A FINITE ELEMENT FORMULATION FOR ANALYSIS OF COMPOSITE LAMINATE SHELLS

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RESUMEN

En este artículo se presenta la formulación de un nuevo elemento triangular de lámina plana para el análisis de estructuras laminares de materiales compuestos. La formulación del elemento está basada en una combinación de la teoría degenerada de láminas y el uso de un supuesto campo de tensiones de corte. La interpolación sobre el espesor está basada en una aproximación lineal "leyer-wise". Los grados de libertad sobre el espesor son eliminados en los niveles de ensamblaje usando una técnica de subestructuración. Se presenta un ejemplo del buen comportamiento del elemento.

SUMMARY

In this paper the formulation of a new triangular facet shell element for analysis of composite laminate shell structures is presented. The element formulation is based on a combination of degenerate shell theory and the use of an assumed shear strain field. The thickness interpolation is based on a linear layer-wise approximation. The thickness degrees of freedom are eliminated at assembly level using a substructuring technique. An example of the good behaviour of the element is presented.

INTRODUCTION

Composite laminates are nowdays widely used for a variety of structures in automobile, aerospace, building and medicine equipment industries, amongst many others. The analysis of such structures is performed today via numerical techniques and in particular using the finite element method (FEM) [17,18]. A classification of the most popular theories for the analysis of laminated plates and shells in the context of FEM could be the following:

1) 3D elasticity theory
2) Single layer theory
3) Layer-wise 2D theory

Obviously the use of 3D elasticity models introduces little simplifications in the analysis. However, 3D FE models for real laminated structures are nowdays still costly due to large number of unknowns involved plus the difficulties intrinsic to pre and postprocessing.

Single layer (SL) theory provides the simplest approach for analysis of laminates. Displacements in SL theory are expanded along the thickness direction in the form [1]

$$u_i(x,y,z) = \sum_{j=0}^{m_i} u_i^j(x,y) z^j \qquad (i = 1,2,3) \tag{1}$$

where m_i are the number of terms of the expansion for the ith displacement component. Note that (1) leads to a continous shear strain field along the thickness direction which in turn produces a discontinuous shear stress field at the laminate interfaces if each laminate has different material properties. Eq.(1) is the basis for deriving first order ($m_1 = m_2 = 1$ and $m_3 = 0$) [1] and higher order, quadratic [3,16] or cubic [2,11] SL theories. A recent survey of different finite element models based on Sl theory can be found in [14].

The layer-wise 2D theory was proposed by Reddy [12,13] to overcome the stress continuity limitations of SL models. In layer-wise theory the 3D displacement field is first expanded as a linear combination of the thickness coordinate as

$$u_i(x,y,z) = u_i^0(x,y) + \sum_{j=1}^{n_i} u_i^j(x,y) z^j \Phi_j(z) \tag{2}$$

where n_i is the number of divisions across the thickness. The displacements $u_i^j(x,y)$ are now interpolated over each layer interface using a standard finite element approximation. The thickness interpolating functions Φ_j are piece-wise constant across the thickness direction. This implies that displacements are continuous across the thickness but the shear strains are discontinuous. This allows to obtain a continuous field of transverse shear strains at the layer interfaces.

In this paper a triangular facet shell element for the analysis of laminate shells based on layer-wise 2D theory is presented. The element can be considered as an extension of the linear/quadratic triangular Reissner-Mindlin plate element based on an assumed shear strain formulation presented by Zienkiewicz *et al.* [17], Papadopoulus and Taylor [10] Oñate *et al.* [4,5]. The in-plane displacements are linearly interpolated within each layer and they are eliminated during the global assembly process using a substructuring technique. Recent successful applications of this element for analysis of laminated plates carried out by the authors have been the motivations of present work [6]. Details of the element formulation are given in next section.

