Asociación Argentina de Mecánica Computacional

Mecánica Computacional Vol XXVIII, págs. 1477-1488 (artículo completo)
Cristian García Bauza, Pablo Lotito, Lisandro Parente, Marcelo Vénere (Eds.)
Tandil, Argentina, 3-6 Noviembre 2009

GEOMETRIC CONSERVATION LAW IN ALE FORMULATIONS

Luciano Garelli, Rodrigo R. Paz and Mario A. Storti

Centro Internacional de Métodos Computacionales en Ingeniería (CIMEC), (INTEC-CONICET-UNL)
Güemes 3450, (S3000GLN) Santa Fe, Argentina.

Keywords: Geometric Conservation Law, ALE (Arbitrary Lagrangian-Eulerian) formulations, Moving meshes.

Abstract. The aim of this work is to study the influence of the Geometric Conservation Law (GCL) when numerical simulations are performed on deforming domains with an Arbitrary Lagrangian-Eulerian (ALE) formulation. This analysis is carried out in the context of the Finite Element Method (FEM) for the scalar advection-diffusion equation defined on a moving domain.

Solving the problem on a moving mesh using an ALE formulation needs the computation of some geometric quantities, such as element volumes and Jacobians, which involve the nodal positions and velocities. The so-called Geometric Conservation Law (GCL) is satisfied if the algorithm can exactly reproduce a constant solution on moving grids.

Not complying with the GCL means that the stability of the time integration is not assured and, thus, the order of convergence could not be preserved. To emphasize the importance of fulfilling the GCL, numerical experiments are performed in 2D using several mesh movements. In these experiments different temporal integration schemes have been used .

1 INTRODUCTION

When dealing with partial differential equations that needs to be solve on moving domains, like Fluid-Structure Interaction problems (Storti et al., 2009; Garelli et al., 2008), one of the most used techniques is the so-called *Arbitrary Lagrangian Eulerian* (ALE) formulation. The key idea in the ALE formulation is the introduction of a computational mesh which can move with a velocity independent of the velocity of the material particles. The ALE method were first proposed in the context of finite difference in works Noh (1964); Hirt et al. (1974), then it was extended to finite elements in Donea (1983); Hughes et al. (1978) and to finite volumes in Trepanier et al. (1991). The reformulation of the equation in an ALE scheme introduces additional terms, which are related to the grid velocity and deformation.
The *Geometric Conservation Law* (GCL) is directly related with the evolution of the mesh velocity and the change of the elemental area or volume. This law was introduced by Thomas and Lombard (1979) and it is a consistency criterion, being that the numerical method must be able to reproduce exactly a constant solution on a moving domain. In practice the GCL can be violated, but its relationship with the stability and the accuracy of the numerical scheme have not been clarified yet. Several works have been done in this direction (Boffi and Gastaldi, 2004; Formaggia and Nobile, 2004; Étienne et al., 2009) with the aim of establish the importance of the GCL.
In this paper the analysis is carried out in the context of the Finite Element Method (FEM) for the scalar advection-diffusion equation defined on a moving domain. The behavior of two well-know first and second order time advancing schemes, like Implicit Euler and Crank-Nicolson is analyzed. Several types of movements are imposed on the domain with different degrees of regularity in the mesh velocity. In the next section an introduction to the problem on moving domains and to its ALE formulation is given in the context of finite element discretization.

2 ALE FORMULATION OF THE MODEL PROBLEM

The analysis of the GCL compliance can be performed in any kind of problem that is solved on a moving domain, due to the GCL naturally arises in ALE formulations. Let consider, for the sake of clarity, the linear advection diffusion model of the type

$$\begin{aligned} \frac{\partial u}{\partial t} + \nabla_{\boldsymbol{x}} \cdot (\boldsymbol{\beta} u) - \mu \Delta u = g \quad & \text{for } \boldsymbol{x} \in \Omega_{\rm t}, \mathrm{t} \in [0, \mathrm{T}] \\ u = u_0 \quad & \text{for } \boldsymbol{x} \in \Omega_0, \mathrm{t} = 0 \\ u = u_D \quad & \text{for } \boldsymbol{x} \in \partial\Omega_{\rm t}, \mathrm{t} \in [0, \mathrm{T}] \end{aligned} \tag{1}$$

is considered, where $\boldsymbol{\beta}$ is the convective velocity vector, which satisfy $\nabla_{\boldsymbol{x}} \cdot \boldsymbol{\beta} = 0$, μ is the constant diffusivity and Δ is the Laplacian operator.
To recast the original problem (1) to an arbitrary Lagrangian-Eulerian frame, a set of mapping $\mathcal{A}_t$ must be defined. $\mathcal{A}_t$ maps, at each time $t \in]0, T]$, the point $\boldsymbol{\xi} = (\xi, \eta)$ of a reference domain Ω_0 to a point $\boldsymbol{x}$=(x,y) on the current domain Ω_t.

$$\mathcal{A}_t : \Omega_0 \rightarrow \Omega_t, \;\; \boldsymbol{x}(\boldsymbol{\xi}, \mathrm{t}) = \mathcal{A}_{\rm t}(\boldsymbol{\xi}) \tag{2}$$

The ALE mapping must fulfill some requirements, like surjection, bounded and Lipschitz continuity. A more detailed description about these requirements can be founded in Donea and Huerta (2003); Boffi and Gastaldi (2004); Formaggia and Nobile (2004). If the evolution of the boundary domain is known, several techniques can be used to construct the ALE mapping. The

current domain can be obtained from a harmonic extension of the boundary (Nobile, 2001) or it can be considered as an elastic body which is deformed (Gastaldi, 2001).
Now, a brief introduction of the ALE kinematics is done. Therefore, let consider a scalar function $f : \Omega_t \times]0, T]$ defined on the Eulerian frame and the function $\hat{f} := f \circ \mathcal{A}_t$ defined on the ALE frame, where

$$\hat{f}(\boldsymbol{\xi}, t) = f(\mathcal{A}_t(\boldsymbol{\xi}), t), \quad \hat{f} : \Omega_0 \times]0, T] \tag{3}$$

being the time derivative in the ALE frame defined as follows

$$\left.\frac{\partial f}{\partial t}\right|_{\xi} (\boldsymbol{x}, t) = \frac{\partial \hat{f}}{\partial t}(\boldsymbol{\xi}, t), \qquad \xi = \mathcal{A}_t^{-1}(\boldsymbol{x}). \tag{4}$$

