THE METHOD OF PARAMETER DIFFERENTIATION APPLIED TO FLUID MECHANICS PROBLEMS

Marta B. Peirotti
Julio A. Deiber
Instituto de Desarrollo Tecnológico para la Industria Química.
UNL - CONICET
Santa Fe - Argentina

ABSTRACT

This work applies the method of parameter differentiation (MPD) to nonlinear ordinary and partial differential equations of fluid mechanics. It is shown that the differentiation parameter does not need to be a physical parameter of the problem, because it can be arbitrarily selected and placed in any nonlinear term of the differential equation, with the constraint that it takes the value one at the end of the integration procedure.

Emphasis is placed in two numerical aspects:

a) A nonlinear ordinary differential equation with boundary conditions can be transformed into a simpler problem, which consists of linear ordinary differential equations with initial conditions. The solution is then noniterative.

b) The solution of the steady stream function-vorticity scheme through finite differences with the overall iterative procedure (Gupta, 1980, p. 170) can be simplified because inner iterations are eliminated, and initialization functions are solutions of the problem at each previous outer parameter iteration.

RESUMEN

Este trabajo aplica el método de diferenciación paramétrica (MPD) a ecuaciones diferenciales ordinarias y a derivadas parciales no lineales de la mecánica de fluídos. Se muestra que el parámetro de diferenciación no necesita ser un parámetro físico del problema, porque puede ser seleccionado arbitrariamente y colocado en el término no lineal de la ecuación diferencial, con la restricción que tome el valor uno al final del procedimiento de integración.

Se pone énfasis en dos aspectos numéricos:

a) Una ecuación diferencial ordinaria no lineal con condiciones de contorno puede ser transformada en un problema más simple, que consiste en una ecuación diferencial ordinaria lineal con condiciones iniciales. La solución es entonces no iterativa.

b) La solución del esquema función línea de corriente-vorticidad en estado estacionario a través de diferencias finitas con el procedimiento iterativo global (Gupta, 1980, p. 170) puede ser simplificado porque las iteraciones internas son eliminadas y las funciones de inicialización son soluciones del problema en cada iteración paramétrica externa previa.

INTRODUCTION

Many mathematical models generated in all branches of engineering and science involve linear ordinary and partial differential equations that can be solved either analytically (see, for example, Ince, 1956; Courant and Hilbert, 1962) or numerically through methods which are probed to be convergent, consistent and stable (see, for example, Noye, 1978; Smith, 1978). However, this is not necessarily the case of nonlinear differential equations which, in general, require special treatments and approximations to find a solution, depending these upon the nature of the nonlinear terms involved.

The method of parameter differentiation (MPD) also known in the applied mathematical literature as the method of continuation, (see, for example, Wacker, 1978), has been probed to be a great potential tool for solving nonlinear algebraic equations and nonlinear ordinary differential equations of engineering (see, Na, 1979, p. 233). In fact, nonlinear mathematical models involving a physical parameter that appears either in the differential equation or in the boundary conditions, can be solved by integrating the rate of change of the corresponding solution with respect to this parameter. Therefore, to proceed in this way, the starting point is a known solution of the problem for a specified value (zero value) of the physical parameter, i.e., the differentiation parameter.

The MPD can be carried out because the formulation of the mathematical problem has to satisfy the requirement of parameter continuity of the obtained solution. Thus, from a wider mathematical point of view, the differential equation with its boundary conditions must satisfy (see, for example, Street, 1973),

- Existence: at least one solution exists
- Uniqueness: there exists at most one solution
- Continuity: the solution varies continuously in all given data, including parameters.

so that, the formulated problem is well posed.

The main advantage of the MPD applied to two boundary values problems of nonlinear ordinary differential equations is: A systematic procedure for linearization is used. Furthermore, the resulting linear problem with two point boundary conditions can be transformed into an initial value problem, also in a systematic way through the method of superposition, and solved in its turn by noniterative methods such as the Runge-Kutta methods.

This work presents a brief overview of the MPD and its particular case designated here as the method of parameter iteration (MPI). Although it is not done in a generalized procedure, we will show for particular cases the relation between a solution obtained by the MPD and the one obtained through a regular perturbation expansion when the final value to be reached by the differentiation parameter is small. Furthermore, Na (1979, p. 234) has shown that in the application of the MPD to a nonlinear algebraic equation, it is always possible to introduce arbitrarily in any term of this equation, a non physical parameter which has to take the value one at the end of the integration procedure so that the original physical and mathematical problem is recovered. Therefore, we will use this important concept to solve a nonlinear ordinary differential equation, of which a solution to start the MPD is not available neither analytically nor without complex numerical evaluations. To illustrate this aspect of the MPD, we will solve the third order nonlinear ordinary differential equation corresponding to the boundary layer theory applied to a plate (see, Schlichting, 1960, p. 116) which is a particular case of the Falkner-Skan problem when the physical parameter β involving the angle of a wadge is zero.

It is interesting to point out here that Rubbert and Landahl (1967) solved the Falkner-Skan problem through the MPD by starting the procedure with a known numerical solution for $\beta = 0$, i.e., the solution of the boundary layer theory for a plate (Blasius equation). This starting solution was numerical because the Blasius equation is also a nonlinear problem and in addition, it has not any physical parameter available for application of the MPD. These authors also solved the Blasius equation by the MPD through the definition of still a physical parameter. Therefore, it is in this mathematical aspect that our solution of the Blasius equation by the use of an unphysical parameter, will show a qualitative advantage and generalization of the MPD.

