 2010). In recent articles (Filipich and Piovan, 2010; Piovan et al., 2008) some approaches to calculate the aforementioned coefficients are introduced for in-plane problems.

Now substituting Eq. (15) into Eq. (38) and then into Eq. (24), the internal forces can be represented in terms of generalized strains as follows:

$$\bar{\mathbf{Q}} = \mathbf{J}.\bar{\mathbf{D}} \tag{39}$$

where:

$$\begin{aligned} \bar{\mathbf{Q}} &= \{Q_x, M_y, M_z, B, Q_y, Q_z, T_w, T_{sv}\}^T, \\ \bar{\mathbf{D}} &= \{\varepsilon_{D1}, \varepsilon_{D2}, \varepsilon_{D3}, \varepsilon_{D4}, \varepsilon_{D5}, \varepsilon_{D6}, \varepsilon_{D7}, \varepsilon_{D8}\}^T, \end{aligned} \tag{40}$$

$$\bar{\mathbf{J}} = \begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} & 0 & 0 & 0 & 0 \\ J_{12} & J_{22} & J_{23} & J_{24} & 0 & 0 & 0 & 0 \\ J_{13} & J_{23} & J_{33} & J_{34} & 0 & 0 & 0 & 0 \\ J_{14} & J_{24} & J_{34} & J_{44} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & J_{55} & J_{56} & J_{57} & J_{58} \\ 0 & 0 & 0 & 0 & J_{56} & J_{66} & J_{67} & J_{67} \\ 0 & 0 & 0 & 0 & J_{57} & J_{67} & J_{77} & J_{78} \\ 0 & 0 & 0 & 0 & J_{58} & J_{68} & J_{78} & J_{88} \end{bmatrix} \tag{41}$$

$$\begin{aligned} J_{ik} &= \int_A E_{xx} \bar{\mathbf{g}}_i^a \bar{\mathbf{g}}_k^a \mathcal{F} dydz, \quad i,k = 1,2,3,4 \\ J_{hl} &= \int_A \left[G_{xy} \bar{\mathbf{g}}_h^b \bar{\mathbf{g}}_l^b + G_{xz} \bar{\mathbf{g}}_h^c \bar{\mathbf{g}}_l^c \right] \mathcal{F} dydz, \quad h,l = 5,6,7,8 \end{aligned} \tag{42}$$

$$\begin{aligned} \bar{\mathbf{g}}^a &= \{1, z, -y, -\omega\}, \\ \bar{\mathbf{g}}^b &= \left\{1, 0, \frac{\partial \bar{\omega}}{\partial y}, -z - \frac{\partial \bar{\omega}}{\partial y}\right\}, \\ \bar{\mathbf{g}}^c &= \left\{0, 1, \frac{\partial \bar{\omega}}{\partial z}, y - \frac{\partial \bar{\omega}}{\partial z}\right\}. \end{aligned} \tag{43}$$

The Eq. (39) allows to calculate forces in terms of generalized strains. Moreover, it is possible to employ Eq. (39) to calculate the forces associated to initial stresses, if vector $\bar{\mathbf{D}}$ of generalized incremental deformations, is substituted by $\bar{\mathbf{D}}^0$, i.e. the vector of generalized initial deformations.

2.5 An analytical solution

Under certain conditions it is possible to obtain a simple analytical solution of Eq. (36) for a free vibration problem. Thus in absence of a state of initial stresses and for the following boundary conditions at the ends:

$$u_{yc} = u_{zc} = \phi_x = Q_x = M_y = M_z = B = 0 \tag{44}$$

the displacements given in Eq. (5) can be represented by means of the following expression:

$$\begin{aligned} u_{xc} &= C_1 \cos[\Omega t] \cos[k_n x], \\ u_{yc} &= C_2 \cos[\Omega t] \sin[k_n x], \\ \theta_z &= C_3 \cos[\Omega t] \cos[k_n x], \\ u_{zc} &= C_4 \cos[\Omega t] \sin[k_n x], \\ \theta_y &= C_5 \cos[\Omega t] \cos[k_n x], \\ \phi_x &= C_6 \cos[\Omega t] \sin[k_n x], \\ \theta_x &= C_7 \cos[\Omega t] \cos[k_n x], \end{aligned} \tag{45}$$

where Ω is the circular frequency measured in rad/seg, C_i, $i = 1, ..., 7$ are amplitude coefficient and