Likewise the partial time derivative in the Eulerian frame is defined as $\left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{x}}$ and the mesh velocity $\mathbf{w}$ as

$$\mathbf{w}(\boldsymbol{x}, t) = \left.\frac{\partial \boldsymbol{x}}{\partial t}\right|_{\boldsymbol{\xi}} (\mathcal{A}_t^{-1}, t). \tag{5}$$

In order to obtain the ALE counterpart of (1) and assuming that u is regular enough, the chain rule is applied to the time derivatives

$$\left.\frac{\partial u}{\partial t}\right|_{\boldsymbol{\xi}} = \left.\frac{\partial u}{\partial t}\right|_{\boldsymbol{x}} + \left.\frac{\partial \boldsymbol{x}}{\partial t}\right|_{\boldsymbol{\xi}} \cdot \nabla_{\boldsymbol{x}} u = \left.\frac{\partial u}{\partial t}\right|_{\boldsymbol{x}} + \mathbf{w} \cdot \nabla_{\boldsymbol{x}} u, \tag{6}$$

and substituting (6) in (1), the Eulerian time derivative is replaced by the ALE time derivative

$$\begin{aligned} \left.\frac{\partial u}{\partial t}\right|_{\xi} - \mathbf{w} \cdot \nabla_{\boldsymbol{x}} u + \nabla_{\boldsymbol{x}} \cdot (\boldsymbol{\beta} u) - \mu \Delta_{\boldsymbol{x}} u = g \quad & \text{for } \boldsymbol{x} \in \Omega_{\mathrm{t}}, \mathrm{t} \in [0, \mathrm{T}] \\ u = u_0 \quad & \text{for } \boldsymbol{x} \in \Omega_0, \mathrm{t} = 0 \\ u = u_D \quad & \text{for } \boldsymbol{x} \in \partial\Omega_{\mathrm{t}}, \mathrm{t} \in [0, \mathrm{T}]. \end{aligned} \tag{7}$$

To obtain the variational formulation of (7) in a conservative form, the Reynolds transport theorem in employed. Let $f(\boldsymbol{x}, t)$ a scalar function over a arbitrary volume $\mathcal{V}_t$, so the material time derivative of the integral of $f(\boldsymbol{x}, t)$ can be expressed as

$$\frac{d}{dt} \int_{\mathcal{V}_t} f(\boldsymbol{x}, t)\, d\mathcal{V} = \int_{\mathcal{V}_t} \frac{\partial f(\boldsymbol{x}, t)}{\partial t}\, d\mathcal{V} + \int_{\mathcal{S}_t} f(\boldsymbol{x}, t)\, \mathbf{w} \cdot n\, d\mathcal{S}. \tag{8}$$

The first term of the r.h.s of (8) represents the spatial time derivative of the volume integral and the boundary integral represents the flux of the scalar quantity f across the boundary. Now, using the divergence theorem one obtains

$$\frac{d}{dt} \int_{\mathcal{V}_t} f(\boldsymbol{x}, t)\, d\mathcal{V} = \int_{\mathcal{V}_t} (\left.\frac{\partial f}{\partial t}\right|_{\xi} + f\, \nabla_{\boldsymbol{x}} \cdot \mathbf{w})\, d\mathcal{V} = \int_{\mathcal{V}_t} (\left.\frac{\partial f}{\partial t}\right|_{\mathbf{x}} + \nabla_{\boldsymbol{x}} f \cdot \mathbf{w} + f\, \nabla_{\boldsymbol{x}} \cdot \mathbf{w})\, d\mathcal{V}. \tag{9}$$

2.1 Weak formulation in the ALE framework

In order to derive the weak formulation a finite element space compatible with ALE mapping must be defined

$$\mathcal{H}(\Omega_t) = \{\psi : \Omega_t \times]0, T] \to \mathbb{R} : \psi = \hat{\psi} \circ \mathcal{A}_t^{-1}, \hat{\psi} \in \mathcal{H}_0^1(\Omega_0)\}, \tag{10}$$

and with some mathematical manipulation of (7) and (9) the following weak formulation in conservative form is reached and reads as follow:
For all $t \in]0, T]$, find $u \in \mathcal{H}(\Omega_t)$ such that

$$\frac{d}{dt}\int_{\Omega_t} u\psi\, d\Omega + \int_{\Omega_t} \nabla_{\boldsymbol{x}} u \cdot \nabla_{\boldsymbol{x}}\psi\, d\Omega + \int_{\Omega_t} \nabla_{\boldsymbol{x}} \cdot [(\boldsymbol{\beta} - \mathbf{w})u]\psi\, d\Omega = \int_{\Omega_t} g\,\psi\, d\Omega, \qquad \forall\, \psi \in \mathcal{H}(\Omega_t). \tag{11}$$

with the corresponding boundary and initial conditions

$$\begin{aligned} u &= u_0 \quad \text{for } \boldsymbol{x} \in \Omega_0, \mathrm{t} = 0 \\ u &= u_D \quad \text{for } \boldsymbol{x} \in \partial\Omega_{\mathrm{t}}, \mathrm{t} \in]0, \mathrm{T}], \end{aligned}$$

With the problem formulated in an ALE framework the next objective is to solve the problem (11) via the Finite Element Method using a Galerkin formulation. The numerical solution u_h will then be sought in the discrete space $\mathcal{H}_h(\Omega_t)$ and u_h is expressed as linear combination of nodal finite element basis,

$$u_h(\boldsymbol{x}, t) = \sum_{i\in N} \psi_i(\boldsymbol{x}, t)u_i(t). \tag{12}$$