After the analysis of the above mentioned examples, our main target is to try the application of the MPD, to the solution of partial differential equations involving Newtonian fluid flows as a substitution of the classical iteratives methods such as the method of succesive over-relaxation (SOR) (see, for example, Greenspan, 1974, p. 12, 208; Gupta, 1980, p. 148). In fact, it will be shown that the solution of the steady stream function-vorticity scheme through finite differences with the overall iterative procedure (Gupta, 1980, p. 174) can be greatly simplified in two aspects: a) Inner iterations can be avoided since the method of Gauss-Seidel converges in one step if the increment of the differentiation parameter chosen is sufficiently small; b) outer iterations are substituted for a sequence of numerical solutions in which two consecutive solutions differ themselves in a small increment of the differentiation parameter.

II) METHOD OF PARAMETER DIFFERENTIATION

II-1) Ordinary Differential Equations (ODE)

Consider a second order nonlinear ordinary differential operator N_x , a scalar function ϕ and a physical parameter ε (Reynolds number, geometrical ratio, etc.) defining the following physical problem,

$$N_x [\phi, \varepsilon] = 0 \tag{1}$$

subject to boundary conditions,

$$h_a[\phi, \varepsilon] = 0 \quad , \quad x = a \tag{2}$$

$$h_b[\phi,\varepsilon] = 0 \quad , \quad x = b \tag{3}$$

Therefore, the solution $\phi(x,\varepsilon)$ is required for $a \le x \le b$ and $\varepsilon = \varepsilon^*$.

The procedure to apply the MPD is the following:

- Differentiate equations (1) to (3) with respect to ε to obtain,

$$L_x^{\varepsilon}[g,\phi,\varepsilon] = 0 \tag{4}$$

$$\frac{\partial h_a}{\partial \varepsilon} + \frac{\partial h_a}{\partial \phi} g = 0 \quad , \quad x = a \tag{5}$$

$$\frac{\partial h_b}{\partial \varepsilon} + \frac{\partial h_b}{\partial \phi} g = 0 \quad , \quad x = b \tag{6}$$

$$g = \frac{\partial \phi}{\partial \varepsilon} \tag{7}$$

where L_x^{ε} is a linear ordinary differential operator applied to g because ϕ and its differentiations are considered as known variable coefficients in equation (4).

- Find the starting solution ϕ_0 for $\varepsilon = 0$,

$$N_x[\phi_0,0] = 0 \tag{8}$$

$$h_a[\phi_0,0] = 0 \quad , \quad x = a \tag{9}$$

$$h_b[\phi_0,0] = 0 \quad , \quad x = b \tag{10}$$

- Find the starting solution g_0 for $\varepsilon = 0$,

$$L_x^{0}[g_0,\phi_0,0] = 0 \tag{11}$$

$$\frac{\partial h_a}{\partial \varepsilon} + \frac{\partial h_a}{\partial \phi} g_0 = 0 \quad , \quad x = a \quad , \quad \varepsilon = 0 \quad , \quad \phi = \phi_0 \tag{12}$$

$$\frac{\partial h_b}{\partial \varepsilon} + \frac{\partial h_b}{\partial \phi} g_0 = 0 \quad , \quad x = b \quad , \quad \varepsilon = 0 \quad , \quad \phi = \phi_0 \tag{13}$$

- Find ϕ_1 for $\varepsilon = \Delta\varepsilon << 1$ as follows,

$$\phi_1(x,\Delta\varepsilon) - \phi_0(x,0) = g_0(x,0)\,\Delta\varepsilon + O(\Delta\varepsilon^2) \tag{13}$$

- Using equation (4) to (7) find at $\varepsilon = n\,\Delta\varepsilon$,

$$L_x^{\varepsilon}[g_n\,,\phi_n\,,n\,\Delta\varepsilon] = 0 \tag{15}$$

$$\frac{\partial h_a}{\partial \varepsilon} + \frac{\partial h_a}{\partial \phi} g_n = 0 \quad , \quad x = a \quad , \quad \varepsilon = n\,\Delta\varepsilon \, , \quad \phi = \phi_n \tag{16}$$

$$\frac{\partial h_b}{\partial \varepsilon} + \frac{\partial h_b}{\partial \phi} g_n = 0 \quad , \quad x = b \quad , \quad \varepsilon = n\,\Delta\varepsilon \quad , \quad \phi = \phi_n \tag{17}$$

$$\phi_{n+1}(x,(n+1)\Delta\varepsilon) - \phi_n(x,n\,\Delta\varepsilon) = g_n(x,n\,\Delta\varepsilon)\,\Delta\varepsilon + O(\Delta\varepsilon^2) \tag{18}$$

$$n = 1,2,\ldots\, N \tag{19}$$

$$\varepsilon^* = N\,\Delta\varepsilon \tag{20}$$

Therefore when $n = N$ the solution $\phi_N(x,N\Delta\varepsilon) = \phi(x,\varepsilon^*)$ is found.

It should be observed that equations (8) to (10) can frequently be solved analytically and that equations (11) to (13) and (15) to (17) can be transformed to an initial value problem which solutions are directly found through noniterative methods like the Runge-Kutta or Euler methods (see, Na, 1979, p. 13).

