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STABILITY ANALYSIS OF ELASTIC PLATES UNDER NON-UNIFORM STRESS FIELDS BY THE BOUNDARY ELEMENT METHOD

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Abstract.

This paper presents a boundary element formulation to investigate the onset of instability of elastic plates with a wide variety of boundary conditions and arbitrary geometries. Stresses caused by external loads are calculated by the formulation of plane elasticity boundary element method. Then, these stresses are introduced as body forces in the classical formulation of plates. The domain integrals due to body forces are transformed into boundary integrals using the radial integration method. In this method, body forces are approximated by a sum of radial basis functions, called approximation functions, multiplied by coefficients to be determined. Functions used for approximations are known as thin plate splines. Various numerical examples are analyzed in which critical loads, buckling modes, and coefficients of buckling are calculated. The accuracy of the proposed formulation is assessed by comparison with results from literature.

1 INTRODUCTION

An understanding of buckling of structural components under compressive load has become particularly important with the introduction of steel and high-strength alloys in engineering structures, which resulted in more optimized components than those used in previous projects. Buckling analysis of compression panels also is particularly important in aerospace structures. Structures built with these materials and slender members may fail when subjected to compressive loads in your plan. In some cases these failures are not by direct compression, but for lateral buckling. The finite element method (FEM) is currently one of the most used tools by researchers to study the engineering problems of buckling of plates. Potentially powerful and relatively new, the numerical method of boundary elements (BEM), has also shown excellent results in the study of buckling of plates. Syngellakis and Elzein (1994) present solutions for the buckling of plates by boundary element method based on Kirchhoff's theory in different load conditions and support. Nerantzaki and Katsikadelis (1996) developed a boundary element method for analysis of buckling of plates with variable thickness. Linear buckling analysis of plates using the boundary element method also can be found in Lin et al. (1999). Buckling analysis of shear deformable isotropic plates was presented by Purbolaksono and Aliabadi (2005).

In this paper, a boundary element formulation for the stability analysis of general isotropic plates with no domain discretization is presented. Classical plate bending and plane elasticity formulations are used and the domain integrals due to non-uniform body forces are transformed into boundary integrals using the radial integration method. Numerical results are presented to assess the accuracy of the method. Buckling coefficients computed using the proposed formulation is compared with results available in literature.

2 GOVERNING EQUATIONS

Basically, the classic problem of buckling, is a geometrically nonlinear problem described by a set of three differential equations which can be uncoupled and linearized in the case of elastic critical loads. In the absence of body forces, the equations that describe the buckling of plates is given by:

$$\begin{aligned} \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= 0, \\ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} &= 0. \end{aligned} \tag{1}$$

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{1}{D}\left(N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} \right), \tag{2}$$

where w is the displacement in normal directions x and y, that is, displacement in the normal direction of the plate surface; N_{ij} are the stress components; and, D is plate stiffness constants.

2.1 Boundary integral equations

Here, in-plane stress fields, due to stress concentrations in the geometry, are non-uniform, and stress resultants in the domain due to external loads on the boundary is considered to be unknown. Therefore, determination of in-plane stress resultants in the domain is the first step in the solution of plate buckling. Next, the plate buckling equations are derived from the plate bending equations. Critical load factors are are introduced into the equations as multiplication factors of body forces or transverse loads.

The in-plane boundary integral equation for displacements, obtained by applying the reciprocity and Green theorems in equation (1), is given by (Aliabadi, 2002):

$$c_{ij}u_j(Q) + \int_\Gamma t^*_{ik}(Q,P)u_k(P)d\Gamma(P) = \int_\Gamma u^*_{ik}(Q,P)t_k(P)d\Gamma(P), \tag{3}$$

where $t_i = N_{ij}n_j$ is the traction in the boundary of the plate in the plane $x_1 - x_2$, and n_j is the normal at the boundary point; P is the field point; Q is the source point; and asterisks denote fundamental solutions. The constant c_{ij} is introduced in order to take into account the possibility that the point Q can be placed in the domain, on the boundary, or outside the domain.

