A NEW (?) ALTERNATIVE FOR DECISION-MAKING CONCERNING THE COST/ACCURACY CONTROVERSY

J.C. Ferreri*
CNEA, Gerencia de Protección Radiológica y Seguridad
Avda. del Libertador 8250
1429 Buenos Aires, Argentina

RESUMEN

Se presenta una técnica para mejorar la precisión de las soluciones numéricas de una clase de problemas fluidodinámicos. Esta consiste en la consideración explícita de soluciones analíticas locales en zonas seleccionadas del dominio de integración. Esta técnica permite la obtención de soluciones de gran precisión con grillas muy gruesas y ha sido aplicada en un contexto de diferencias finitas. Los ejemplos mostrados son válidos para casos de advección-difusión y para flujo potencial estacionario bidimensional.

ABSTRACT

A technique for the improvement of the accuracy of numerical solutions in a class of fluid dynamic problems is presented. It consists in the explicit consideration of local analytical solutions in selected zones of the integration domain. This technique allows for the obtention of very accurate solutions with very coarse grids and has been applied in a finite-difference context. The examples shown are valid for one-dimensional cases of advection-diffussion and for two-dimensional, steady, potential flow.

* Member of the Carrera del Investigador, CONICET, Argentina

INTRODUCTION

In many problems of engineering significance, the accuracy of the numerical solutions obtained using finite-difference techniques can only be improved through considerable grid refinement in selected zones of the integration domain. This refinement may be obtained in a number of ways, the most recent one being the so-called "adaptive grid refinement technique". The knowledge of the peculiarities of the searched solution is, sometimes, an inevitable pre-requisite in order to specify the difference scheme and the major task of adaptive solvers is the proper identification of the zones showing the steep variations of the dependent variables.

Conventional approaches usually imply the specification of a graded grid, with a higher density of nodes in the aforementioned zones. These zones are usually called "boundary layers" and, in this paper, this name is used in a somewhat loose sense. The thickness of a boundary layer is, tipically, inversely proportionate to the power of a given parameter. In the case of advection-difussion problems this parameter is Peclet's number (Reynolds' number for momentum). Then, the amount of grid refinement is directly proportional (if a constant spacing is considered) to the parameter. When advection dominates the behaviour of the solution, the obtention of a numerical solution free from the so-called "wiggles" may be costly. The meaning of the wiggles in a general context has been discussed in /1/, where the relationship between this spurious behaviour and discretization was clearly shown. The elimination of these wiggles (or in more general terms, the improvement of the accuracy of the solution) without significant grid refinements is the goal of the technique shown herein.

This technique consists in the incorporation of the knowledge on the local behaviour of the solution in the neighbourhood of isolated points of the domain. Sometimes this behaviour is known or, at least, the asymptotic behaviour may be inferred. Depending on the class of problems, this knowledge may be incorporated to the solution technique, leading to significant improvements in its global accuracy. In the author's opinion, this criterion should always be employed and this fact justifies the inclusion of the question mark in the title.

In what follows the technique is presented as applied to specific problems, showing the benefits obtained.

THE TECHNIQUE AND ITS RESULTS

In reference /2/, it was shown that the accuracy of numerical solutions in heat conduction problems associated with almost punctual heat sources could be improved, according with the results obtained by Emery /3/, by using local analytical solutions. Later on, an extension of this concept was applied to the steady one-dimensional advection-diffusion as shown in reference /4/.

In what follows, the results in /4/ are used as an introductory example of the ideas involved in the present paper.

The equation to be solved is:

$$L(u) = - P \frac{du}{dx} + \frac{d^2u}{dx^2} = 0, \qquad (1)$$

subject to $u(0)=0$, $u(1)=1$. This equation has been the base for plenty of literature until very recently because of its interesting behaviour in relation with different discretizations. The most recently published comparison of standard approaches to equation (1) is, perhaps, reference /5/, where equation (1), with the addition of a source term, is solved.

Let us consider the following finite-difference approach for the solution of (1):

$$(I - k L_h) U^{n+1} = U^n - k L_h(F), \qquad (2)$$

where L_h is the discrete approximation of L.