$$k_n = \frac{n\pi}{L}, \quad n = 1, 2, 3, ... \tag{46}$$

Then, substituting Eq. (45) in Eq. (36) and manipulating algebraically it is possible to arrive to the following frequency equation:

$$\left| k_n^2 \mathbf{M}_1 \mathbf{J} \mathbf{M}_1^T - k_n \left(\mathbf{M}_1 \mathbf{J} \mathbf{M}_2^T - \mathbf{M}_2 \mathbf{J} \mathbf{M}_1^T \right) \mathbf{M}_0 + \mathbf{M}_2 \mathbf{J} \mathbf{M}_2^T - \Omega^2 \mathbf{M}_m \right| = 0 \tag{47}$$

where, $\mathbf{M}_1$, $\mathbf{M}_2$ and $\mathbf{M}_0$ are defined as follows:

$$\mathbf{M}_1 = \begin{bmatrix} 1 & 0 & -1/R & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/R & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{48}$$

$$\mathbf{M}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/R & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1/R \\ 0 & -1/R & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \tag{49}$$

$$\mathbf{M}_0 = Diag\left[-1, 1, -1, 1, -1, 1, -1\right] \tag{50}$$

On the other hand matrices $\mathbf{M}_m$ and $\mathbf{J}$ are the ones defined in Eq. (32) in Eq. (41), respectively.

3 FINITE ELEMENT FORMULATION

In order to solve problems of static, dynamic and buckling with several boundary conditions, off-axis loading and arbitrary material gradation; iso-parametric finite elements with five nodes are employed. It is clear that five-nodes elements imply approximation of quadric order. The vector of nodal displacements $\bar{\mathbf{U}}_e$ can be arranged in the following form:

$$\bar{\mathbf{U}}_e = \left\{ \bar{\mathbf{U}}_e^{(1)}, ..., \bar{\mathbf{U}}_e^{(5)} \right\} \tag{51}$$

where:

$$\bar{\mathbf{U}}_e^{(j)} = \left\{ u_{xc_j}, u_{yc_j}, \theta_{z_j}, u_{zc_j}, \theta_{y_j}, \phi_{x_j}, \theta_{x_j} \right\}, \quad j = 1, ..., 5 \tag{52}$$

Then, the variables u_{xc}, u_{yc}, θ_z, u_{zc}, θ_y, ϕ_x and θ_x can be interpolated along the element in the following compact form as:

$$U_i = \sum_{j=1}^{5} f_j^{(5)} \bar{\mathbf{U}}_{ei}^{(j)}, \quad j = 1, ..., 5, \quad i = 1, ..., 7 \tag{53}$$

Where $f_j^{(5)}$ is the shape function of the j^{th}-node of the element, whereas $\bar{\mathbf{U}}_{ei}^{(j)}$ is the corresponding nodal displacement in the j^{th}-node. The iso-parametric shape functions $f_j^{(5)}$, $j = 1, ..., 5$, can be found in every finite element textbook (Bathe, 1982). From the previous equations, it is clear that $U_1 = u_{xc}$, $U_2 = u_{yc}$, $U_3 = \theta_z$, $U_4 = u_{zc}$, $U_5 = \theta_y$, $U_6 = \phi_x$ and $U_7 = \theta_x$.

Now, substituting Eq. (51) into Eq. (18) and applying the conventional steps of finite element procedures, it is possible to arrive to the following general finite element equation:

$$(\mathbf{K} + \mathbf{K}_G)\,\bar{\mathbf{W}} + \mathbf{M}\ddot{\bar{\mathbf{U}}} = \bar{\mathbf{P}} \tag{54}$$

where $\mathbf{K}$, $\mathbf{K}_G$ and $\mathbf{M}$ are global matrices of elastic stiffness, geometric stiffness, and mass, respectively; whereas $\bar{\mathbf{W}}$, $\ddot{\bar{\mathbf{U}}}$ and $\bar{\mathbf{P}}$ are the global vectors of nodal displacements, nodal accelerations and nodal forces. In order to obtain the initial stresses, equation Eq. (55) has to be employed before any other calculation. This equation corresponds to the finite element form of the self-equilibrium condition of initial stresses, initial volume and surface forces given in Eq. (3):

$$\mathbf{K}\bar{\mathbf{W}}^0 = \bar{\mathbf{P}}^0, \tag{55}$$

In the previous Eq. (55), $\bar{\mathbf{W}}^0$ and $\bar{\mathbf{P}}^0$ are the global vector of initial nodal displacements and the global vector of initial volume and surface forces, respectively.