ELEMENT FORMULATION

Figure 1 shows the element geometry and the local coordinate system x', y', z' defining the local displacements u', v', w', respectively. Axes x' and y' are contained in the element plane, whereas z' is normal to such a plane. The element has n_l layers with $n_l + 1$ interfaces. The in-plane local displacements u', v' in local coordinates within the kth layer are interpolated as

$$\begin{Bmatrix} u' \\ v' \end{Bmatrix} = \sum_{i=1}^{3} N_i(\xi,\eta) \left[\begin{Bmatrix} u'_{i,o} \\ v'_{i,o} \end{Bmatrix} + N^k(\zeta) \begin{Bmatrix} u'^k_i \\ v'^k_i \end{Bmatrix} + N^k(\zeta) \begin{Bmatrix} u'^{k+1}_i \\ v'^{k+1}_i \end{Bmatrix} \right]$$

$$+ \sum_{i=4}^{6} N_i(\xi,\eta) \mathbf{e}_{i-3} \left[N^k(\zeta) \Delta u^k_{t_i} + N^{k+1}(\zeta) \Delta u^{k+1}_{t_i} \right] \qquad (3)$$

where $\begin{Bmatrix} u'_{i,o} \\ v'_{i,o} \end{Bmatrix}$ are constant in-plane displacement through the laminate thickness, $\begin{Bmatrix} u'^k_i \\ v'^k_i \end{Bmatrix}$ are variable in-plane displacements through the laminate thickness and $\Delta u^k_{t_i}$ are displacement increments in the mid-side nodes and in the direction defined by the unit vectors $\mathbf{e}_{i-3}$ (see Figure 1).

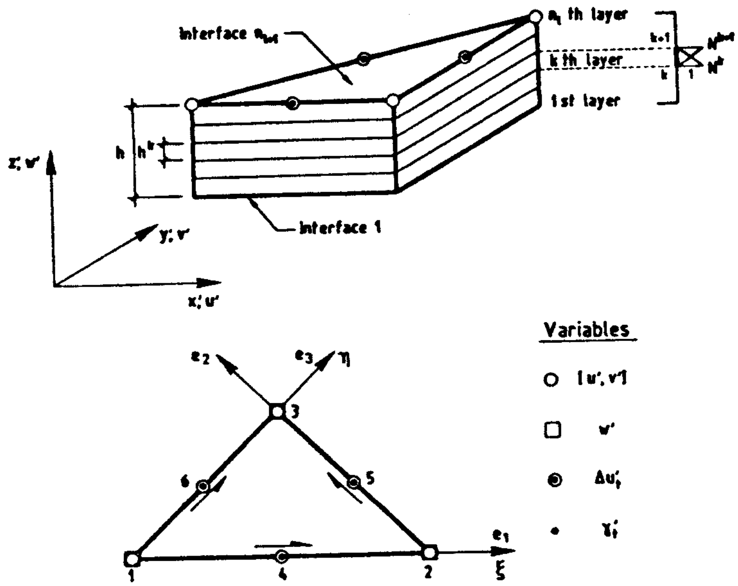


Figure 1 Finite element definition.

The normal displacement w' is assumed to be constant through the thickness. Following this assumption we can write

$$w' = \sum_{i=1}^{3} N_i(\xi,\eta) w'_i \qquad (4)$$

In (3) and (4) we have

$$
\begin{aligned}
&N_i(\xi,\eta) = L_i \qquad \text{for} \qquad i = 1,2,3 \\
&N_4(\xi,\eta) = 4L_1L_2 \quad , \quad N_5(\xi,\eta) = 4L_2L_3 \quad , \quad N_6(\xi,\eta) = 4L_1L_3
\end{aligned}
\tag{5a}
$$

where L_i are the standard shape functions of the 3 nodes triangle [18] and

$$
N^k(\zeta) = \frac{1-\zeta}{2} \qquad\qquad N^{k+1}(\zeta) = \frac{1+\zeta}{2} \tag{5b}
$$

Eqs.(3) and (4) imply a hierarchical quadratic interpolation for the horizontal displacements u' and v' over each interface whereas a linear interpolation for w is used. Note that for a single layer the element simplifies in its flat form to the linear/quadratic triangle based on Reissner-Mindlin plate theory proposed by Zienkiewicz *et al.* [17], Papadopoulus and Taylor [10] and Oñate *et al.* [4,5].

It has been shown that the undesirable defect of "locking" in thick plate elements when used for thin plate analysis can be avoided by imposing "a priori" a shear strain field compatible with the discretized displacement field [5,6]. In the element presented a linear shear strain field with constant values of the tangential shear strains along each side is imposed. Further details of the element formulation, including a discussion of the compatibility conditions to be satisfied by the displacement, rotations and shear strain fields can be found in [4,5,17,18].