If a steady state exists, then, the exact solution of (2) leads to $L_h(U) = L_h(F)$, where U is the numerical value of u at any point in the grid. When $F \equiv 0$, the solution corresponds to the standard approach. In /6/, the exact solution to the problem $L_h(U) = 0$ is given in relation to a number of discrete approximations and the corresponding behaviour is discussed.

Let us now consider that F is the analytical solution of (1). Then, $U \equiv F$ in all the grid points (as obviously expected), regardless the grid size. When $F \equiv 0$ and L_h is a centered approximation, then, for a sufficiently high value of P and a given discretization, the solution becomes oscillatory (i.e., when $P.h > 2$, where h is the constant grid spacing).

When:

$$F = e^{-P(1-x)} , \qquad (3)$$

{which corresponds to:

$$\lim_{P\to\infty} \frac{1-e^{Px}}{1-e^{P}} \qquad (4)$$

(i.e. the limiting form of the analytical solution of (1)}, the effect of incorporating (3) in (2) for the numerical solution of (1) is the one shown in /4/. This effect was the complete elimination of the wiggles for any value of P with $h = 0.1$ (i.e., 10 grid intervals!). Further algebraic details can be found in said reference.

It must be noted that (3) is a solution of (1), but it does not satisfy both boundary conditions; it just satisfies the one in the boundary layer zone. An inspection of figure 1, taken from /4/, gives an idea of the results just mentioned.

Let us now consider the case of a steady, two-dimensional, incompressible potential flow. In this case the governing equations are:

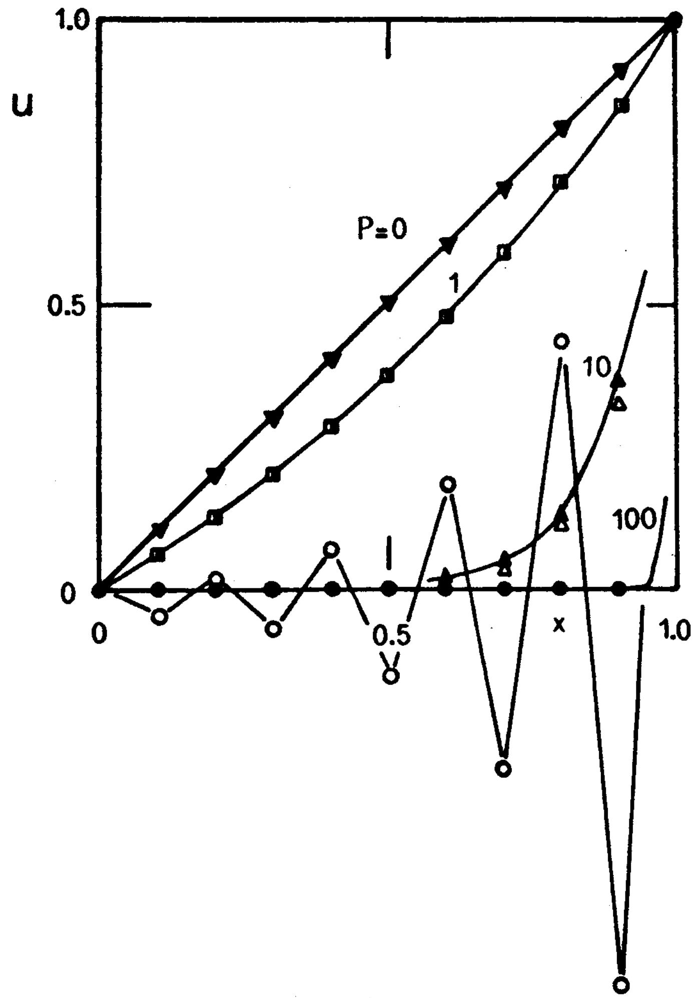

Figure 1 - The effect of considering F = exp(-P(1-x))
in the solution of equation (1).
Taken from reference /4/

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad (5)$$

$$u = k\,\frac{\partial p}{\partial x} \quad \text{and} \qquad (6a)$$

$$v = k\,\frac{\partial p}{\partial y} \qquad (6b)$$

which state, respectively, the conservation of mass and the variation of the two components of momentum. In equations 5-6, u and v are the components of the velocity vector and p is a reduced pressure; x and y are the space coordinates.