The in-plane stress resultants at a point $Q \in \Omega$ are written as:

$$c_{ik}N_{kj}(Q) + \int_\Gamma S^*_{ikj}(Q,P)u_k(P)d\Gamma(P) = \int_\Gamma D^*_{ijk}(Q,P)t_k(P)d\Gamma(P), \tag{4}$$

where D_{ikj} and S_{ikj} are linear combinations of the plane-elasticity fundamental solutions. The integral equation for the plate buckling formulation, obtained by applying reciprocity and Green theorems at equation (2), is given by:

$$\begin{aligned} Ku_3(Q) &+ \int_\Gamma \left[V^*_n(Q,P)w(P) - m^*_n(Q,P)\frac{\partial w(P)}{\partial n} \right] d\Gamma(P) + \sum_{i=1}^{N_c} R^*_{c_i}(Q,P)u_{3_{ci}}(P) \\ &= \sum_{i=1}^{N_c} R_{c_i}(P)u^*_{3_{ci}}(Q,P) + \int_\Gamma \left[V_n(P)u^*_3(Q,P) - m_n(P)\frac{\partial u^*_3}{\partial n}(Q,P) \right] d\Gamma(P) \\ &+ \lambda \left[\int_\Omega u_3 N_{ij} u^*_{3,ij}\, d\Omega + \int_\Gamma \left(t_i u^*_3 u_{3,i} - t_i u_3 u^*_{3,i} \right) d\Gamma \right], \end{aligned} \tag{5}$$

where $\frac{\partial()}{\partial n}$ is the derivative in the direction of the outward vector $\mathbf{n}$ that is normal to the boundary Γ; m_n e V_n are, respectively, the normal bending moment and the Kirchhoff equivalent shear force on the boundary Γ; R_c is the thin-plate reaction of corners; $u^*_{3_{ci}}$ is the transverse displacement of corners; λ is the critical load factor; the constant K is introduced in order to take into account the possibility that the point Q can be placed in the domain, on the boundary, or outside the domain. As in the previous equation, an asterisk denotes a fundamental solution.

A second integral equation is necessary in order to obtain the thin plate buckling boundary element formulation. This equation is given by:

$$\begin{aligned} K\frac{\partial u_3}{\partial m}(Q) &+ \int_\Gamma \left[\frac{\partial V^*_n}{\partial m}(Q,P)w(P) - \frac{\partial M^*_n}{\partial m}(Q,P)\frac{\partial u_3(P)}{\partial n} \right] d\Gamma(P) + \sum_{i=1}^{N_c} \frac{\partial R^*_{c_i}}{\partial m}(Q,P)u_{3_{ci}}(P) \\ &= \sum_{i=1}^{N_c} R_{c_i}(P)\frac{\partial u^*_{3_{ci}}}{\partial m}(Q,P) + \int_\Gamma \left[V_n(P)\frac{\partial u^*_3(Q,P)}{\partial m} - m_n(P)\frac{\partial^2 u^*_3}{\partial n \partial m}(Q,P) \right] d\Gamma(P) \\ &+ \lambda \left[\int_\Omega u_3 N_{ij}\frac{\partial u^*_{3,ij}}{\partial m}\, d\Omega + \int_\Gamma \left(t_i u^*_3 \frac{\partial u_{3,i}}{\partial m} - t_i u_3 \frac{\partial u^*_{3,i}}{\partial m} \right) d\Gamma \right], \end{aligned} \tag{6}$$

where $\frac{\partial()}{\partial m}$ is the derivative in the direction of the outward vector $\mathbf{m}$ that is normal to the boundary Γ, at the source point Q.

Domain integrals arise in the formulation owing to the contribution of in-plane stresses to the out of plane direction. In order to transform these integrals into boundary integrals, consider that a body force b is approximated over the domain as a sum of M products between approximation functions f_m and unknown coefficients γ_m, that is:

$$b(P) \cong \sum_{m=1}^{M} \gamma_m f_m. \tag{7}$$

The approximation function used in this work is:

$$f_m = r^2 \log r, \tag{8}$$

Equation (7) can be written in a matrix form, considering all boundary and domain source points, as:

$$\mathbf{b} = \mathbf{F}\gamma \tag{9}$$

Thus, γ can be computed as:

$$\gamma = \mathbf{F^{-1}b}. \tag{10}$$

Body forces of integral equations (5) and (6) depend on displacements. So, using equation (10) and following the procedure presented by Albuquerque et al. (2007), domain integrals that come from these body forces can be transformed into boundary integrals.