Eq. (54) can be modified in order to account for "a posteriori" structural proportional Rayleigh damping given by:

$$\mathbf{C}_{RD} = \alpha\mathbf{M} + \eta\mathbf{K}. \tag{56}$$

The coefficients α and η in Eq. (23) can be computed from two experimental modal damping coefficients (namely, ξ_1 and ξ_2) for the first and second frequencies according to the common methodology presented in the bibliography related to finite element procedures (Bathe, 1982) and vibration analysis (Meirovith, 1997). Remember that $\mathbf{M}$ is the global mass matrix and $\mathbf{K}$ is the global elastic stiffness matrix.

For the case of free vibration analysis, the general Eq. (54) can be reduced to the following eigenvalue equation when damping effects are neglected and harmonic motion is prescribed.

$$\left(\mathbf{K} + \lambda\mathbf{K}_G - \Omega^2\mathbf{M}\right)\bar{\mathbf{W}}^* = \bar{\mathbf{O}} \tag{57}$$

where $\Omega = 2\pi f$, f is the natural frequency measured in hertz and λ is a parameter appropriately defined, in terms of beam-stress-resultants, for the characterization of initial stresses. It is possible to see that Eq. (54) allows the computation of natural frequencies (Ω or f) of beams subjected or not (this implies $\lambda = 1$ or $\lambda = 0$) to arbitrary initial stresses. On the other hand, the same equation can be utilized to calculate buckling loads when the condition $f = 0$ is imposed.

4 COMPUTATIONAL STUDIES

In the present section a numerical testing of the procedure as well as parametric studies are performed in order to check the validity and usefulness of the beam model and its finite element approach.

Tab. 1 shows the properties of different metallic (Steel SUS302 and aluminium) and ceramic (Alumina Al_2O_3 and Silicon carbide SiC) materials to be employed in the next sections. The hypothesis of proportionality between longitudinal modulus and transversal modulus is employed in some cases to calculate the remaining elastic properties, according to recent works

Properties	Steel	Alumina	Aluminium	Silicon Carbide
Longitudinal Modulus of elasticity (GPa)	214	390	67	302
Transversal Modulus of elasticity (GPa)	80.0	137	—	—
Poisson's coefficient	—	—	0.33	0.17
Density (Kg/m^3)	7800	3200	2700	3200

Table 1: Material properties for metallic and ceramic components.

in the technical literature (Chakraborty et al., 2003; Filipich and Piovan, 2010; Kapuria et al., 2008).

4.1 Convergence check

The first example corresponds to a convergence test of the finite element developed. The curved beam properties are such that $R = 1.0\ m$, $L = 1.0\ m$, with a rectangular cross section of $b = 0.05\ m$, $h = 0.01\ m$. The material properties are varying from a metallic surface (SUS302 at $z = -h/2$) to a ceramic surface (Alumina at $z = h/2$) according to the following exponential law:

$$E_{xx} = E_c e^{\left[-\Lambda_1\left(\frac{1}{2}-\frac{z}{h}\right)\right]}, \quad G_{xy} = G_{xyc} e^{\left[-\Lambda_2\left(\frac{1}{2}-\frac{z}{h}\right)\right]}, \quad G_{xz} = G_{xzc} e^{\left[-\Lambda_3\left(\frac{1}{2}-\frac{z}{h}\right)\right]}, \tag{58}$$

where:

$$\Lambda_1 = Ln\left[\frac{E_c}{E_m}\right], \quad \Lambda_2 = Ln\left[\frac{G_{xyc}}{G_{xym}}\right], \quad \Lambda_3 = Ln\left[\frac{G_{xzc}}{G_{xzm}}\right]. \tag{59}$$

In the previous equations, E_c and E_m are the longitudinal modulus of elasticity of the ceramic and the metallic materials, whereas G_{xyc} and G_{xzc} are the shear transverse modulus of elasticity of the ceramic material and G_{xym} and G_{xzm} are the shear transverse modulus of elasticity of the metallic material.