Steep variations of u, v or p may arise for several reasons; usually, the imposition of boundary conditions of the Dirichlet's type or abrupt variations in k give rise to these situations. If these steep variations are considered jointly with a coarse grid, then, the computed solution may show a spurious behaviour.

As a first example, let us consider the flow from an almost punctual source (e.g., radial flow from a well). The case of a flow toward a sink is similar to the latter but the flow direction is reversed. The computer code is the one quoted in reference /6/ and the grid for the calculations is shown in figure 2a, where the ratio between the outer and inner radii is 50. The grid spacing is uniform and periodicity was imposed along line A-B.

The boundary conditions were of the Dirichlet's type, i.e. imposing velocities as given by the analytic solution. The computed solution is the one shown in figure 2b. The solution (in terms of velocities) has a strong "boundary layer" because the ratio of modulii in the first and second row of nodes is in the order of 6. As may be observed, the direction of the velocity vectors is definitely spurious with respect to the expected one. The oscillations in the modulii cannot be observed in this figure but they do exist, as well as in the pressure, and may be eliminated, under standard finite-difference approaches, by suitable grid refining.

Let us now consider the following version of the algorithm in /6/:

i) The discrete analogue of the divergence in a cell (Dij) is evaluated as in /6/;

i') Redefine Dij as

$$Dij \leftarrow Dij - D_{hij} ,$$

where

D_{hij} is the divergence of the velocity field obtained from the velocities calculated with the discrete analogue of 6 a-b and an analytic distribution of pressure.

ii) the pressure is adjusted as in /6/;

iii) the velocities are modified as in /6/.

As may be seen from the above sketchy definition, the only modification consists in introducing the known behaviour of the solution into the computing algorithm.

In order to obtain a useful tool, the analytic solution may be a local one and this fact may be simulated by considering the application of the correction in i') in selected zones of the integration domain. Figure 2c shows the results obtained through the correction applied in the first row of cells around the source. In fact, all oscillations dissappeared and the solution agreed fairly well with the analytic one all over the domain.

Another interesting example may be taken from the results in /7/. In this case, the algorithm in /6/ was allowed to include discrete fractures. Thus, the flowfield may be abruptly disturbed by the presence of zones which short-circuit the fluid paths.

Figure 3 is an interesting example of such situation, which consists in predicting the flow in a homogeneous-porous medium with an isolated fracture aligned with the flow at infinite. Figure 3a shows the grid adopted and Figure 3b is a close up of the vector plot in the neighbourhood of the fracture. Spurious oscillations may be seen and the errors, with respect to the analytical solution, were large both in modulii and angles. Simply by including step i') in the algorithm of reference /7/, the flow became civilized and errors, with a coarse grid of 9 x 24 nodes, are shown in Figure 3c, where the results of Baca at al. /8/ are shown as a reference. Errors were in the order of 2% in the fracture and reached 9% in the porous media. It must be noted that the analytical solution was employed to calculate the correction in the four velocity nodes sorrounding the beginning (and the end) of the fracture.

It may be argued that the correction influences the whole flowfield because the grid was coarse. In order to test the sen-