In particular, if equations (8) to (10) cannot be solved analytically, we can introduce a nonphysical parameter λ in any nonlinear term, provided $\lambda = m\,\Delta\lambda$, $\Delta\lambda \ll 1$, $m = 1,2\,\ldots\, M$ and $M\,\Delta\lambda = 1$. Therefore, the application of the MPD yields,

$$N_x^\lambda[\phi_0,\lambda,0] = 0 \tag{21}$$

$$h_a^\lambda[\phi_0,\lambda,0] = 0 \tag{22}$$

$$h_b^\lambda[\phi_0,\lambda,0] = 0 \tag{23}$$

The corresponding starting solution $\phi_{0,0}$ satisfies,

$$N_x^0[\phi_{0,0},0,0] = 0 \tag{24}$$

$$h_a^0[\phi_{0,0},0,0] = 0 \tag{25}$$

$$h_b^0[\phi_{0,0},0,0] = 0 \tag{26}$$

Therefore at $\lambda = m\,\Delta\lambda$ and for $f = \dfrac{\partial\phi_0}{\partial\lambda}$,

$$L_x^\lambda[f_m,\phi_{0,m},m\,\Delta\lambda] = 0 \tag{27}$$

$$\frac{\partial h_a^\lambda}{\partial \lambda} + \frac{\partial h_a^\lambda}{\partial \phi_0} f_m = 0 \quad , \quad x = a \quad , \quad \lambda = m\,\Delta\lambda \quad , \quad \phi_0 = \phi_{0,m} \tag{28}$$

$$\frac{\partial h_b^\lambda}{\partial \lambda} + \frac{\partial h_b^\lambda}{\partial \phi_0} f_m = 0 \quad , \quad x = b \quad , \quad \lambda = m\,\Delta\lambda \quad , \quad \phi_0 = \phi_{0,m} \tag{29}$$

$$\phi_{0,m+1}(x,(m+1)\Delta\lambda) - \phi_{0,m}(x,m\,\Delta\lambda) = f_m(x,\,m\,\Delta\lambda)\Delta\lambda + O(\Delta\lambda^2) \tag{30}$$

$$n = 1,2\,\ldots\, M \tag{31}$$

$$\lambda^* = M\,\Delta\lambda = 1 \tag{32}$$

When $n = M$, the solution $\phi_{0,M}(x, M\,\Delta\lambda) = \phi_0(x)$ is found.

It is also clear that for $\varepsilon = 0$ and $\lambda = 1$,

$$N_x^1 = N_x \tag{33}$$

$$h_a^1 = h_a \tag{34}$$

$$h_b^1 = h_b \tag{35}$$

and that λ has to be placed into N_x so that N_x^0 is a differential operator of easy analytical solution.

The method of parameter iteration (MPI) is a particular case of the MPD and it can be formulated as follows:

- The nonlinear operator N_x is decomposed in two parts,

$$N_x = L_x + N_x^\lambda \tag{36}$$

where L_x is again a linear ordinary differential operator and N_x^λ is the remaining nonlinear part of N_x. Therefore, equation (1) is now expressed,

$$L_x[\phi,\varepsilon] + \lambda\, N_x^\lambda[\phi,\varepsilon] = 0 \tag{37}$$

where λ can also be taken as equal to the physical parameter ε if the nature of the nonlinear problem is appropriate, i.e., sometimes it is possible to make $\lambda = \varepsilon$ and obtain,

$$L_x[\phi,\varepsilon] + \varepsilon\, N_x^\varepsilon[\phi,\varepsilon] = 0 \tag{38}$$

however, this decomposition is not the general case.

- Solve equation (37) for $\lambda = 0$ with its boundary conditions (equations (2) and (3)).

$$L_x[\phi_0,\varepsilon] = 0 \tag{39}$$

- Find the sequence of solution ϕ_n solving L_x in the following equation,

$$L_x[\phi_n(x,n\Delta\lambda,\varepsilon),\varepsilon] + n\,\Delta\lambda\, N_x^\lambda[\phi_{n-1}(x,(n-1)\Delta\lambda,\varepsilon),\varepsilon] = 0 \tag{40}$$

$$n = 1,2\ \ldots\ N \tag{41}$$

$$N\,\Delta\lambda = 1 \tag{42}$$

Consequently $\phi_N(x,N\Delta\lambda,\varepsilon) = \phi(x,\varepsilon)$ is the approximate solution of equations (1) to (3).

Next, we present four examples to illustrate the use of the MPD and MPI. The fourth one is a nonlinear ordinary differential equation involving the boundary layer theory of fluid flows.

Example 1

Consider the simple case of a first order linear differential equation,

$$y' + \varepsilon\, y = 0 \tag{43}$$

$$y(0) = 1 \tag{44}$$

which exact solution is,

$$y^{EX} = e^{-\varepsilon x} \tag{45}$$

In addition, if $\varepsilon \ll 1$, a regular perturbation solution (RPS) of equations (43) and (44) is equivalent to the series expansion of equation (45) and it can be written,

$$y_3^{RPS} = 1 - \varepsilon x + (\varepsilon x)^2/2! - (\varepsilon x)^3/3! \ \ldots \tag{46}$$

The MPI implies,

$$y_n' + n\,\Delta\varepsilon\, y_{n-1} = 0 \tag{47}$$

$$y_n(0) = 1 \tag{48}$$

$$\varepsilon = n\,\Delta\varepsilon \tag{49}$$

and we readily obtain for $n = 3$,

$$y_3^{MPI} = 1 - \varepsilon x + (\varepsilon x)^2/3 - (\varepsilon x)^3/27 \tag{50}$$

The MPD however implies,

$$g_n' + n\,\Delta\varepsilon\, g_n + y_n = 0 \tag{51}$$

$$g_n(0) = 0 \tag{52}$$

$$y_{n+1} - y_n = g_n\,\Delta\varepsilon + O(\Delta\varepsilon^2) \tag{53}$$

and also equation (49), which is still valid.