The local strains for the kth layer can be obtained as

$$
\begin{aligned}
\varepsilon'_b &= \mathbf{B}_b \mathbf{a}' \\
\varepsilon'_s &= \mathbf{B}_s \mathbf{a}'
\end{aligned}
\tag{6}
$$

where ε'_b and ε'_s are defined as

$$
\varepsilon'_b = \left[\frac{\partial u'}{\partial x'}, \frac{\partial v'}{\partial y'}, \left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right)\right]^T \tag{7a}
$$

$$
\varepsilon'_s = \left[\frac{\partial w'}{\partial x'}, \frac{\partial u'}{\partial x'}, \left(\frac{\partial w'}{\partial y'} + \frac{\partial v'}{\partial x'}\right)\right]^T \tag{7b}
$$

Note that ε'_b are the local strains due to the combination of membrane and bending effects and ε'_s are the transversal shear strains. The form of matrices $\mathbf{B}_b$ and $\mathbf{B}_s$ is given in Table I.

For convenience we write the local displacement vector $\mathbf{a}'$ for the kth layer as

$$
\mathbf{a}' = [\mathbf{a}'^k, \mathbf{a}'^{k+1}, \mathbf{a}'_0]^T \tag{8a}
$$

where

$$\begin{aligned} \mathbf{a}'^{k} &= [u'^{k}_{1}, v'^{k}_{1}, w'^{k}_{1}, u'^{k}_{2}, v'^{k}_{2}, w'^{k}_{2}, u'^{k}_{3}, v'^{k}_{3}, w'^{k}_{3}, \Delta u^{k}_{14}, \Delta u^{k}_{15}, \Delta u^{k}_{16}]^{T} \\ \mathbf{a}'_{0} &= [u'_{01}, v'_{01}, w'_{01}, u'_{02}, v'_{02}, w'_{02}, u'_{03}, v'_{03}, w'_{03}]^{T} \end{aligned} \tag{8b}$$

The nodal displacement $\mathbf{a}'$ are transformed to global axes by

$$\mathbf{a}' = \bar{\mathbf{T}}\mathbf{a} \tag{9}$$

and

$$\mathbf{a} = [\mathbf{a}^{k}\mathbf{a}^{k+1}, \mathbf{a}^{0}]$$

$$\begin{aligned} \mathbf{a}^{k} &= [u^{k}_{1}, v^{k}_{1}, w^{k}_{1}, u^{k}_{2}, v^{k}_{2}, w^{k}_{2}, u^{k}_{3}, v^{k}_{3}, w^{k}_{3}, \Delta u^{k}_{14}, \Delta u^{k}_{15}, \Delta u^{k}_{16}]^{T} \\ \mathbf{a}_{0} &= [u_{01}, v_{01}, w_{01}, u_{02}, v_{02}, w_{02}, u_{03}, v_{03}, w_{03}]^{T} \end{aligned}$$

where the transformation matrix is given by

$$\bar{\mathbf{T}} = \begin{bmatrix} \hat{\mathbf{T}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \hat{\mathbf{T}}' \end{bmatrix} \quad \underset{(12\times 12)}{\hat{\mathbf{T}}} \quad \underset{(9\times 9)}{\hat{\mathbf{T}}'} \tag{10a}$$

with

$$\hat{\mathbf{T}} = \begin{bmatrix} \hat{\mathbf{T}}' & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3 \end{bmatrix} \tag{10b}$$

where $\mathbf{I}_3$ as the 3×3 unit matrix and

$$\hat{\mathbf{T}}' = \begin{bmatrix} \mathbf{T} & & \mathbf{0} \\ & \mathbf{T} & \\ \mathbf{0} & & \mathbf{T} \end{bmatrix} \quad , \quad \mathbf{T} = \begin{bmatrix} \lambda_{x'x} & \lambda_{x'y} & \lambda_{x'z} \\ \lambda_{y'x} & \lambda_{y'y} & \lambda_{y'z} \\ \lambda_{z'x} & \lambda_{z'y} & \lambda_{z'z} \end{bmatrix} \tag{10c}$$

and $\lambda_{x'x}$ the cosine which the local axe x' forms with the global axe x, etc (Figure 2).