An important difference with the classical finite element formulation for time-dependent problems is that here the basis are also time-dependent.
So, the finite element semi-discrete approximation of (11) is written as:
For all $t \in]0, T]$, find $u_h \in \mathcal{H}_h(\Omega_t)$, such that

$$\frac{d}{dt}\int_{\Omega_t} u_h\psi_h\, d\Omega + \int_{\Omega_t} \nabla_{\boldsymbol{x}} u_h \cdot \nabla_{\boldsymbol{x}}\psi_h\, d\Omega + \int_{\Omega_t} \nabla_{\boldsymbol{x}} \cdot [(\boldsymbol{\beta} - \mathbf{w})u_h]\psi_h\, d\Omega = \int_{\Omega_t} g\,\psi_h\, d\Omega, \qquad \forall\, \psi_h \in \mathcal{H}_h(\Omega_t). \tag{13}$$

with the corresponding discrete boundary and initial conditions

$$\begin{aligned} u_h &= u_{0h} \quad \text{for } \boldsymbol{x} \in \Omega_0, \mathrm{t} = 0 \\ u_h &= u_{Dh} \quad \text{for } \boldsymbol{x} \in \partial\Omega_{\mathrm{t}}, \mathrm{t} \in]0, \mathrm{T}]. \end{aligned}$$

Finally, it must be chosen a suitable time discretization scheme. But, before to do this, the approximation (13) can be rewritten in a compact form (14), which helps to understand the term involved in the compliance of the GCL, so that

$$\frac{d}{dt}(\mathbf{M}_{(t)}\mathbf{U}_{(t)}) + (\mathbf{K}_{(t)} - \mathbf{A}_{(t,\mathbf{w})} - \mathbf{B}_{(t,\mathbf{w})})\mathbf{U}_{(t)} = \mathbf{G}_{(t)}, \tag{14}$$

where

$$\begin{aligned}
\mathbf{M}_{(t)} &= \int_{\Omega_t} \psi_i(t)\psi_j(t)\, d\Omega \\
\mathbf{K}_{(t)} &= \int_{\Omega_t} \nabla_{\boldsymbol{x}}\psi_j(t) \cdot \nabla_{\boldsymbol{x}}\psi_i(t)\, d\Omega + \int_{\Omega_t} \nabla_{\boldsymbol{x}} \cdot (\boldsymbol{\beta}\psi_j(t))\psi_i(t)\, d\Omega \\
\mathbf{A}_{(t,\mathbf{w})} &= \int_{\Omega_t} \nabla_{\boldsymbol{x}} \cdot \mathbf{w}(t)\psi_j(t)\psi_i(t)\, d\Omega \\
\mathbf{B}_{(t,\mathbf{w})} &= \int_{\Omega_t} \mathbf{w}(t) \cdot \nabla_{\boldsymbol{x}}\psi_j(t)\psi_i(t)\, d\Omega \\
\mathbf{G}_{(t)} &= \int_{\Omega_t} g(t)\psi_i(t)\, d\Omega
\end{aligned}$$

2.2 Temporal discretization of the semi-discrete system.

Taken into account the system (14) in a semi-discrete form, a time integration method must be used in order to get the fully discrete system. In this work the so-called $\theta-$family approximation is used, obtaining

$$\frac{\mathbf{M}^{n+1}\mathbf{U}^{n+1}-\mathbf{M}^{n}\mathbf{U}^{n}}{\Delta t}+(\mathbf{K}^{n+\theta}-\mathbf{A}^{n+\theta}-\mathbf{B}^{n+\theta})\mathbf{U}^{n+\theta}=\mathbf{G}^{n+\theta} \tag{15}$$

where for example the evaluation of $\mathbf{F}^{n+\theta}=(1-\theta)\mathbf{F}^{n}+\theta\mathbf{F}^{n+1}$.
For $\theta=1$ we get the *Backward Euler* method, which is unconditionally stable in fixed domains and first-order accurate in time (i.e.,$\mathcal{O}(\Delta t)$). For $\theta=0.5$ we get the *Crank-Nicolson* method, which is unconditionally stable in fixed domains and second-order accurate in time (i.e.,$\mathcal{O}(\Delta t^2)$). Though, the *Crank-Nicolson* scheme has no time step restriction for stability, but it could suffer nonphysical oscillatory response in the solution when the time step exceed a critical value which is given for the largest eigenvalue of the system.

3 THE GEOMETRIC CONSERVATION LAW

As stated in the section §1 the GCL compliance is interpreted as a consistency criterion for the numerical method. When the GCL is satisfied the algorithm is able to reproduce a constant solution on a moving domain independently of the velocity and distortion of the mesh, as it is shown in next sections.
At the continuous level the ALE formulation is equivalent to the original problem stated in a fixed domain; but in the discrete form that is not longer true. If in eq. (13) the source term and the advective velocity are null ($g=0,\boldsymbol{\beta}=0$) and considering a constant field solution, the GCL can be written as

$$\int_{\Omega_{n+1}}\psi_i^{n+1}\psi_j^{n+1}\,d\Omega-\int_{\Omega_n}\psi_i^{n}\psi_j^{n}\,d\Omega=\mathcal{Q}\left(\int_{\Omega_t}\psi_i(t)\psi_j(t)\nabla_{\boldsymbol{x}}\cdot\mathbf{w}_h(t)\,d\Omega\right)\qquad\forall i,j\in N. \tag{16}$$

So, a sufficient condition to satisfy the equality (16) is to use a time integration scheme $\mathcal{Q}$ (at least for the ALE term) with degree $d\cdot s-1$, where d is the space dimension and s is the order of the polynomial used to represent the time evolution of the nodal displacement. In this work a piecewise linear polynomial is used and two-dimensional problem are solved, therefore the time integration rule $\mathcal{Q}$ should integrate exactly a linear polynomial. Demonstrations of this proposition can be found in Boffi and Gastaldi (2004); Formaggia and Nobile (2004).
As an example, the *Backward Euler* and the *Crank-Nicolson* methods are used to integrate the r.h.s of (16) for a triangle which is deformed between t^n and t^{n+1}. The node 1 of the triangle is fixed, the node 2 has a velocity $(w_{2,x},w_{2,y})=(1,0)$ and the node 3 has a velocity $(w_{3,x},w_{3,y})=(0,1)$. The length of the sides in t^n are $L_x^n=L_y^n=1$ and in $t^{n+1}=t^n+\Delta t$ are $L_x^{n+1}=L_y^{n+1}=1.1$, as shown in Figure (1).
The evaluation of the l.h.s is independent of the integration rule $\mathcal{Q}$, and for the case in which $i=j=1$ in eq. (16) (i.e., only one node integrals are computed for the sake of simplicity),

$$\int_{A_{n+1}}(\psi_1^{n+1})^2\,dA-\int_{A_n}(\psi_1^{n})^2\,dA=\frac{1}{6}A^{n+1}-\frac{1}{6}A^{n}=\frac{0.605}{6}-\frac{0.5}{6}=0.0175. \tag{17}$$