It is then readily probed that $y_0 = 1$ and $g_0 = -x$, which are the starting solutions of the MPD. However, it is not necessary to start from g_0 since g_1 can be obtained without difficulty using y_0 and without g_0. We can find directly g_1 as follows,

$$g_1' + \Delta\epsilon\ g_1 + y_0 = 0 \tag{54}$$

$$g_1(0) = 0 \tag{55}$$

and,

$$g_1 = \frac{1}{\Delta\epsilon}\ (e^{-\Delta\epsilon x} - 1) \tag{56}$$

From equation (53) and since $\epsilon = \Delta\epsilon$ for $n = 1$,

$$y_1^{MPD} = e^{-\epsilon x} \tag{57}$$

Thus, the MPD gives the exact solution in one step.

The following results for $\epsilon = 0.5$ and $x = 0.5$ are interesting to compare: $y^{EX} = y_1^{MPD} = 0.7788$, $y_2^{RPS} = 0.7812$ and $y_3^{MPI} = 0.7703$.

It is here appropriate to place emphasis on that the MPD can also be applied by starting with $g_0 = -x$ to obtain,

$$y_1 = 1 - \Delta\epsilon\ x \tag{58}$$

$$g_1 = x - \frac{2}{\Delta\epsilon}\ (e^{-\Delta\epsilon x} - 1) \tag{59}$$

then,

$$y_2 = 1 + 2(e^{-\Delta\epsilon x} - 1) \tag{60}$$

what is the same as,

$$y_2^{MPD} = 1 + 2\ (e^{-\frac{\epsilon x}{2}} - 1) \tag{61}$$

and for $\epsilon \ll 1$,

$$y_2^{MPD} \approx 1 - \epsilon x + (\epsilon x)^2/4 - (\epsilon x)^3/24 \tag{62}$$

Although this example is very simple, it can easily show the relationship between y^{MPD} and y^{RPS} in a clear procedure. To place emphasis on the relation between y^{MPD} and y^{MPI} , Table I shows numerical solutions of equations (43) and (44) obtained through the MPD and MPI in 1000 and 10000 steps respectively, for $\epsilon = 1$ and $\Delta x = 1/2000$

<u>Example 2</u>

Consider the case of a nonlinear first order differential equation,

$$y' + \epsilon\ y^2 = 0 \tag{63}$$

$$y(0) = 1 \tag{64}$$

which exact solution is:

$$y^{EX} = \frac{1}{1+\epsilon x}$$

TABLE I

$$y' + \varepsilon y = 0 \quad , \quad y(0) = 1$$

$\varepsilon = 1$, $\Delta x = \frac{1}{2000}$, $\Delta\varepsilon = \frac{1}{1000}$			
x	y^{EX}	y^{MPD}	y^{MPI}
0.00000	1.00000	1.00000	1.00000
0.10000	0.90484	0.90472	0.90473
0.20000	0.81873	0.81851	0.81853
0.30000	0.74082	0.74050	0.74054
0.40000	0.67032	0.66992	0.66999
0.50000	0.60653	0.60606	0.60616
0.60000	0.54881	0.54828	0.54841
0.70000	0.49659	0.49599	0.49616
0.80000	0.44933	0.44869	0.44889
0.90000	0.40657	0.40588	0.40612
1.00000	0.36788	0.36715	0.36743

$\varepsilon = 1$, $\Delta x = \frac{1}{2000}$, $\Delta\varepsilon = \frac{1}{10000}$			
x	y^{EX}	y^{MPD}	y^{MPI}
0.00000	1.00000	1.00000	1.00000
0.10000	0.90484	0.90482	0.90481
0.20000	0.81873	0.81870	0.81868
0.30000	0.74082	0.74079	0.74076
0.40000	0.67032	0.67030	0.67025
0.50000	0.60653	0.60648	0.60645
0.60000	0.54881	0.54879	0.54872
0.70000	0.49659	0.49652	0.49649
0.80000	0.44933	0.44930	0.44923
0.90000	0.40657	0.40651	0.40647
1.00000	0.36788	0.36782	0.36778

The following approximate solutions are found when $\varepsilon \ll 1$,

$$y_2^{RPS} = 1 - \varepsilon x + (\varepsilon x)^2 + O(\varepsilon^3) \tag{65}$$

$$y_4^{MPI} = 1 - \varepsilon x + \frac{3}{4}(\varepsilon x)^2 - \frac{3}{16}(\varepsilon x)^3 \tag{66}$$

$$y_2^{MPD} \sim e^{-\varepsilon x} + O(\Delta\varepsilon^2) \tag{67}$$

The last solution y_2^{MPD} has been obtained by dropping terms of $O(\Delta\varepsilon^2)$ to avoid heavy algebraic manipulations in the determination of g_1 .

For $\varepsilon = 0.1$ and $x = 1$, it is interesting to compare the following results,

$$y_2^{RPS} = 0.9100$$

$$y_2^{MPD} = 0.9048 \quad , \quad \Delta\varepsilon = 0.05$$

$$y_1^{MPI} = 0.9000 \quad , \quad \Delta\varepsilon = 0.1$$

$$y_2^{MPI} = 0.9049 \quad , \quad \Delta\varepsilon = 0.05$$

$$y_3^{MPI} = 0.9065 \quad , \quad \Delta\varepsilon = 0.0333...$$

$$y_4^{MPI} = 0.9073 \quad , \quad \Delta\varepsilon = 0.025$$

Therefore, it is observed how the y^{MPI} approximates to the y^{RPS} by increasing the number of steps used to reach the final value $\varepsilon = 0.1$.