The element stiffness matrix for the kth layer can be written as

$$\mathbf{K}^{(e)} = \bar{\mathbf{T}}^{T}\mathbf{K}'^{(e)}\bar{\mathbf{T}} \tag{11}$$

with

$$\mathbf{K}'^{(e)} = \int_{A^{(e)}} \mathbf{B}^{t}\mathbf{D}\mathbf{B}\,dV \tag{12}$$

where $\mathbf{B} = \begin{Bmatrix} \mathbf{B}_b \\ \mathbf{B}_s \end{Bmatrix}$ (see Table I) and $\mathbf{D}$ is the constitutive matrix for orthotropic laminates [11-14].

$$\underset{(3\times 33)}{\mathbf{B}_b} = [\ \underset{(3\times 12)}{\mathbf{B}_b^k},\quad \underset{(3\times 12)}{\mathbf{B}_b^{k+1}},\quad \underset{(3\times 9)}{\mathbf{B}_b^o}]$$

$$\mathbf{B}_b^k = [\mathbf{B}_{b_1}^k, \mathbf{B}_{b_2}^k, \mathbf{B}_{b_3}^k, \bar{\mathbf{B}}_{b_4}^k, \bar{\mathbf{B}}_{b_5}^k, \bar{\mathbf{B}}_{b_6}^k]$$

$$\mathbf{B}_b^o = [\mathbf{B}_{b_1}^o, \mathbf{B}_{b_2}^o, \mathbf{B}_{b_3}^o]$$

with

$$\mathbf{B}_{b_i}^o = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & 0 \end{bmatrix} \quad , \quad \mathbf{B}_{b_i}^k = N^k \mathbf{B}_{b_i}^o \quad i = 1,2,3$$

$$\bar{\mathbf{B}}_{b_i}^k = \mathbf{B}_{b_{i-3}}^k \mathbf{e}_{i-3} \qquad i = 4,5,6$$

$$\underset{(2\times 33)}{\mathbf{B}_s^k} = \mathbf{J}^{-1}\mathbf{M}\ [\underset{(3\times 12)}{\mathbf{B}_s^k}\ ,\ \underset{(3\times 12)}{\mathbf{B}_s^{k+1}}\ ,\ \underset{(3\times 9)}{\mathbf{B}_w}]$$

$$\mathbf{B}_s^k = \begin{bmatrix} a_{12} & b_{12} & 0 & \vdots & a_{12} & b_{12} & 0 & \vdots & 0 & 0 & 0 & \vdots & c_{12} & 0 & 0 \\ 0 & 0 & 0 & \vdots & \frac{a_{23}}{\sqrt{2}} & \frac{b_{23}}{\sqrt{2}} & 0 & \vdots & \frac{a_{23}}{\sqrt{2}} & \frac{b_{23}}{\sqrt{2}} & 0 & \vdots & 0 & \frac{c_{23}}{\sqrt{2}} & 0 \\ a_{13} & b_{13} & 0 & \vdots & 0 & 0 & 0 & \vdots & a_{32} & b_{32} & 0 & \vdots & 0 & 0 & c_{32} \end{bmatrix}$$

$$\mathbf{B}_s^{k+1} = -\mathbf{B}_s^k$$

$$\mathbf{B}_w = \begin{bmatrix} 0 & 0 & -1 & \vdots & 0 & 0 & 1 & \vdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 & -\frac{1}{\sqrt{2}} & \vdots & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & -1 & \vdots & 0 & 0 & 0 & \vdots & 0 & 0 & 1 \end{bmatrix}$$

$$M = \begin{bmatrix} 1-\eta & -\sqrt{2}\eta & \eta \\ \zeta & \sqrt{2}\zeta & 1-\zeta \end{bmatrix}$$

$$a_{ij} = -\frac{c_i L^{ij}}{2h^k} \quad ; \quad b_{ij} = -\frac{s_i L^{ij}}{2h^k} \quad ; \quad c_{ij} = \frac{2L^{ij}}{3h^k}$$

c_i, s_i = components of vector $e_i = [c_i, s_i]^T \qquad i = 1,2,3$

L^{ij} = length of side ij

h^k = kth layer

J = Jacobian matrix

Table I Local strain matrices for the triangular facet shell element of Figure 1.