Now introducing the *Backward Euler* method, the r.h.s of (16) is approximated as

$$\int_{tn}^{tn+1}\int_{\Omega_t}\psi_i(t)\psi_j(t)\nabla_{\boldsymbol{x}}\cdot\mathbf{w}_h(t)\,d\Omega\approx\Delta t\int_{\Omega_{tn+1}}\psi_i^{n+1}\psi_j^{n+1}\nabla_{\boldsymbol{x}}\cdot\mathbf{w}_h^{n+1}\,d\Omega, \tag{18}$$

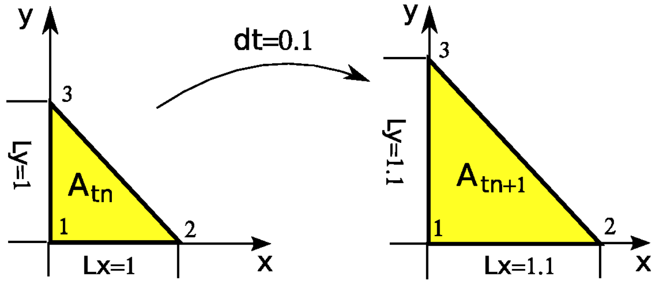

Figure 1: Deformation of a triangle during a time step.

where the $\nabla_{\boldsymbol{x}} \cdot \mathbf{w}_h$ is approximated with the same basis used in u_h,

$$\nabla_{\boldsymbol{x}} \cdot \mathbf{w}_h = \sum_{i=1}^{3} \frac{\partial \psi_i}{\partial x_j}(\boldsymbol{x}, t) w_{i,j}(t), \; j = 1..2, \tag{19}$$

being for this case

$$\nabla_{\boldsymbol{x}} \cdot \mathbf{w}_h^n = \sum_{i=1}^{3} \frac{\partial \psi_i}{\partial x_j}^n w_{i,j}^n = (1+1) = 2$$

$$\nabla_{\boldsymbol{x}} \cdot \mathbf{w}_h^{n+1} = \sum_{i=1}^{3} \frac{\partial \psi_i}{\partial x_j}^{n+1} w_{i,j}^{n+1} = \left(\frac{1}{1.1} + \frac{1}{1.1}\right) = 1.8\,\widehat{18}\;.$$

wherewith

$$\Delta t \int_{A_{tn+1}} (\psi_1^{n+1})^2 \nabla_{\boldsymbol{x}} \cdot \mathbf{w}_h^{n+1} \, dA = \Delta t \frac{A^{n+1}}{6} \nabla_{\boldsymbol{x}} \cdot \mathbf{w}_h^{n+1} = 0.1 \frac{0.605}{6} 1.818 = 0.01833. \tag{20}$$

If the *Crank-Nicolson* method is used as integration rule for the r.h.s of (16)

$$\begin{aligned} &\int_{tn}^{tn+1} \int_{\Omega_t} \psi_i(t)\psi_j(t) \nabla_{\boldsymbol{x}} \cdot \mathbf{w}_h(t) \, d\Omega \approx \\ &\approx \frac{\Delta t}{2} \left(\int_{\Omega_{tn+1}} \psi_i^{n+1}\psi_j^{n+1} \nabla_{\boldsymbol{x}} \cdot \mathbf{w}_h^{n+1} \, d\Omega + \int_{\Omega_{tn}} \psi_i^n \psi_j^n \nabla_{\boldsymbol{x}} \cdot \mathbf{w}_h^n \, d\Omega \right) = \\ &= \frac{\Delta t}{2} \left(\frac{A^{n+1}}{6} \nabla_{\boldsymbol{x}} \cdot \mathbf{w}_h^{n+1} + \frac{A^n}{6} \nabla_{\boldsymbol{x}} \cdot \mathbf{w}_h^n \right) = \frac{0.1}{2} \left(\frac{0.605}{6} 1.8\,\widehat{18} + \frac{0.5}{6} 2 \right) = 0.0175. \end{aligned} \tag{21}$$

In this example it is evident that the *Backward Euler* time integrator do not satisfy the identity (16). So, an error in the temporal integration (in the ALE term) is introduced and it is due to the lack of compliance with the GCL. Moreover, it behaves as an elemental source or sink when solving transient problems and depending on whether the element expands or shrinks. The magnitude of this error decreases with the reduction of the time step and the mesh velocity (see section §4.1).

Several propositions have been made in other works in order to have temporal θ-integrators complying the GCL. One of these modifications consists of using a temporal integration rule of higher order for ALE terms. But this alternative is less desirable from the programming point of view and it may be more convenient to treat all terms equally.

In the next section the problem introduced in §2 is used to verify the GCL with the two temporal schemes proposed. Then, a numerical analysis is carried out regarding the effect in the FEM solutions when complying or not the GCL. Finally, using the method of 'Manufactured Solution' (Roache, 1998; Roy, 2004) a convergence analysis is performed on a moving domain.