R/L	Approach	Number of Elements	f_1	f_2	f_3	f_4
0.5	Analytical		18.695	118.727	286.438	520.956
	FEM	2	18.695	118.771	289.763	529.443
		5	18.695	118.726	286.434	520.866
		10	18.695	118.726	286.43	520.812
1.0	Analytical		29.735	130.674	298.776	533.684
	FEM	2	29.742	130.714	301.352	570.91
		5	29.742	130.673	298.749	533.705
		10	29.742	130.673	298.746	533.659
1.5	Analytical		31.919	132.929	301.083	536.057
	FEM	2	31.938	132.968	303.476	571.626
		5	31.938	132.928	301.02	536.077
		10	31.938	132.928	301.017	536.034

Table 2: Convergence test of the first four frequencies [Hz] of a simply supported curved beam.

In Tab. 2 the convergence of the first four frequencies is presented. Thus, at least with five element one can guarantee differences lower than 1.00% in the first four frequencies, when they are compared with the test values. In this case the test values were the frequencies calculated with the analytical solution of the subsection 2.5.

4.2 Comparisons with experimental data:the case of a straight beam

The curved beam model developed in Section 2 can be reduced to the case of a straight beam if the condition $R \to \infty$ is imposed. Then as a first example the values of free vibration frequencies obtained experimentally by Kapuria et al. (2008) are compared with the frequencies calculated with the present model reduced to the straight beam case. Thus, Fig. 2 shows a cantilever beam with five layers composed by different mixtures of aluminium and silicon carbide (see Tab. 2 for properties). The thickness of each layer is 2 mm whereas the depth of the cross section is 15 mm. The constitutive law to calculate the longitudinal modulus of elasticity is given in Eq. (60), which is a variety of the two-constituents rule of mixtures (Finot et al., 1996), whereas the variation of the density and Poisson's coefficient can be characterized with the classic linear rule of mixtures given by Eq. (61).

$$E_{xx} = \frac{[V_m E_m (q_\sigma + E_c) + (1 - V_m)(q_\sigma + E_m) E_c]}{[V_m (q_\sigma + E_c) + (1 - V_m)(q_\sigma + E_m)]} \tag{60}$$

$$p = V_m p_m + V_c p_c \tag{61}$$

where V_m and V_c are the volumetric proportions of metallic and ceramic constituents, respectively. E_m and E_c are the longitudinal modulus of elasticity of metallic and ceramic constituents, respectively. p, p_m and p_c identify, in a generic sense, the property of a FGM, the property of metallic constituent and the property of the ceramic constituent, respectively. Finally, q_σ is the ratio of stress to strain transfer between the metallic and ceramic phases:

$$q_\sigma = -\frac{\sigma_c - \sigma_m}{\varepsilon_c - \varepsilon_m} \tag{62}$$

If $q_\sigma = \infty$, which means equal strain transfer, Eq. (60) can be reduced to Eq. (61). The value of q_σ for the material Al/SiC has been experimentally determined to be 91.6 GPa (Kapuria et al., 2008).

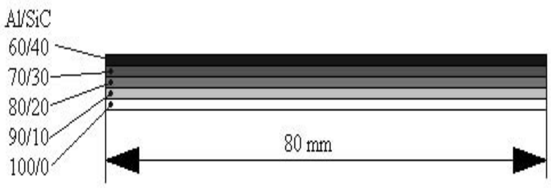

Figure 2: Experimental sample of the layered Al/SiC beam.

Tab. 3 shows the comparison of the first four natural frequencies obtained experimentally by Kapuria et al. (2008) and the finite element approach introduced in the present paper. The experimental data of three samples and their media are contrasted with a numerical model of 4 finite element of 5 nodes. It is possible to see a quite good correlation of the finite element approach with the experimental data.

Freq	Sample 1	Sample 2	Sample 3	Experiments Media	Present FE Approach
1	1433	1365	1441	1413	1424
2	8422	8075	8462	8320	8339
3	18388	18142	18411	18314	17737
4	21589	20970	21882	21480	21369

Table 3: Natural frequencies [Hz] of a Al/SiC straight layered beam. Comparison of experimental values (Kapuria et al., 2008) and the present approach.