As can be seen in equations (5) and (6), the body force that generates domain integrals is given by:

$$b = N_{ij} u_3. \tag{11}$$

So, we need to compute N_{ij} in each integration points. However, we have only the values of N_{ij} at nodes and internal points. Values of N_{ij} in integration points is computed by:

$$\mathbf{N_i(x_1, x_2) = f(r)F^{-1}N_{ij}}. \tag{12}$$

2.2 Matrix Equations

After the discretization of equations (5) and (6) into boundary elements and collocation of the source points in all boundary nodes, a linear system is generated. It is worth notice that the only loads considered in the linear buckling equations are that related to the in-plane stress N_{ij} and tractions t_i that are multiplied by the critical load factor λ. Furthermore, all the known values of u_3, $\partial u_3/\partial n$, M_n, V_n, w_{ci}, R_{ci} (boundary conditions) are set to zero. Dividing the boundary into Γ_1 e Γ_2 (Figure 1), this linear system can be written as:

$$\begin{bmatrix} \mathbf{H_{11}} & \mathbf{H_{12}} \\ \mathbf{H_{21}} & \mathbf{H_{22}} \end{bmatrix} \begin{Bmatrix} \mathbf{w_1} \\ \mathbf{w_2} \end{Bmatrix} - \begin{bmatrix} \mathbf{G_{11}} & \mathbf{G_{12}} \\ \mathbf{G_{21}} & \mathbf{G_{22}} \end{bmatrix} \begin{Bmatrix} \mathbf{V_1} \\ \mathbf{V_2} \end{Bmatrix} = \lambda \begin{bmatrix} \mathbf{M_{11}} & \mathbf{M_{12}} \\ \mathbf{M_{21}} & \mathbf{M_{22}} \end{bmatrix} \begin{Bmatrix} \mathbf{w_1} \\ \mathbf{w_2} \end{Bmatrix}, \tag{13}$$

where Γ_1 stands for the part of the boundary where displacements and rotations are zero and Γ_2 stands for the part of the boundary where bending moment and tractions are zero. Indices 1 and

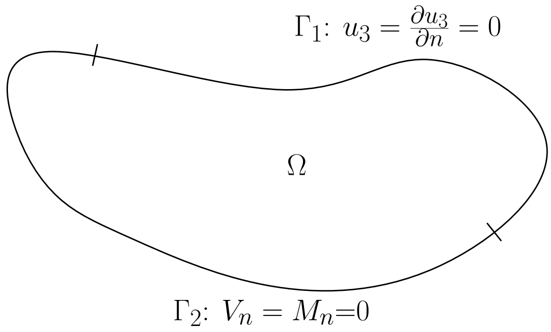

Figure 1: Domain with constrained and free degrees of freedom.

2 stand for boundaries Γ_1 and Γ_2, respectively. Matrices **H, G**, and **M** are influence matrices of the boundary element method due to integral terms of equations (5) and (6).

As $\mathbf{w_1} = \mathbf{0}$ and $\mathbf{V_2} = \mathbf{0}$, equation (13) can be written as:

$$\begin{aligned} \mathbf{H_{12}w_2} - \mathbf{G_{11}V_1} &= \lambda \mathbf{M_{12}w_2}, \\ \mathbf{H_{22}w_2} - \mathbf{G_{21}V_1} &= \lambda \mathbf{M_{22}w_2} \end{aligned} \tag{14}$$

or,

$$\hat{\mathbf{H}}\mathbf{w_2} = \lambda \hat{\mathbf{M}}\mathbf{w_2}, \tag{15}$$

where, $\hat{\mathbf{H}}$ e $\hat{\mathbf{M}}$, are given by:

$$\begin{aligned} \hat{\mathbf{H}} &= \mathbf{H_{22}} - \mathbf{G_{21}G_{11}^{-1}H_{12}}, \\ \hat{\mathbf{M}} &= \mathbf{M_{22}} - \mathbf{G_{21}G_{11}^{-1}M_{12}}. \end{aligned} \tag{16}$$