Table II shows numerical solutions of equations (63) and (64) obtained through the MPD and MPI for different values of $\Delta\varepsilon$ when $\varepsilon = 1$. Solutions y^{MPD} and y^{MPI} coincide in four digits when $\Delta\varepsilon = 1/1000$ for almost all values of x .

<u>Example 3</u>

Consider the case of a second order nonlinear differential equation with two points boundary conditions,

$$y'' + \varepsilon A y' y = 0 \tag{68}$$

$$y(0) = 1 \tag{69}$$

$$y'(1) = 1 \tag{70}$$

In the particular case of $\varepsilon \ll 1$, the approximate solutions are,

TABLE II

$$y' + \varepsilon y^2 = 0 \quad , \quad y(0) = 1$$

$$\Delta x = 1/100 \qquad \varepsilon = 1$$

	x	y^{MPD}	y^{MPI}
$\Delta\varepsilon = \frac{1}{10}$	0.00000	1.00000	1.00000
	0.10000	0.90737	0.90745
	0.20000	0.83036	0.83065
	0.30000	0.76531	0.76587
	0.40000	0.70962	0.71050
	0.50000	0.66139	0.66261
	0.60000	0.61921	0.62080
	0.70000	0.58201	0.58396
	0.80000	0.54893	0.55126
	0.90000	0.51934	0.52203
	1.00000	0.49270	0.49576
$\Delta\varepsilon = \frac{1}{100}$	0.00000	1.00000	1.00000
	0.10000	0.90820	0.90821
	0.20000	0.83188	0.83191
	0.30000	0.76743	0.76748
	0.40000	0.71226	0.71235
	0.50000	0.66450	0.66463
	0.60000	0.62276	0.62292
	0.70000	0.58595	0.58615
	0.80000	0.55325	0.55348
	0.90000	0.52401	0.52428
	1.00000	0.49770	0.49801
$\Delta\varepsilon = \frac{1}{1000}$	0.00000	1.00000	1.00000
	0.10000	0.90829	0.90829
	0.20000	0.83204	0.83205
	0.30000	0.76765	0.76766
	0.40000	0.71254	0.71254
	0.50000	0.66484	0.66484
	0.60000	0.62314	0.62314
	0.70000	0.58637	0.58637
	0.80000	0.55372	0.55371
	0.90000	0.52451	0.52450
	1.00000	0.49824	0.49823

$$y_2^{RPS} = 1 + x + \varepsilon A[3x/2 - x^2/2 - x^3/6] +$$

$$+ \varepsilon^2 A^2[5x/6 - 3x^2/4 - x^3/3 + x^4/6 + x^5/30] + O(\Delta\varepsilon^3) \qquad (71)$$

$$y_2^{MPI} = 1 + x + \varepsilon A[3x/2 - x^2/2 - x^3/6] \qquad (72)$$

$$y_1^{MPD} = y_2^{MPI} \qquad (73)$$

Furthermore, comparison of the above three solutions shows that,

$$y_1^{RPS} = y_1^{MPD} = y_2^{MPI} \qquad (74)$$

Thus, y^{MPD} approximates y^{RPS} in less steps than y^{MPI} does it.

The next step in this example is to reduce this nonlinear problem with boundary conditions to a linear problem with initial conditions, through the combination of the MPD and the method of superposition.

Differentiation of equations (68) to (70) with respect to ε and application of the MPD yields,

$$g_n'' + n\ \Delta\varepsilon\ A\ y_n\ g_n' + n\ \Delta\varepsilon\ A\ y_n'\ g_n = -A\ y_n\ y_n' \qquad (75)$$

$$g_n(0) = 0 \qquad (76)$$

$$g_n'(1) = 0 \qquad (77)$$

$$g_n = \frac{\partial y_n}{\partial \varepsilon} \qquad (78)$$

The starting solutions are,

$$y_0 = 1 + x \qquad (79)$$

$$g_0 = -A(x^2/2 + x^3/6) + 3Ax/2 \qquad (80)$$

The method of superposition (see Na, 1979, p. 13) can then be applied. Define,

$$g_n = F_n + \mu_n\ G_n \qquad (81)$$

where μ_n is a constant.

Combining equations (75) and (81) we obtain,

$$F_n'' + n\ \Delta\varepsilon\ A\ y_n\ F_n' + n\ \Delta\varepsilon\ A\ y_n'\ F_n = -\ A\ y_n\ y_n' \qquad (82)$$

$$G_n'' + n\ \Delta\varepsilon\ A\ y_n\ G_n' + n\ \Delta\varepsilon\ A\ y_n'\ G_n = 0 \qquad (83)$$

since equation (76) has to be satisfied,

$$F_n(0) = G_n(0) = 0 \tag{84}$$

It is also possible to choose arbitrarily an additional initial condition for F_n and G_n as follows,

$$F_n'(0) = G_n'(0) = 0 \tag{85}$$

because equation (77), the remaining boundary condition, is used only to evaluate μ_n. Thus,

$$g_n'(1) = F_n'(1) + \mu_n G_n'(1) = 0 \tag{86}$$

and

$$\mu_n = - F_n'(1)/G_n'(1) \tag{87}$$

Once equations (82) to (85) have been solved through the Runge-Kutta or Euler methods at each step $\Delta\varepsilon$, equations (87), (81) and (78) are used to find g_n and y_{n+1} until $n = N$ and $N\Delta\varepsilon = \varepsilon^*$.