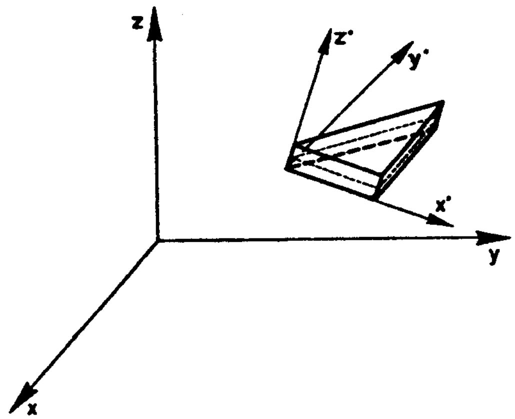


Figure 2 Transformation from local to global axes.

A more explicit form of the local element matrix $\mathbf{K}'^{(e)}$ is given in Table II.

It is worth noting that the tangential shear must be defined by an unique direction on each edge of contiguous elements. The signs in matrix $\mathbf{B}_w$ of Table I correspond to a definition of the direction of $\mathbf{e}_i$ in the directions of increasing (global) node numbers for the end points of each element edge [4,5,17].

The volume integral (12) involves integration over the layer thickness h^k and the area $A^{(e)}$ of each layer interface. The simplicity of the linear thickness shape functions N^k allows to perform the thickness integration explicity whereas a 2×2 Gaussian quadrature must be used for the interface area integral.

The global assembly process has the following steps. First the element equations for each layer are assembled at the interface level giving a global stiffness equation for each individual layer. Then the different layer equations are assembled through the thickness to give the total global equation system.

ELIMINATION OF LAYER DEGREES OF FREEDOM

The assembly of the stiffness matrices of the different layers ressembles the assembly of 1D bar elements (see Figure 3). This allows to eliminate the degrees of freedom, $\mathbf{a}^k$, of a layer after they have been assembled the global stiffness matrix. From Figure 3 we deduce that after assembly of the stiffness equations of the first layer, the variables $\mathbf{a}^1$ can be eliminated as

$$\mathbf{a}^1 = [\mathbf{K}_{11}^{(1)}]^{-1}[\mathbf{f}^1 - \mathbf{K}_{12}^{(1)}\mathbf{a}^2 - \mathbf{K}_{13}^{(1)}\mathbf{a}_o] \tag{13}$$

If the stiffness equations of the second layer are now assembled the global equations can be written in terms of $\mathbf{a}^2$, $\mathbf{a}^3$ and $\mathbf{a}_o$ variables using (13) as

$\mathbf{K}'^{(e)} = \mathbf{K}'^{(e)}_b + \mathbf{K}'^{(e)}_s$
$\mathbf{K}'^{(e)}_b = \frac{h^k}{6}\begin{bmatrix} 2\mathbf{K}'_{bb} & \mathbf{K}'_{bb} & 0 \\ \mathbf{K}'_{bb} & 2\mathbf{K}'_{bb} & 0 \\ 0 & 0 & 6\mathbf{K}'_{oo} \end{bmatrix}$ where $\underset{(12 \times 12)}{\mathbf{K}'_{bb}} = \int_{A^{(e)}} \hat{\mathbf{B}}_b^T \mathbf{D} \hat{\mathbf{B}}_b dA$ with $\hat{\mathbf{B}}_b = [\mathbf{B}_b^o, \mathbf{B}_t]$ $\mathbf{B}_t = [\bar{B}_{b4}, \bar{B}_{b5}, \bar{B}_{b6}]$ and $\underset{(9 \times 9)}{\mathbf{K}'_{oo}} = \int_{A^{(e)}} \mathbf{B}_b^{o^T} \mathbf{D} \mathbf{B}_b^o dA$
$\mathbf{K}'^{(e)}_s = h^k \int_{A^{(e)}} \mathbf{B}_s^T \mathbf{D} \mathbf{B}_s dA$

TABLE II. Local element stiffness matrix for a single layer.