4.3 Comparisons with other approaches

This section describes the comparison of the present model with other curved beam model (Malekzadeh, 2009; Tufekci and Yasar Dogruer, 2006). The beam is isotropic with equal height and depth ($b = h$), such that the Poisson's coefficient is $\mu = 0.3$, the shear stiffness is affected by a shear correction coefficient $k_s = 0.85$ and the geometrical features of the beam are such that $S_r = R/h\sqrt{12} = 100$ (Tufekci and Yasar Dogruer, 2006). In the solved examples, the following non-dimensional frequency parameter is used,

$$\bar{\Omega}_i = \Omega_1 R^2 \sqrt{\frac{I_{44}}{J_{22}}}, i = 1, 2, 3. \tag{63}$$

Angle	Approach	$\bar{\Omega}_1$	$\bar{\Omega}_2$	$\bar{\Omega}_3$
60	Malekzadek (2010)	19.398	54.014	105.611
	Tufekci (2006)	19.402	54.031	105.651
	present	19.442	54.093	105.707
120	Malekzadek (2010)	4.452	12.825	25.984
	Tufekci (2006)	4.451	12.826	25.989
	present	4.471	12.885	26.064
180	Malekzadek (2010)	1.805	5.198	10.918
	Tufekci (2006)	1.804	5.198	10.918
	present	1.817	5.239	10.984

Table 4: Comparison of the first three non-dimensional natural frequency parameter.

Tab. 4 shows the non-dimensional natural frequency parameters of a clamped-clamped curved beam. It is possible to see a quite good agreement of the three approaches; in fact differences in percentage not higher than 0.6% have been observed.

The second example corresponds to the comparison of the present model with a full 3D approach in curved beam with graded properties varying from an inner metallic core (steel) to a ceramic (alumina) phase at both external surfaces (i.e. $z = \pm z/2$). The curved beam is such that $R = 1\ m$ and $h = 2b = 0.02\ m$. The variation of properties has the following form:

$$p = p_m + (p_c - p_m)\left|\frac{2z}{h}\right|^n \tag{64}$$

where as in the previous section, p, p_m and p_c identify, in a generic sense, the property of a FGM, the property of metallic constituent and the property of the ceramic constituent, respectively. The term generic property, means indistinctly the longitudinal modulus of elasticity E_{xx} or the density ρ or the Poisson's coefficient μ, etc; n is the power index of the variation law.

n	h/L	Approach	f_1	f_2	f_3	f_4
1	0.025	1D FEM	20.37	117.27	341.79	674.63
		3D FEM	20.42	118.15	343.46	680.43
	0.1	1D FEM	320.19	1969.96	5402.62	10334.6
		3D FEM	322.31	1980.57	5469.64	10401.7
10	0.05	1D FEM	59.8	366.66	1026.53	2000.33
		3D FEM	59.37	362.97	1020.52	2012.39
	0.1	1D FEM	237.25	1461.48	4015.65	7669.83
		3D FEM	237.74	1462.73	4046.95	7713.62

Table 5: Comparison between 3D and curved beam approaches of the first four out-of-plane natural frequencies

Tab. 5 shows the comparison of the full 3D approach and the present curved beam model. The full 3D calculation is performed with a flexible 3D general solver (called FlexPDE) of partial differential equations with in the context of the finite element method. In this solver one can easily cope with the complex material laws to be included in the structural model as well as the model it self (see http://www.pdesolutions.com and Ramirez and Piovan (2009) for further explanations). The agreement is good and the differences in percentage no higher than 2% have been observed.

4.4 Dynamics of curved beams constructed with FGM

In this section parametric studies of the dynamics of curved beams constructed with FGM are carried out. All the numerical computations are performed with a curved beam having $b = 5h = 0.05\ m$, $L = 1\ m$ and with graded properties varying in the exponential form given in Eq. (58), from a metallic phase (steel, at $z = -h/2$) to a ceramic phase (alumina, at $z = h/2$).