Table III shows numerical results of equations (68) to (70) when equations (75) to (87) are applied for two different step sizes $\Delta\varepsilon$. The two runs are coincident in the three first digits, hence further refinements in $\Delta\varepsilon$ and Δx are considered unnecessary

Example 4

The experience gained in Example 3 is now applied systematically to solve the boundary layer flow problem for a plate, (Blasius problem). The corresponding equations are, (see Schlichting, 1960, p. 117)

$$2f''' + \lambda f'' f = 0 \tag{88}$$

$$\eta = 0 \quad , \quad f' = 0 \quad , \quad f = 0 \tag{89}$$

$$\eta \to \infty \quad , \quad f' = 1 \tag{90}$$

where λ is an unphysical parameter such that for $\lambda = 1$ the physical problem involving the boundary layer theory is recovered (see also equation (37)).

To solve this problem through the MPD the starting solution f_0 is necessary; then for $\lambda = 0$,

$$f_0''' = 0 \tag{91}$$

$$\eta = 0 \quad , \quad f_0' = 0 \quad , \quad f_0 = 0 \tag{92}$$

$$\eta \to \infty \, , \, f_0' = 1 \tag{93}$$

However, the solution f_0 thus formulated does not exist, the MPD cannot be applied. This difficulty is overcame by adding nonhomogeneous terms to equation (88) which are multiplied by $(1-\lambda)$ so that they are zero

TABLE III

$$y'' + \varepsilon A y' y = 0 \quad , \quad y(0) = 1 \quad , \quad y'(1) = 1$$

$$\Delta x = 0.05 \quad , \quad \varepsilon = 1 \quad , \quad A = 0.1$$

	x	y^{MPD}
$\Delta\varepsilon = \frac{1}{1000}$	0.00000	1.00000
	0.05000	1.05849
	0.10000	1.11667
	0.15000	1.17453
	0.20000	1.23205
	0.25000	1.28922
	0.30000	1.34603
	0.35000	1.40245
	0.40000	1.45848
	0.45000	1.51410
	0.50000	1.56930
	0.55000	1.62408
	0.60000	1.67841
	0.65000	1.73229
	0.70000	1.78570
	0.75000	1.83864
	0.80000	1.89110
	0.85000	1.94306
	0.90000	1.99452
	0.95000	2.04548
	1.00000	2.09548
$\Delta\varepsilon = \frac{1}{100}$	0.00000	1.00000
	0.05000	1.05856
	0.10000	1.11682
	0.15000	1.17475
	0.20000	1.23234
	0.25000	1.28958
	0.30000	1.34644
	0.35000	1.40292
	0.40000	1.45900
	0.45000	1.51468
	0.50000	1.56993
	0.55000	1.62474
	0.60000	1.67911
	0.65000	1.73303
	0.70000	1.78647
	0.75000	1.83944
	0.80000	1.89192
	0.85000	1.94390
	0.90000	1.99538
	0.95000	2.04634
	1.00000	2.09634

for $\lambda = 1$. In fact, Blasius problem can be rewritten as follows,

$$2f''' + \lambda f'' f' = -2(1-\lambda) e^{-\eta} \tag{94}$$

where the nonhomogeneous term is an obvious choice if the boundary conditions at $\eta \to \infty$ has to be satisfied. For $\lambda = 1$, equation (94) reduces to the expected physical problem.

Now, the starting solution can be found by placing $\lambda = 0$, in equation (94) to obtain,

$$f_0''' = -e^{-\eta} \tag{95}$$

which solution is,

$$f_0 = e^{-\eta} + \eta - 1 \tag{96}$$

Also, it is readily shown that the MPD implies,

$$2g_n''' + n \Delta\lambda \, (f_n'' g_n + f g_n'') = -n \Delta\lambda \, e^{-\eta} - f_n'' f_n \tag{97}$$

$$g_n(0) = g_n'(0) = 0 \tag{98}$$

$$g_n'(\infty) = 0 \tag{99}$$

$$f_{n+1} - f_n = g_n \Delta\lambda \tag{100}$$

Using the method of superposition as in Example 3 we obtain,

$$g_n = G_n + \mu_n H_n \tag{101}$$

$$2G_n''' + n \Delta\lambda \, (f_n'' G_n + f_n G_n'') = -n \Delta\lambda \, e^{-\eta} - f_n'' f_n \tag{102}$$

$$G_n(0) = 0 \quad , \quad G_n'(0) = 0 \quad , \quad G_n''(0) = 1 \tag{103}$$

$$2H_n''' + n \Delta\lambda \, (f_n'' H_n + f_n H_n'') = 0 \tag{104}$$

$$H_n(0) = 0 \quad , \quad H_n'(0) = 0 \quad , \quad H_n''(0) = 1 \tag{105}$$

where $G''(0)$ and $H_n''(0)$ are arbitrarily imposed. Therefore, it is clear that,

$$\mu_n = [g_n(\infty) - G_n(\infty)] / H_n(\infty) = -G_n(\infty) / H_n(\infty) \tag{106}$$

μ_n can be obtained at each value of n with solutions G_n and H_n , which are in their turns evaluated from equations (102) to (105) through a noniterative method as in Example 3. Once μ_n is known, g_n is a result of equation (101).

The integration procedure is performed in N steps such that $N\,\Delta\lambda = 1$; also $\Delta\lambda \to 0$ as $N \to \infty$.

It should be observed that

$$g_n''(0) = 1 + \mu_n \tag{107}$$

which is designated as the missing second derivative in the classical shootting method that have been used to solve the Blasius problem (White, 1974, p. 261).