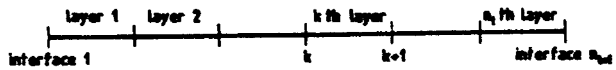
$$\begin{bmatrix} \begin{matrix}(\mathbf{K}_{22}^{(1)} + \mathbf{K}_{11}^{(2)} - \\ -\mathbf{K}_{21}^{(1)}[\mathbf{K}_{11}^{(1)}]^{-1}\mathbf{K}_{12}^{(1)})\end{matrix} & \mathbf{K}_{12}^{(2)} & \begin{matrix}(\mathbf{K}_{23}^{(1)} + \mathbf{K}_{13}^{(2)} - \\ -\mathbf{K}_{21}^{(1)}[\mathbf{K}_{11}^{(1)}]^{-1}\mathbf{K}_{13}^{(1)})\end{matrix} \\ \text{Simmetry} & \mathbf{K}_{22}^{(2)} & \mathbf{K}_{23}^{(2)} \\ & & \begin{matrix}(\mathbf{K}_{33}^{(2)} - \\ \mathbf{K}_{31}^{(1)}[\mathbf{K}_{11}^{(1)}]^{-1}\mathbf{K}_{13}^{(1)})\end{matrix} \end{bmatrix} \begin{Bmatrix} \mathbf{a}^2 \\ \mathbf{a}^3 \\ \mathbf{a}_o \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}^2 + \mathbf{K}_{21}^{(1)}[\mathbf{K}_{11}^{(1)}]^{-1}\mathbf{f}^1 \\ \mathbf{f}^2 \\ \mathbf{f}^3 \\ \mathbf{f}_o + \mathbf{K}_{31}^{(1)}[\mathbf{K}_{11}^{(1)}]^{-1}\mathbf{f}^1 \end{Bmatrix} \tag{14}$$

The variables of the second interface $\mathbf{a}^2$ can now be eliminated by an equation similar to (13). The procedure is repeated by subsequently assembling the equation of a new layer and eliminating the variables of the kth interface, $\mathbf{a}^k$, in terms of those of the $k+1$ interface, $\mathbf{a}^{k+1}$, and the displacements $\mathbf{a}_o$.

This elimination technique yields a final systems of equations involving only the variables of the last (upper) interface $\mathbf{a}^{n_l+1}$ and the others variables $\mathbf{a}_o$ i.e.

$$\begin{bmatrix} \bar{\mathbf{K}}_{11} & \bar{\mathbf{K}}_{12} \\ \bar{\mathbf{K}}_{21} & \bar{\mathbf{K}}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{a}^{n_l+1} \\ \mathbf{a}_o \end{Bmatrix} = \begin{Bmatrix} \bar{\mathbf{f}}^{n_l+1} \\ \bar{\mathbf{f}}_o \end{Bmatrix} \tag{15}$$

where $(\bar{\cdot})$ means that the coefficients have been adequately modified by the elimination process. Solution of (15) allows to recover all the variables of the lower interfaces in terms of those of the upper ones and the thickness independent variables $\mathbf{a}_o$. This elimination technique was first



$$\begin{bmatrix} K^{(1)}_{11} & K^{(2)}_{12} & 0 & 0 & \cdots & K^{(1)}_{13} \\ K^{(1)}_{21} & (K^{(12)}_{22} + K^{(2)}_{11}) & K^{(2)}_{12} & 0 & \cdots & (K^{(1)}_{23} + K^{(2)}_{13}) \\ 0 & K^{(2)}_{21} & (K^{(2)}_{22} + K^{(3)}_{11}) & K^{(3)}_{12} & \cdots & (K^{(3)}_{23} + K^{(1)}_{13}) \\ & & \ddots & \vdots & & \vdots \\ & & & K^{n_l}_{22} & & K^{n_l}_{23} \\ & & & & & K^{n_l}_{33} \end{bmatrix} \begin{Bmatrix} a^1 \\ a^2 \\ a^3 \\ \vdots \\ \vdots \\ a^{n_l} \\ a_o \end{Bmatrix} = \begin{Bmatrix} f^1 \\ f^2 \\ f^3 \\ \vdots \\ \vdots \\ f^{n_l} \\ f_o \end{Bmatrix}$$

$K^{(k)}$ = stiffness matrix of kth layer

$a^{(k)}$ = nodal displacements associated to kth interface

a_o = other nodal displacements

Figure 3. Form of the global stiffness equations in the analysis of a laminated plate with n_l layers.

suggested in the context of laminated plate analysis by Owen and Li [7], and it can be easily extended to vibration a non linear analysis [7,8,9].