4 NUMERICAL TEST

4.1 Constant solution test

As stated in the previous section, a code that satisfy the GCL must be able to reproduce a constant solution. The problem (13) is solved in an unit square domain with a source term $g = 0$, $\boldsymbol{\beta} = 0$ and $\mu = 0.1$, so that

$$\begin{aligned} u_t - 0.1\Delta u &= 0 \quad \text{for } \boldsymbol{x} \in \Omega_{\mathrm{t}}, \mathrm{t} \in [0, \mathrm{T}], \\ u_0 &= 1 \quad \text{for } \boldsymbol{x} \in \Omega_0, \mathrm{t} = 0, \\ u &= 1 \quad \text{for } \boldsymbol{x} \in \partial\Omega_{\mathrm{t}}, \mathrm{t} \in [0, \mathrm{T}], \end{aligned} \tag{22}$$

being the domain deformed according to the following rule

$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^n + 0.125 \ \sin(\pi t) \ \sin(2\pi \, \boldsymbol{x}^n). \tag{23}$$

Figure (2) shows the reference domain and the deformed domain for $t = 0.5$ where the maximum deformation occurs. The problem is solved using piecewise linear triangles for the spatial

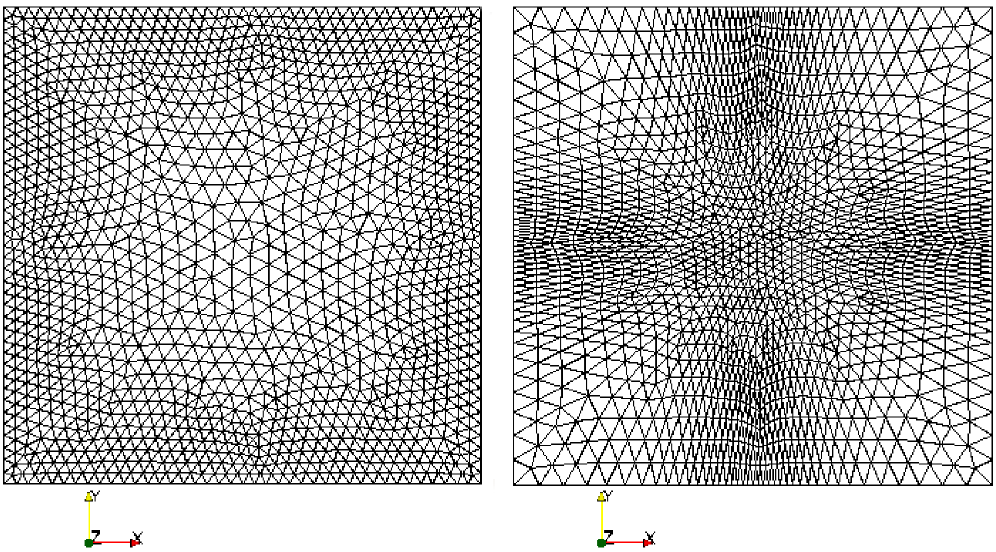

Figure 2: Reference and deformed domains.

discretization, a piecewise linear interpolation of the mesh movement and for the time integration the *Backward Euler* and *Crank-Nicolson* schemes are considered with Δt =0.2, 0.1, 0.05, 0.025. Figure (3) reports the error $||u_h - u||_{L^2(\Omega_n)}$ for three periods of oscillation, which must remain close to zero (in machine precision ($\approx 10^{-14}$)) over time for a GCL compliant scheme. An error is introduced when using the Backward Euler scheme due to lack in GCL compliance. We can see in Figure (3) that the error $||u_h - u||_{L^2(\Omega_n)}$ grows linearly in time and roughly behaves proportional to time step and mesh velocity. This test was performed with different mesh movements with similar results. In Figure (4) the solution in $t = 6$ $[s]$ is shown for both integration schemes. The error with respect to the constant solution are localized in the zones of the domain where the element deformation is higher, like in the center and the corners.

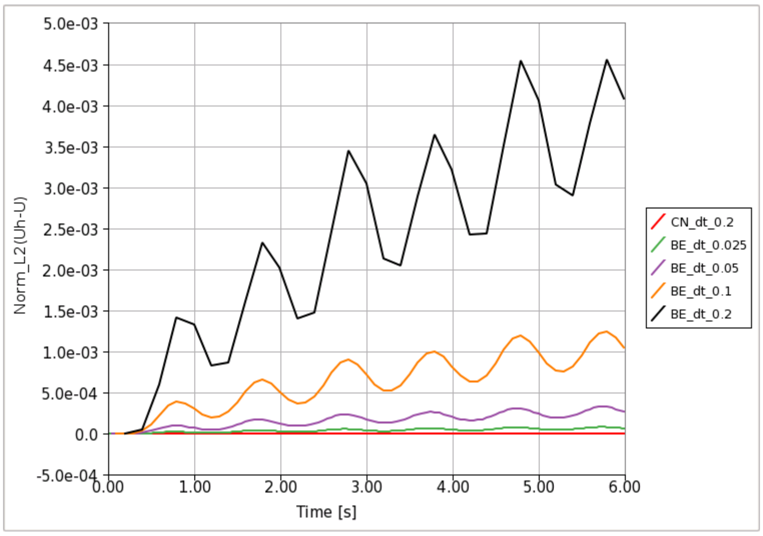

Figure 3: L^2-norm of the error for *Backward Euler* (BE) and *Crank-Nicolson* (CN) schemes.

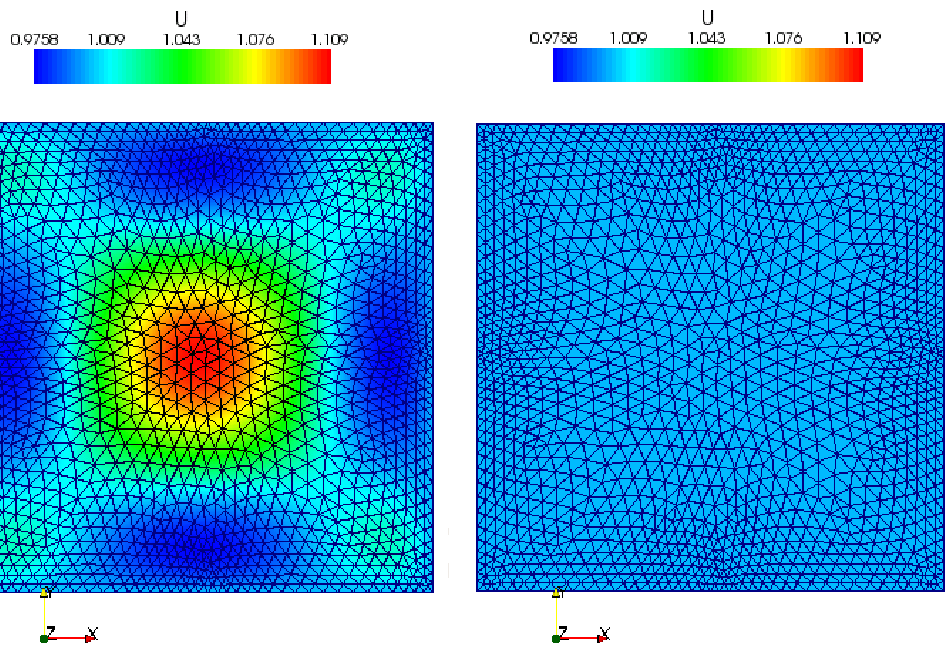

Figure 4: The solution in $t = 6[s]$ for the *Backward Euler* (BE) and *Crank-Nicolson* (CN) schemes.