The matrix equation (15) can be rewritten as an eigen vector problem

$$\mathbf{A}\mathbf{w_2} = \frac{1}{\lambda}\mathbf{w_2}, \tag{17}$$

where,

$$\mathbf{A} = \hat{\mathbf{H}}^{-1}\hat{\mathbf{M}}. \tag{18}$$

Provided that $\mathbf{A}$ is non-symmetric, eigenvalues and eigenvectors of equation (17) can be found using standard numerical procedures for non symmetric matrices.

2.3 Numerical results

The numerical results are presented in terms of the dimensionless parameter K_{cr} which is given by:

$$K_{cr} = \frac{N_{cr}a^2}{\pi^2 D} \tag{19}$$

where, N_{cr} is the critical load and a is the edge length of the square plate.

In this work it was considered a square plate with a square hole under different boundary conditions. The ratio between length a and thickness h of the square plate is $a/h = 100$.

The ratio between the edge length of the plate and the edge length of the hole is a/b = 5. The material properties are: elastic moduli $E = 210\ GPa$ and Poisson ratio $\nu = 0,25$. The mesh used has 24 quadratic discontinuous boundary elements (12 elements of equal length at the external boundary and 12 elements of equal length at the hole) with uniformly distributed internal points.

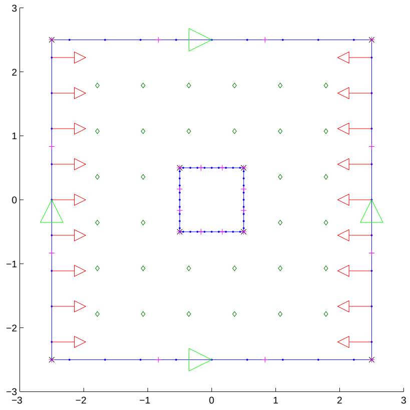

Figure 2: Boundary element model (24 discontinuous boundary element and 32 internal points).

The plate is under uniformly uniaxial compression and the critical load parameter K_{cr} is computed considering all edges simply-supported (SSSS) and all edges clamped (CCCC).

Table 1: Critical load parameter K_{cr} for a square plate with a square hole in the center (simply-supported).

No.	Boundary elements	Internal points	K BEM	K Analytical	Error. %
1	24	32	3.620	3.720	2,60
2	24	60	3.769	3.720	1.30
3	24	96	3.694	3.720	0.07

Table 2: Critical load parameter K_{cr} for a square plate with a square hole in the center (clamped).

No.	Boundary elements	Internal points	K BEM	K Analytical	Error. %
1	24	32	7.892	8.766	9.97
2	24	60	8.495	8.766	3.10
3	24	96	8.841	8.766	0.08

The first buckling mode are shown in figure 4 and 5. Critical load parameters K_{cr} obtained by the boundary element formulation using different number of internal points are shown in table 1 and table 2, for simply-supported and clamped edges, respectively, together with analytical results presented by Hayash et al. (1971). As it can be seen, errors decrease with increasing number of internal points.

As it can be seen, there is a good agreement between the results obtained in this work and those presented in literature.

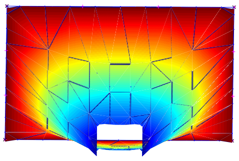

Figure 3: First buckling mode - all edges SSSS

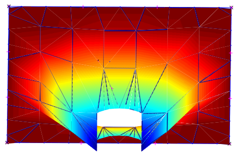

Figure 4: First buckling mode - all edges CCCC

3 CONCLUSIONS

This paper presented a boundary element formulation for the stability analysis of plates with non-uniform stress field. Domain integrals are transformed into boundary integrals by the radial integration method. As the radial integration method does not demand particular solutions, it is easier to implement than the dual reciprocity boundary element method. The formulation is applied for a square plate with a square hole. Results obtained with the proposed formulation are in good agreement with results presented in literature. It was shown that errors decrease with an increasing number of internal points.

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