Table IV shows our results for $\Delta\lambda = 10^{-4}$ and $\Delta\eta = 0.2$ and they are compared with those obtained by Howard (Schlichting, 1960, p. 121).

TABLE IV

$$2f''' + \lambda f'' f = -2(1-\lambda)\, e^{-\eta} \quad , \quad f(0) = 0 \quad , \quad f'(0) = 0 \quad , \quad f'(\infty) = 1$$

$$\lambda = 1 \quad , \quad \Delta\lambda = 1/10000 \quad , \quad \Delta\eta = 0.2$$

η	f	f'	f"	f"(Howard)
0.00000	0.00000	0.00000	0.33199	0.33206
0.20000	0.00661	0.06637	0.33190	0.33199
0.40000	0.02655	0.13275	0.33125	0.33147
0.60000	0.05971	0.19900	0.32850	0.33008
0.80000	0.10615	0.26470	0.32335	0.32739
1.00000	0.16559	0.32937	0.32015	0.32301
1.20000	0.23790	0.39340	0.31335	0.31659
1.40000	0.32295	0.45607	0.30150	0.30787
1.60000	0.42033	0.51637	0.28875	0.29667
1.80000	0.52950	0.57412	0.27675	0.28293
2.00000	0.64998	0.62947	0.25740	0.26675
2.20000	0.78129	0.68095	0.23625	0.24835
2.40000	0.92236	0.72820	0.21655	0.22809
2.60000	1.07257	0.77151	0.19600	0.20646
2.80000	1.23096	0.81071	0.17420	0.18401
3.00000	1.39685	0.84555	0.14810	0.16136
3.20000	1.56918	0.87517	0.12841	0.13913
3.40000	1.74692	0.90085	0.11035	0.11788
3.60000	1.92952	0.92292	0.08765	0.09809
3.80000	2.11609	0.94045	0.07260	0.08013
4.00000	2.30570	0.95497	0.05841	0.06424
4.20000	2.49808	0.96665	0.04335	0.05052
4.40000	2.69236	0.97532	0.03590	0.03897
4.60000	2.88821	0.98250	0.02485	0.02948
4.80000	3.08536	0.98747	0.01925	0.02187
5.00000	3.28320	0.99132	0.01399	0.01591
5.20000	3.48189	0.99412	0.00940	0.01134
5.40000	3.68085	0.99600	0.00855	0.00793
5.60000	3.88029	0.99771	0.00541	0.00543
5.80000	4.07989	0.99939	0.00304	0.00365
6.00000	4.27960	1.00000	0.00000	0.00240
6.20000	4.47948	1.00000	0.00000	0.00155

II-2) Partial Differential Equations

In this section, the MPD is used to generate a procedure that solves the steady Navier-Stokes equation applied to two directional fluid flows. This equation can be written in dimensionless form in terms of the stream function ψ as follows (see, for example, the review work of Gupta, 1980, p. 163).

$$BRe\,[\psi_Y\,(\psi_{YYX} + B^2\,\psi_{XXX}) - \psi_X\,(\psi_{YYY} + B^2\,\psi_{XXY})] =$$
$$= B^4\,\psi_{XXXX} + 2\,B^2\,\psi_{XXYY} + \psi_{YYYY} \tag{108}$$

where Re is the Reynolds number. Since $X = x/a$ and $Y = y/b$, and (a,b) are the characteristic lengths of the dimensional coordinates (x,y), it is clear that $B = b/a$, $0 \le X \le 1$ and $0 \le Y \le 1$.

Equation (108) is a difficult nonlinear partial differential equation to solve and hence it can be linearized systematically through the MPD using for example B as the differentiation parameter. The resulting linear partial differential equation for $f(X,Y)$ defined as $f = \frac{\partial\psi}{\partial B}$, with variable coefficients that depend on $\psi(X,Y)$, is still very complex and, therefore, this linearization procedure is not recommended (see also comments of Gupta, 1980, p. 163, on computing directly equation (108) through finite difference methods). Instead, it is possible to introduce the definition of the vorticity Ω in order to reduce equation (108) to two coupled linear partial differential equations as follows,

$$\Omega = B^2\,\psi_{XX} + \psi_{YY} \tag{109}$$

$$BRe(\psi_Y\,\Omega_X - \psi_X\,\Omega_Y) = B^2\,\Omega_{XX} + \Omega_{YY} \tag{110}$$

Following Gupta (1980, p. 171) (see also Greenspan, 1974, Chapter VII) equations (109) and (110) can be solved through finite differences with the overall iterative procedure (outer-inner iterations) described in the following steps:

a) Start with some initial approximations for ψ_0 and Ω_0.

b) Solve the discrete form of equation (104) to obtain ψ_1. This implies inner iterations if a direct method is not used to solve equation (104).

c) Obtain the boundary values of Ω_1 using ψ_1.

d) Solve the discrete form of equation (105) to obtain Ω_1. This implies inner iteration as in (b).

e) Repeat steps (b), (c) and (d) with new values of ψ_n and Ω_n for $n = 1,2,\ldots$

The outer iterations (steps (b) to (e)) are terminated when,

f) (ψ_n, Ω_n) and $(\psi_{n+1}, \Omega_{n+1})$ are close to a given norm; say:

$$\max\,|\psi_n^{ij} - \psi_{n-1}^{ij}|\;/\;|\psi_n^{ij}| < \delta_1$$

$$\max |\Omega_n^{ij} - \Omega_{n-1}^{ij}| \; / \; |\Omega_n^{ij}| < \delta_2$$

where i,j indicates a mesh point and δ_1 and δ_2 are the allowed tolerances to the norm.

g) The outer or inner iteration procedures diverges.

h) The pre-assigned maximum value of computing time is exceeded.