4.2 Stability analysis

With the aim of check the stability of the schemes under analysis the following problem is considered

$$\begin{aligned} &u_t - 0.01\Delta u = 0 && \text{for } \boldsymbol{x} \in \Omega_\text{t}, \text{t} \in [0, \text{T}], \\ &u_0 = 1600x(1-x)y(1-y) && \text{for } \boldsymbol{x} \in \Omega_0, \text{t} = 0, \\ &u = 0 && \text{for } \boldsymbol{x} \in \partial\Omega_\text{t}, \text{t} \in [0, \text{T}], \end{aligned} \tag{24}$$

which is deformed according to the following rule

$$\mathcal{A}_t(\boldsymbol{\xi}) = \boldsymbol{x}(\boldsymbol{\xi}, t) = (2 - \cos(20\pi t))\boldsymbol{\xi}. \tag{25}$$

This problem was first proposed by Boffi and Gastaldi (2004); Formaggia and Nobile (2004), where it is demonstrated that the solution has a monotone decreasing L^2-norm in time for the continuous case. The problem is solved for Δt =0.02, 0.01, 0.001, 0.0001 and the L^2-norm plotted as a function of time for a simulation of 0.4 [s]. When the *Backward Euler* is used

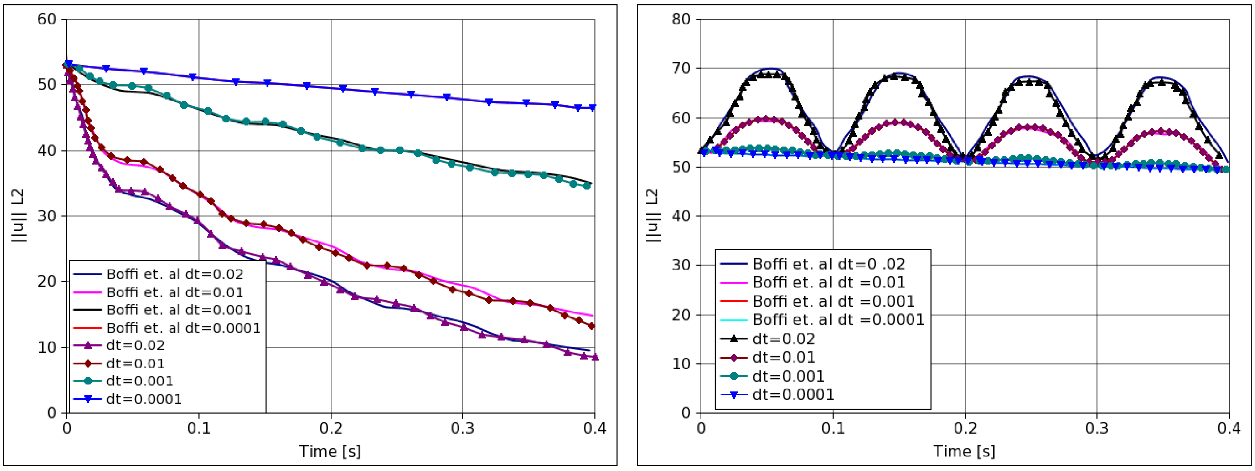

Figure 5: The L^2-norm of the solution u_h for the *Backward Euler* (BE) and *Crank-Nicolson* (CN) schemes.

the L^2-norm of the solution u_h has a strict monotone decay, but the method introduce an extra numerical diffusion increasing the rate of decay. If the *Crank-Nicolson* method is used, no extra diffusion is added but it produces oscillations in the L^2-norm of the solution u_h. These oscillations appear when the advective-like ALE term is introduced in the pure diffusion problem stated in a fixed domain (24). That can be easily seen if the mesh Péclet number is computed based on the mesh velocity, which can be derived from (25),

$$\mathbf{w}_t(\boldsymbol{\xi}) = \frac{d\boldsymbol{x}}{dt} = 20\pi\ \sin(20\pi t)\boldsymbol{\xi}. \tag{26}$$

Therefore, the maximum velocity is obtained in $t = (\frac{n}{10} + \frac{1}{40})$[s], with $n = 0, 1, 2...$ and for the nodes near the top right corner of the domain. Taking the maximum mesh size for this time, the mesh Péclet number is,

$$Pe_h = \frac{|\mathbf{w}_{max}| \cdot h}{\mu} = \frac{2 \cdot \sqrt{2} \cdot 20\pi\ \cdot 0.05}{0.01} = 888.6 \tag{27}$$

For this Péclet, it is necessary the stabilization of ALE equations, for instance using SUPG formulations.