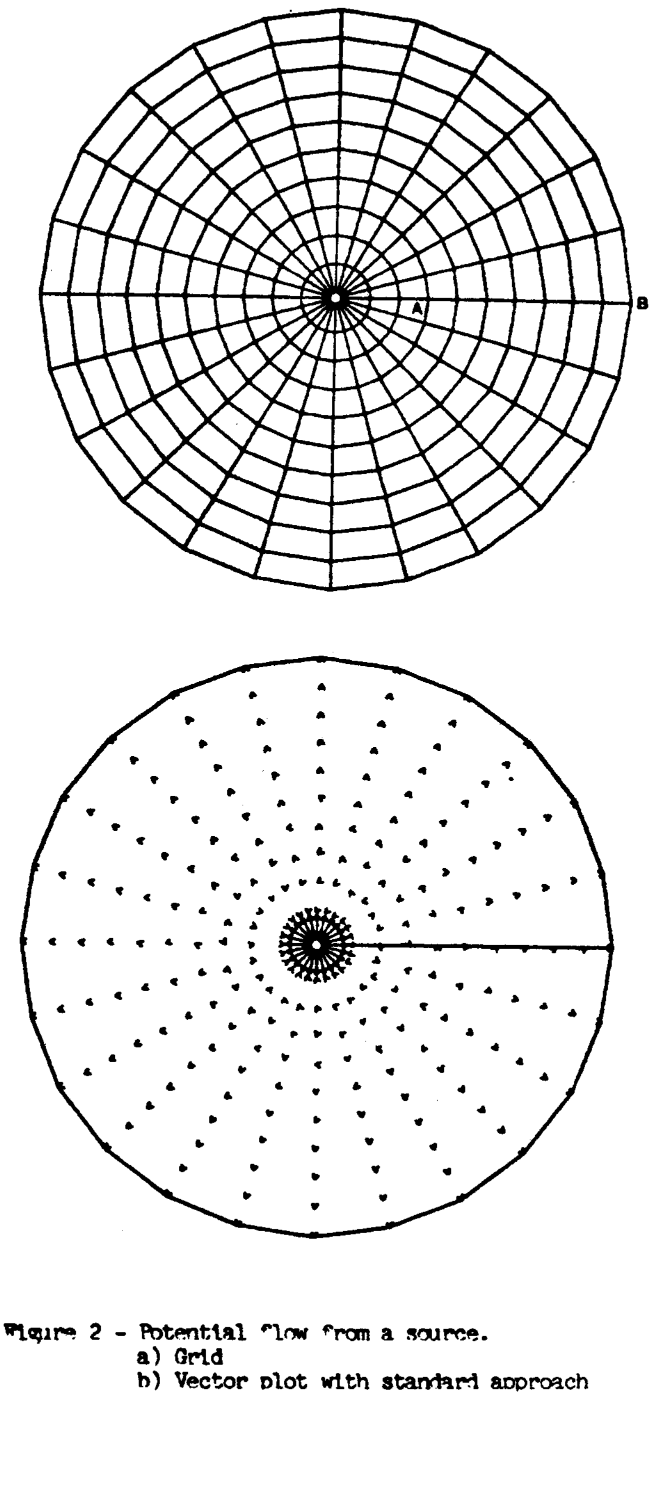

Figure 2 - Potential flow from a source.
a) Grid
b) Vector plot with standard approach