The use of the MPD can improve the above procedure in two important numerical aspects; first, at each step, the numerical problem is initialized with a numerical solution of the previous step instead of iterative trial numbers; second, inner iterations can be avoided since the method of Gauss-Seidel converges in one step, if the increment of the differentiation parameter is chosen sufficiently small and if diagonal dominance of the coefficient matrix is assured (see Upwind discretization, Gupta 1980, p. 154). It should be also observed that direct methods to solve the discrete partial differential equations of ψ and Ω are more difficult to implement than the simple Gauss-Seidel method.

Consequently, the starting solutions in the MPD for $B = 0$ are,

$$\Omega_0 = \psi_{0YY} \tag{111}$$

$$\Omega_{0YY} = 0 \tag{112}$$

which are easy to solve. Then, differentiation of equations (109) and (110) with respect to B yields,

$$g_n = (n \Delta B)^2 \; f_{nXX} + f_{nYY} + 2n \Delta B \; \psi_{nXX} \tag{113}$$

$$n \Delta B Re \; (f_{nY} \; \Omega_{nX} - f_{nX} \; \Omega_{nY} + \psi_{nY} \; g_{nX} - \psi_{nX} \; g_{nY}) + Re \; (\psi_{nY} \; \Omega_{nX} - \psi_{nX} \; \Omega_{nY}) =$$

$$= (n \Delta B)^2 \; g_{nXX} + g_{nYY} + 2n \Delta B \; \Omega_{nXX} \tag{114}$$

$$\Omega_{n+1} - \Omega_n = g_n \Delta B \tag{115}$$

$$\psi_{n+1} - \psi_n = f_n \Delta B \tag{116}$$

$$n = 1,2,\ldots N \tag{117}$$

$$B^* = N \Delta B \tag{118}$$

Therefore the starting solutions g_0 and f_0 are,

$$g_0 = f_{0YY} \tag{119}$$

$$g_{0YY} = 0 \tag{120}$$

which can also be readily solved.

It should be observed that boundary conditions for ψ and Ω have also to be differentiated with respect to B . Therefore, it is also expected that this boundary parameter differentiation has a damping effect upon the over and under estimates of vorticity at the boundary (see Gupta, 1980, pp. 172 and references) frequently found at step c) in the overall iterative procedure described above.

Although we do not present here an example involving the application of the MPD to the solution of partial differential equations, the reader is referred to the proceeding of MECOM' 85 where the authors have evaluated the thermal efficiency of a hot water geothermal reservoir through the MPD.

Finally, the MPI is still easier to apply to the solution of equation (109) and (110). Thus, the following equations are readily obtained,

$$\Omega_n = n \Delta B \, \psi_{nXX} + \psi_{nYY} \tag{121}$$

$$n \Delta BRe \, (\psi_{nY} \, \Omega_{nX} - \psi_{nX} \, \Omega_{nY}) = (n \Delta B)^2 \, \Omega_{nXX} + \Omega_{nYY} \tag{122}$$

which have as starting solution those obtained from equation (111) and (112).

CONCLUSION

Along the previous applications of the MPD to nonlinear problems of fluid mechanics, we conclude the following remarkable aspects:

- The MPD offers a systematic procedure for linearization of ordinary and partial differential equations.
- The MPD allows to initialize with analytical solutions the numerical procedure for solving differential equations.
- The differentiation parameter does not need to be a physical parameter of the problem, because it can be arbitrarily selected and placed in any nonlinear term of the differential equation, with the constraint that it takes the value one at the end of the integration procedure.
- The solution of the steady stream function-vorticity scheme through finite differences with the overall iterative procedure can be simplified because inner iterations are eliminated and initialization functions are solutions of the problem at each previous outer parameter iterations. Although the MPD has also been applied to stream function-vorticity in this work, we believe that the MPD still requires an intensive research to better understand its application to partial differential equations.

REFERENCES

[1] Courant, R. and Hilbert, D., "Methods of Mathematical Physics", Vol. II, Interscience, N.Y. (1962).

[2] Greenspan, D., "Discrete Numerical Methods in Physics and Engineering", Academic Press, Inc. N.Y., (1974).

[3] Gupta, M.M., "Numerical Methods for Viscous Flow Problems", Adv. in Transport Processes, I, (1980).

[4] Ince, E.L., "Ordinary Differential Equations", Dover, N.Y. (1956).

[5] Na, T.Y., "Computational Methods in Engineering Boundary Value Problems", Academic Press, N.Y. (1979).

[6] Noye, J., "An Introduction to Finite Differences Techniques", Numerical Simulation of Fluid Motion, 1 (1978).

[7] Rubbert, P.E. and Landahl, M.T.: "Solution of Nonlinear Flow Problems through Parameter Differentiation", Phys. Fluids, 10, 831-835 (1967).

[8] Schlichting, H., "Boundary Layer Theory", Mc Graw-Hill, Inc., (1960).

[9] Smith, G.D., "Numerical Solution of Partial Differential Equations", Oxford University Press, (1978).

[10] Street, R.L., "The Analysis and Solution of Partial Differential Equations. Monterrey, Brooks/Cole, (1973).

[11] Wacker, H. "Continuation Methods", Academic Press, N.Y. (1978).

[12] White, F.M., "Viscous Fluid Flow", Mc Graw-Hill, Inc., N.Y. (1974).