4.3 Convergence analysis

In the next numerical test the accuracy of the *Backward Euler* and the *Crank-Nicolson* schemes is analyzed. The following problem is solved in an unit square

$$\begin{aligned} &u_t - 0.1\Delta u = f && \text{for } \boldsymbol{x} \in \Omega_\mathrm{t}, \mathrm{t} \in [0, \pi], \\ &u_0 = \sin(\pi x) \cdot \sin(\pi y) && \text{for } \boldsymbol{x} \in \Omega_0, \mathrm{t} = 0, \\ &u = 0 && \text{for } \boldsymbol{x} \in \partial\Omega_\mathrm{t}, \mathrm{t} \in [0, \pi], \end{aligned} \tag{28}$$

being the domain deformed according to the following rule:

$$\mathcal{A}_t(\boldsymbol{\xi}) = \boldsymbol{x}(\boldsymbol{\xi}, t) = (1.25 - 0.25\cos(t))\boldsymbol{\xi} \tag{29}$$

The forcing term f is determined using the method of manufactured solutions (MMS), which is a general procedure that can be used to construct analytical solutions. For this case, the forcing term is chosen in such a way that the exact solution $u(\boldsymbol{\xi}, t)$ is

$$u(\boldsymbol{\xi}, t) = (1 + 2\sin(t)) \cdot \sin(\pi\xi) \cdot \sin(\pi\eta). \tag{30}$$

The initial solution of the problem is shown in Figure (6). The rate of convergence of the schemes is computed by taking a sequence of decreasing time steps, ranging from $\Delta t = 0.4$ to $\Delta t = 0.003125$ and the L^2-norm of the error is computed at time $t = 2.0$ [s]. In Figure (7) the rate of convergence for the *Backward Euler* and *Crank-Nicolson* is plotted. It is clearly shown in Figure (7) that both schemes have a rate of convergence similar to those expected in fixed domains. Figure (8) shows a sequences of images corresponding to the solution at different times.

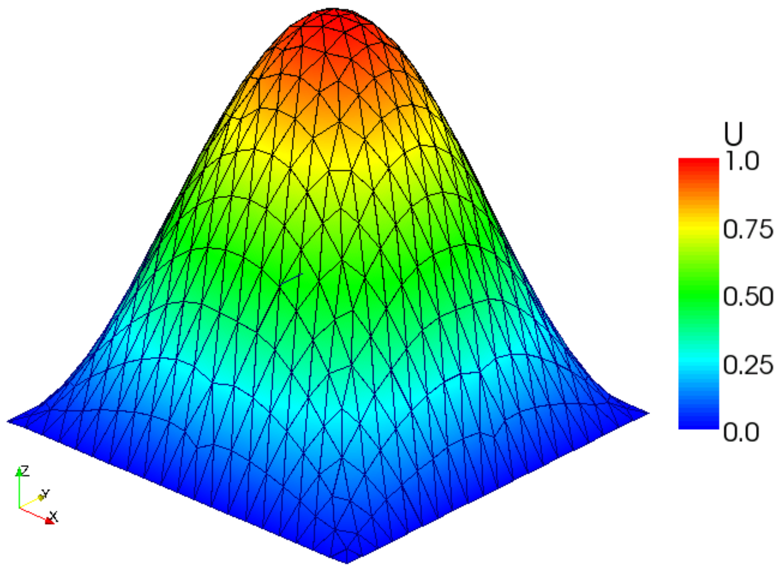

Figure 6: Initial solution of the problem (28).

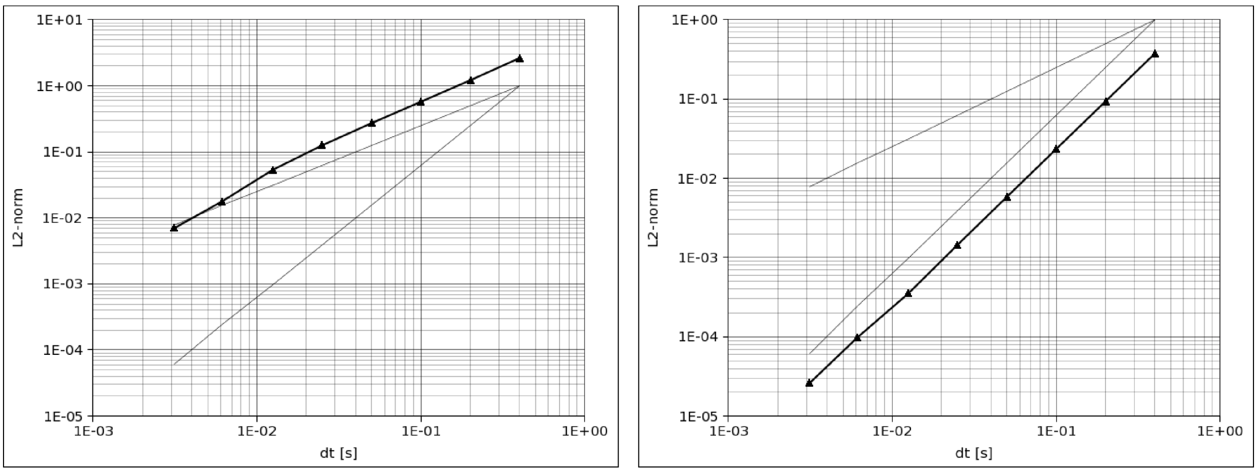

Figure 7: Rate of convergence of the *Backward Euler* and *Crank-Nicolson*.