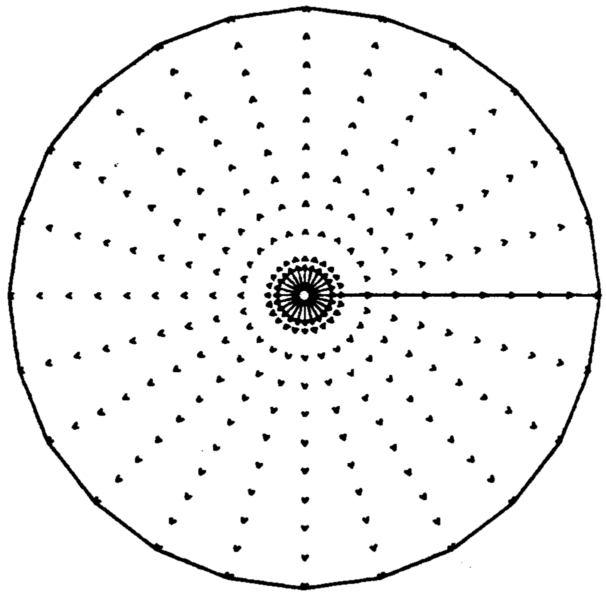

Figure 2 - continued
c) Vector plot with the proposed technique

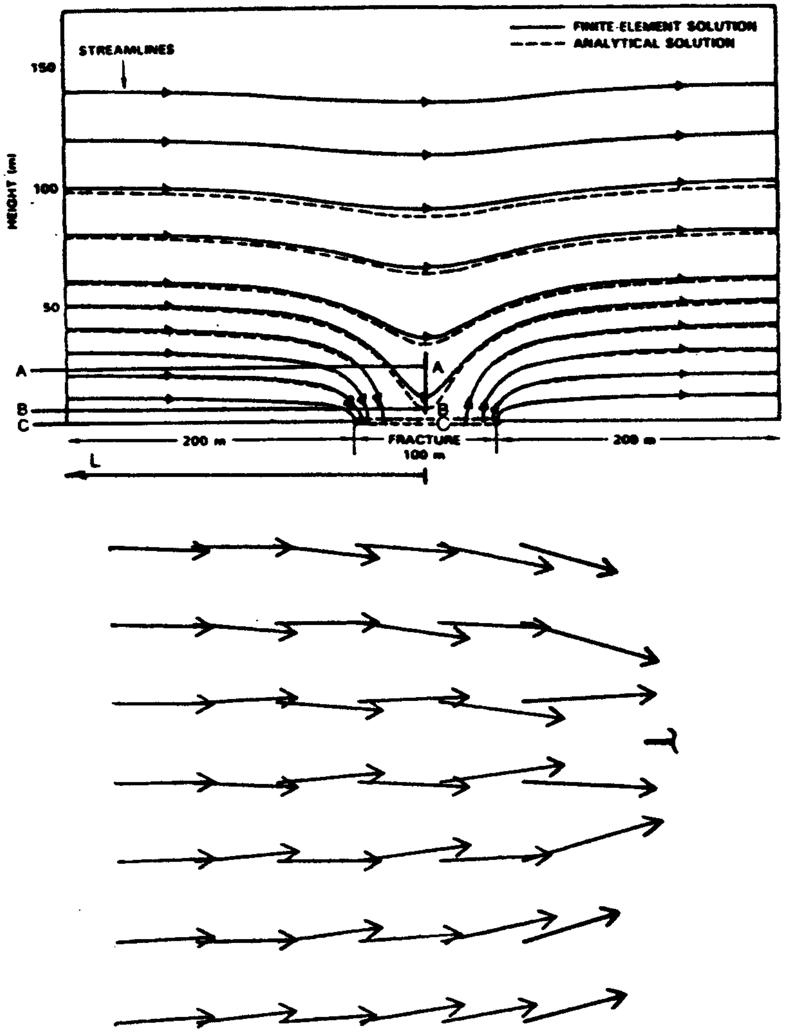

Figure 3 - Flow past an isolated fracture in a homogeneous porous rock
a) The flowfield, adapted from reference /8/
b) Close-up of the velocity field

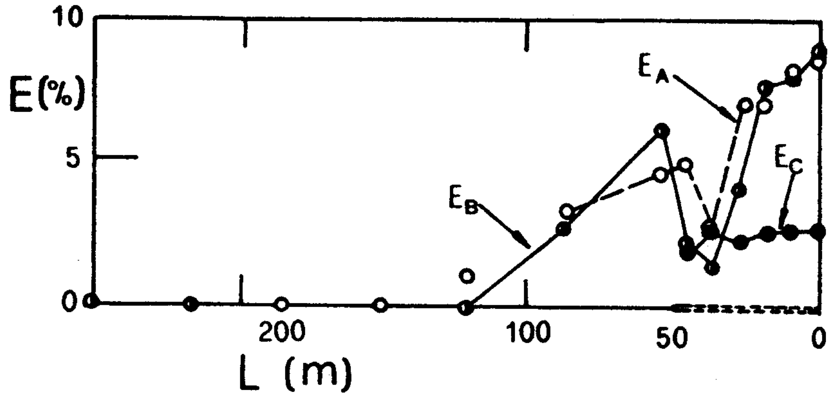

Figure 3 - continued
c) Errors in the computed solution

sitivity of the method to the cell size, smaller cells were used and the correction was applied to the same number of cells. Results were identical.

CONCLUSIONS

It was shown that the correct behaviour of the numerical solution of some problems of fluid flow may be recovered through the use of local analytical solutions in isolated zones of the integration domain. This fact, in turn, implies the obtention of wiggle-free solutions with very coarse grids. This approach proved to be valid in cases of one-dimensional advection-diffusion and in steady potential flow in two dimensions (this case being solved in terms of primitive variables).

The application of this technique to the full Navier-Stokes equations is still under research.

ACKNOWLEDGMENT

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