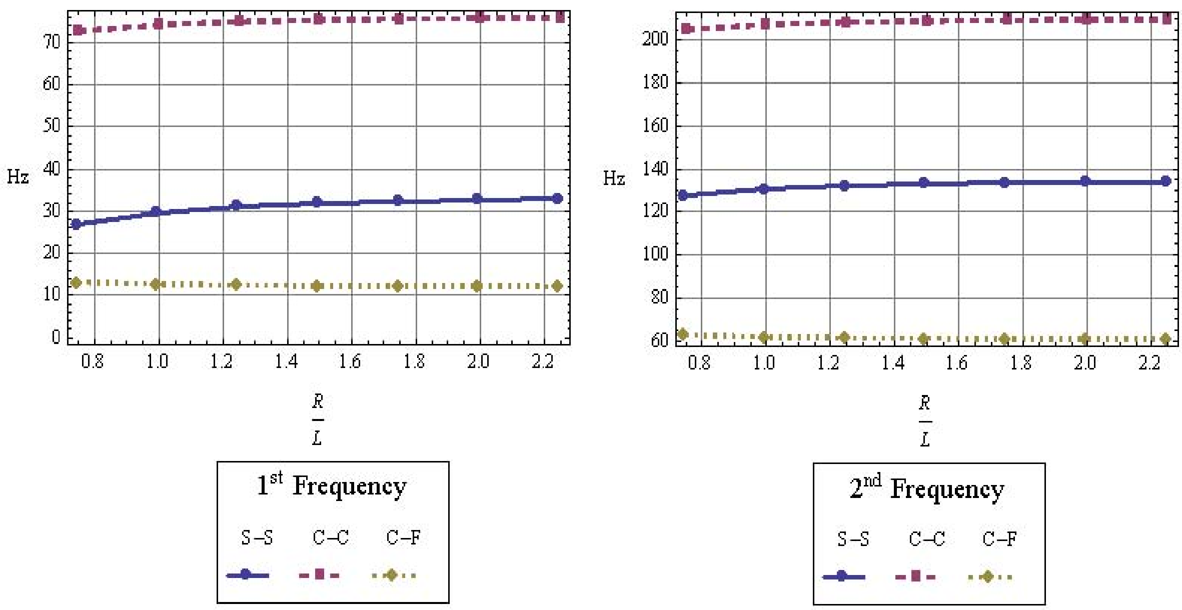

Figure 3: Variation of the first and second frequencies with the ratio R/L

In Fig. 3 and Fig. 4 one can see the variation of the first to the fourth frequencies of curved beams with doubly simply supported, doubly clamped and clamped-free boundaries, with respect to the ratio R/L. This implies a variation from a very curved beam to a straight beam as

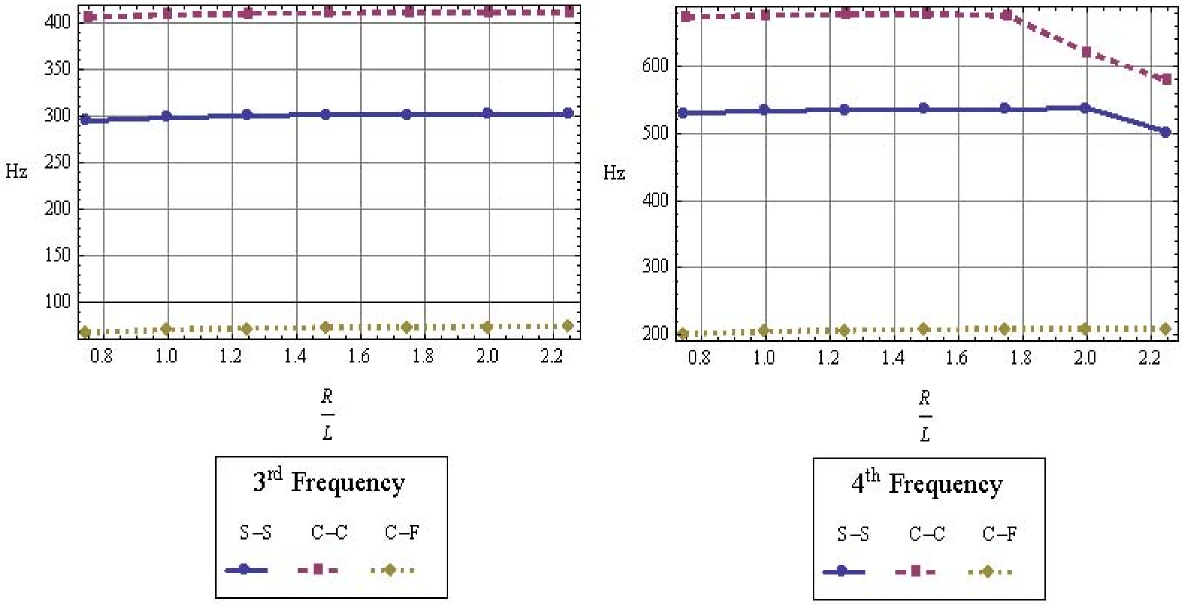

Figure 4: Variation of the third and fourth frequencies with the ratio R/L

$R/L \rightarrow \infty$. In Fig. 4 one can see a different behavior in the variation of the 4^{th} mode, this is due to the cross-over phenomenon that occurs in curved beams due to a different geometry (i.e. stiffness, R and L), changing the mode shape to another.