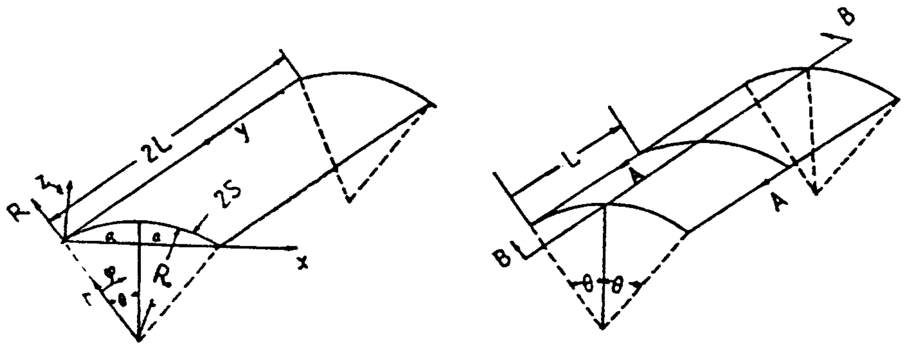
EXAMPLE

The example studied is the analysis of a laminate cylindrical shell simply supported across its boundary. The laminate is composed of three layers of Graphite-Epoxi composites with orientations 90/0/90 with respect to the global axe y of Figure 4 where the geometrical and material properties are also shown.

The analysis has been performed using three meshes of 4×4, 6×6 and 8×8 elements. The thickness direction has been divided in 3, 6 and 24 layers for the 4×4 mesh and 24 layers for the other two meshes. Table III shows some of the numerical results obtained. Also the in-plane displacement across the thickness direction are given in Figure 5. Finally Figure 6 shows the stress σ_{yz} and σ_{xz} distribution across the thickness in the indicated coordinates x, y.

CONCLUSION

The triangular facet shell element proposed combines the advantages of the 2D layer-wise solid model with those of assumed shear strain models to deal with thin shell situations. The linear



boundary conditions

load

$v_\beta = w_\gamma = 0$ $\quad$ *at* $\varphi = 0$ *and* $\varphi = 2\theta$

$u_\alpha = w_\gamma = 0$ $\quad$ *at* $y = 0$ *and* $y = 2L$

$$q(x,y) = q_o sin(\pi x/2a) sin(\pi y/2L)$$

properties

$E_l = 25 \times 10^6 psi$

$E_t = 1 \times 10^6 psi$

$G_l = 5 \times 10^5 psi$

$G_t = 2 \times 10^5 psi$

$\nu_{tl} = \nu_{tt} = 0.25$

$q_o = 1.0lb$

$h = 2,5in$

$L = 5,0in$

$a = 5,0in$

Mesh

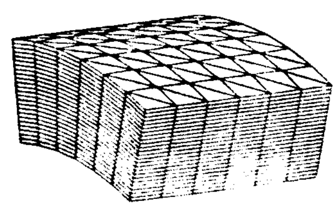


Figure 4 Simple supported square laminated plate (3 layers of Graphite-Epoxi composites 90/0/90 whith respect to axe y) Geometry and material properties (θ 30°

					$\theta = 30°$					
MESH	LAMIN.		DISP X		DISP Y		σ_{xx}		σ_{yy}	
			point	value	point	value	point	value	point	value
4 × 4	3	MAX	$(a, \frac{L}{2}, h)$	0.257E-6	$(a, 0, h)$	0.325E-5	$(a, \frac{L}{2}, 0)$	6.86	$(a, \frac{L}{2}, \frac{h}{3})$	3.15
		MIN	$(0, \frac{L}{2}, h)$	-0.457E-5	$(a, 0, 0)$	-0.282E-5	$(a, \frac{L}{2}, h)$	-7.67	$(a, \frac{L}{2}, \frac{2h}{3})$	-6.15
	6	MAX	$(a, \frac{L}{2}, h)$	0.334E-6	$(a, 0, h)$	0.337E-5	$(a, \frac{L}{2}, 0)$	9.18	$(a, \frac{L}{2}, \frac{h}{3})$	3.50
		MIN	$(0, \frac{L}{2}, 0)$	-0.507E-5	$(a, 0, 0)$	-0.294E-5	$(a, \frac{L}{2}, h)$	-9.47	$(a, \frac{L}{2}, \frac{2h}{3})$	-6.56
	24	MAX	$(a, \frac{L}{2}, h)$	0.379E-6	$(a, 0, h)$	0.346E-5	$(a, \frac{L}{2}, 0)$	10.90	$(a, \frac{L}{2}, \frac{h}{3})$	3.90
		MIN	$(0, \frac{L}{2}, 0)$	-0.525E-5	$(a, 0, 0)$	-0.303E-5	$(a, \frac{L}{2}, h)$	-10.60	$(a, \frac{L}{2}, \frac{2h}{3})$	-7.00
6 × 6	24	MAX	$(a, \frac{L}{2}, h)$	0.374E-6	$(a, 0, h)$	0.358E-5	$(a, \frac{L}{2}, 0)$	10.70	$(a, \frac{L}{2}, \frac{h}{3})$	3.94
		MIN	$(0, \frac{L}{2}, 0)$	-0.535E-5	$(a, 0, 0)$	-0.312E-5	$(a, \frac{L}{2}, h)$	-10.30	$(a, \frac{L}{2}, \frac{2h}{3})$	-7.31
8 × 8	24	MAX	$(a, \frac{L}{2}, h)$	0.373E-6	$(a, 0, h)$	0.361E-5	$(a, \frac{L}{2}, 0)$	10.80	$(a, \frac{L}{2}, \frac{h}{3})$	4.00
		MIN	$(0, \frac{L}{2}, 0)$	-0.539E-5	$(a, 0, 0)$	-0.317E-5	$(a, \frac{L}{2}, h)$	-10.20	$(a, \frac{L}{2}, \frac{2h}{3})$	-7.34