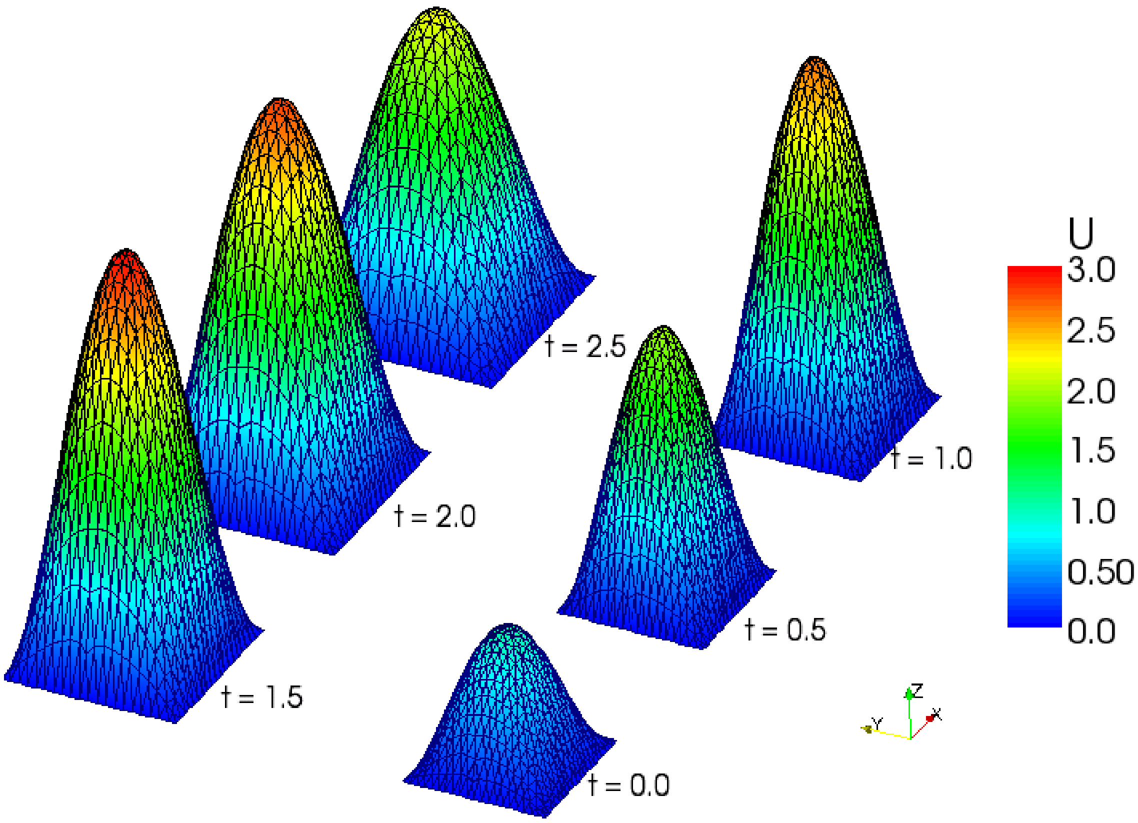

Figure 8: Solution of the problem (28).

5 CONCLUSIONS

In this paper an introduction of formulation of linear advection diffusion problem in an ALE framework is given. In this context, the Geometric Conservation Law arises and states the possibility of a code to simulate a constant solution on a moving domain. When this law is not satisfied an extra numerical error is introduced. This error manifests as elemental sources or sinks in the domain and it is closely related to the time step and the mesh velocity.
Also, it is important to consider the effect of the advective term added to the problem when using an ALE frame of reference. This term can be unsettling or cause nonphysical oscillations of the solution. Finally, a numerical convergence test is carried out for two θ-time integration rules.
This work will be extended to three-dimensional problems where high order schemes like two point Gaussian quadrature rule or multistep method are needed to obtain a GCL compliant code.

ACKNOWLEDGMENTS

This work has received financial support from Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET, Argentina, grant PIP 5271/05), Universidad Nacional del Litoral (UNL, Argentina, grants CAI+D 2009 65/334) and Agencia Nacional de Promoción Científica y Tecnológica (ANPCyT, Argentina, grants PICT 01141/2007, PICT PME 209/2003, PICT-1506/2006, PICTO-23295/2004). Authors made extensive use of freely distributed software as GNU/Linux OS, MPI, PETSc, GCC compilers, Octave, ParaView, Python, Perl, VTK, among many others.

REFERENCES

D. Boffi and L. Gastaldi. Stability and geometric conservation laws for ALE formulations. *Computer Methods in Applied Mechanics and Engineering.*, 193:42-44 (2004):4717–4739, 2004.

J. Donea. *Arbitrary Lagrangian-Eulerian finite elements method.* Belytschko, T. and Hughes, J.R., Amsterdam, 1th edition edition, 1983.

J. Donea and A. Huerta. *Finite Element Methods for Flow Problems.* Jhon Wiley and Sons., 1st edition edition, 2003.

L. Formaggia and F. Nobile. Stability analysis of second-order time accurate schemes for ale-fem. *Computer Methods in Applied Mechanics and Engineering.*, 193:39-41 (2004):4097–4116, 2004.

L. Garelli, R.R. Paz, and M. Storti. Interacción fluido estructura en flujo compresible en régimen supersónico. In *XVII Congreso sobre Métodos Numéricos y sus Aplicaciones*, 2008.

L. Gastaldi. A priori error estimates for the arbitrary lagrangian eulerian formulation with finite elements. *Journal of Numerical Math.*, 9:2 (2001):123–156, 2001.

C.W. Hirt, A.A. Amsden, and J.L. Cook. An Arbitrary Lagrangian-Eulerian computing method for all flow speeds. *Journal of Computational Physics*, 14:3 (1974) 227—253:227–253, 1974.

T.J.R. Hughes, W.K. Liu, and T.K. Zimmermann. Lagrangian-Eulerian finite elements formulations for incompressible viscous flows. In *US-Japan Interdisciplinary Finite Element Analysis.*, 1978.

F. Nobile. *Numerical approximation of fluid-structure interaction problems with application to haemodynamics.* PhD thesis, Department of Mathematics, École Polytechnique Fédérale de Lausanne, 2001.

W.F. Noh. A time-dependent, two space dimensional, coupled Eulerian- Lagrange code. *Methods in Computational Physics.*, 3 (1964) 117—179:117–179, 1964.

P.J. Roache. Verification of codes and calculations. *AIAA*, 36:5 (1998):696–702, 1998.

C.J. Roy. Verification of euler/navier-stokes codes using the method of manufactured solutions. *J. Numer. Methods Fluids*, 44:6 (2004):599–620, 2004.

M. Storti, N. Nigro, R.R. Paz, and D.L. Dalcín. Strong coupling strategy for fluid structure interaction problems in supersonic regime via fixed point iteration. *Journal of Sound and Vibration*, 30:859–877, 2009.

P.D. Thomas and C.K. Lombard. Geometric conservation law and its applications to flow computations on moving grids. *AIAA*, 17:1030–1037, 1979.

J.Y. Trepanier, M. Reggio, H. Zhang, and R. Camarero. A finite-volume method for the euler equations on arbitrary lagrangian-eulerian grids. *Computers and fluids.*, 20:4 (1991) 399—409:399–409, 1991.

S. Étienne, A. Garon, and D. Pelletier. Perspective on the geometric conservation law and finite element methods for ale simulations of incompressible flow. *Journal of Computational Physics*, 228:7 (2009):2313–2333, 2009.