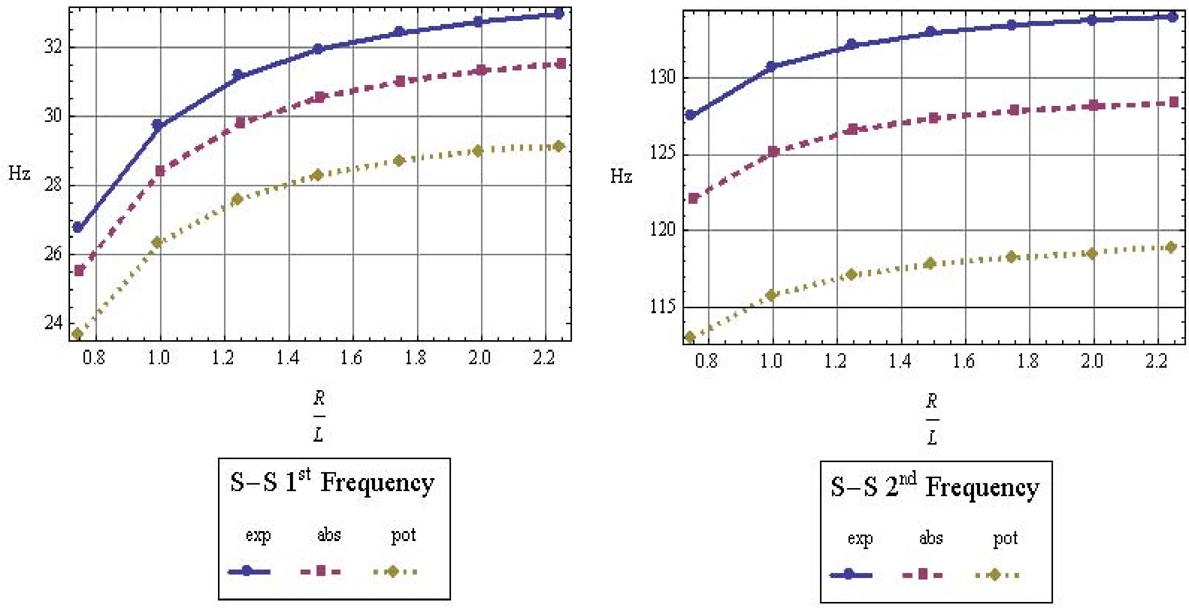

Figure 5: Variation of the 1^{st} and 2^{nd} frequencies with the ratio R/L

Fig. 5 shows the variation of the 1^{st} and 2^{nd} frequencies with respect to the ratio R/L of a simply supported curved beam. A comparison of the three laws of functionally graded materials given in Eq. (58), Eq. (64) and Eq. (65) is performed.

$$p = p_m + (p_c - p_m)\left(\frac{1}{2} + \frac{z}{h}\right)^n \quad (65)$$

As in the previous sections, p, p_m and p_c identify, in a generic sense, the property of a FGM, the property of metallic constituent and the property of the ceramic constituent, respectively.

5 CONCLUSIONS

A general model for curved beams constructed with functionally graded materials was derived. The model has been deduced applying the linearized Principle of Virtual Work based on a displacement field with first- and second-order terms. The displacement has been conceived to take into account shear flexibility in a full form. In the Principle of Virtual Work, arbitrary states of initial stresses and initial volume and surface forces, general initial off-axis forces have been considered. The present model can be employed for dealing with general dynamic and stability problems as well as general static problems of functionally graded curved beams. The model can be decoupled if appropriate restrictions in the geometry and the gradation of material properties are settled. Also the curved beam model can be reduced to a straight beam model. The model is quite efficient and predicts very well experimental results as well as results of full 3D finite element approaches. This point is very important if time cost is crucial, especially in active control and structural optimization which is the topic of the next development.

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