TABLE III. Some displacement and stress results for the example of Figure 4.

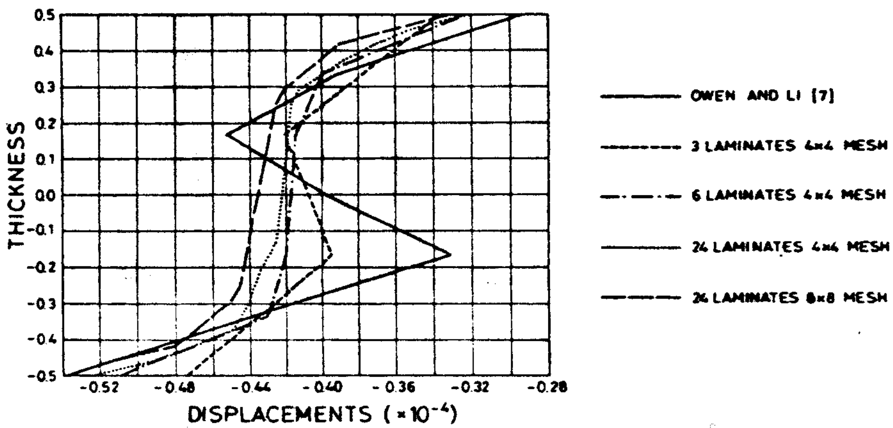


Figure 5 Comparison of in-plane displacement (in $x = 0$, $y = L$) across the thickness for different meshes and layer discretisations.

thickness interpolation used allows to elliminate the thickness variables at assembly level, thus reducing considerably the computational effort. The example presented shows the capability of the element for the analysis of laminate composite shells.

Current research on this topic by the authors includes the extension of the element formulation to account for geometrically and material non linear effects.

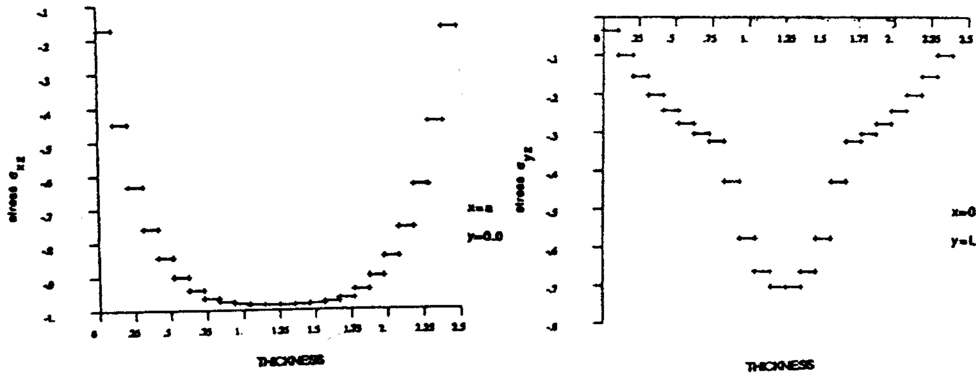


Figure 6 Tickness variation of stresses σ_{yz} and σ_{xz}.

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