DYNAMIC ELASTIC-PERFECTLY PLASTICITY IMPLICIT SCHEME

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RESUMEN

En este trabajo presentamos un esquema implícito con respecto al tiempo para el problema de un cuerpo sujeto a la ley constitutiva elástica perfectamente plástica.
Probamos la convergencia de la solución discreta si las condiciones de contorno impuestas para las tensiones no dependen del tiempo.

ABSTRACT

In this work we present an implicit scheme on time for the dynamical problem of the body subject to an elastic-perfectly plastic constitutive law. We prove the convergence of the discret solution if the imposed boundary conditions on the stress do not depend on time.

SOME NOTATIONS

We suppose that the body occupies a bounded region Ω in $\mathbb{R}^3$.

- We shall denote v the velocity field, $v(x,t) \in \mathbb{R}^3$, σ the stress tensor $\sigma(x,t) \in \mathbb{R}^9_s$ and f the body forces $f(x,t) \in \mathbb{R}^3$ with $x \in \Omega$ $t \in [0,T]$, $T > 0$.
- $\partial\Omega = \partial_v\Omega \cup \partial_F\Omega$ we impose the forces in $\partial_F\Omega$ and the velocity on $\partial_v\Omega$.
- $\mathbf{L}^2(\Omega) = L^2(\Omega)^9_s$, $\mathbf{L}^\infty(\Omega) = L^\infty(\Omega)^9_s$, $\mathbf{L}^1(\Omega) = L^1(\Omega)^9_s$ and $H = \mathbf{L}^2(\Omega) \times L^2(\Omega)^3$
- Let $\omega \in H^1(\Omega)^3$, we define $\epsilon(\omega)$ the strain rate associated with ω

$$\epsilon_{ij}(\omega) = \frac{1}{2}\left(\frac{\partial\omega_i}{\partial x_j} + \frac{\partial\omega_j}{\partial x_i}\right)$$

- $A = (A_{ijhk})$ is the 4th order tensor of elastic compliance of the material exhibiting the usual properties of symmetry, boundedness and coercivity.

$$\begin{cases} A_{ijhk} = A_{ijhk} = A_{ijhk} \\ \alpha\xi_{ij}\xi_{ij} \le A_{ijhk}\xi_{hk}\xi_{ij} \le \beta\xi_{ij}\xi_{ij} \quad \text{with } \alpha > 0 \text{ and } \beta > 0 \end{cases}$$

- We define the scalar product $[.,.]_A$ on H such that $\left[\begin{pmatrix}\tau_1\\ \omega_1\end{pmatrix}, \begin{pmatrix}\tau_2\\ \omega_2\end{pmatrix}\right]_A = \int_\Omega \tau_1 A \tau_2 \, dx + \int_\Omega \omega_1\omega_2 \, dx$

with A symmetric, bounded and coercive.

- We define

$$Y = \{\tau \in \mathbf{L}^2(\Omega) \text{ with } \operatorname{div} \tau \in L^2(\Omega)^3\}$$

REGULARITY ASSUMPTIONS

- $f \in W^{1,\infty}(0,T; L^2(\Omega)^3)$
- We consider mixed boundary conditions. We assume that the forces are given on a part $\partial\Omega_F$ and the velocity is imposed on $\partial\Omega_v = \partial\Omega - \partial\Omega_F$

$$\sigma.\eta = F^d \quad \text{on } \partial\Omega_F$$
$$v = v^d \quad \text{on } \partial\Omega_v$$

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- The initial state of the material is defined by σ_0 the initial stress, and v_0 the initial velocity.
- We assume the existence of (σ^*, v^*) such that

$$\begin{cases} \sigma^*(x,t) \in K \text{ p.p.} \\ \sigma^*.\eta = F^d & \text{if } x \in \partial_F\Omega \text{ and } t \in [0,T] \\ v^* = v^d & \text{if } x \in \partial_v\Omega \text{ and } t \in [0,T] \\ \sigma^*(0) = \sigma_0 \\ v^*(0) = v_0 \end{cases}$$

with the following regularity conditions

$$\begin{cases} v^* \in W^{1,\infty}(0,T; L^2(\Omega)^3) \\ \sigma^* \in W^{2,\infty}(0,T; L^\infty(\Omega)) \\ \text{div } \sigma^* \in L^\infty(0,T; L^2(\Omega)^3) \\ \epsilon(v^*) \in W^{1,\infty}(0,T; L^2(\Omega)) \end{cases}$$

FORMULATION OF THE DYNAMIC ELASTIC-PERFECTLY PLASTIC PROBLEM

The constitutive law of an ideal perfectly elasto plastic material reads as follows:

$$\epsilon(v) = A\dot{\sigma} + \partial I_K(\sigma) \tag{1}$$

where
- K is the closed convex set in $\mathbb{R}^9_s$ which delimits the set of physically admissible stress states.
- I_K is the indicator function of K.

We have in addition the movement equation

$$\text{div } \sigma + f = \dot{v} \tag{2}$$

where we assume $\rho = 1$ for simplicity.
with initial conditions

$$\begin{cases} \sigma(0) = \sigma_0 \\ v(0) = v_0 \end{cases} \tag{3}$$

and boundary conditions

$$\begin{cases} \sigma.\eta = F^d & \text{if } x \in \Omega_F,\ t \in [0,T] \\ v = v^d & \text{if } x \in \Omega_v,\ t \in [0,T] \end{cases} \tag{4}$$

In fact like in the quasi static case (see [5]) the solution will satisfy only a weak form of the constitutive law (see [1]) however, v and σ will be unique.

WEAK CONSTITUTIVE LAW

$$\int_\Omega A\dot{\sigma}(t)(\sigma(t) - \tau(t))\, dx + \int_\Omega (v(t) - v^*(t)) \text{ div } (\sigma(t) - \tau(t))\, dx - \int_\Omega \epsilon(v^*(t))(\sigma(t) - \tau(t))\, dx \le 0 \tag{5}$$

Note: In [1] it is proved the existence of a solution for the problem defined by (2), (3), (4) and (5).

We set

$$\bar{v} = v - v^*, \quad \bar{f} = f - \dot{v}^*, \quad h = \epsilon(v^*)$$

Therefore we obtain from (2), (3), (4) and (5) the equivalent problem.
Find $(\sigma, \bar{v})$, $\sigma(t) \in \mathbf{K}$ with $\mathbf{K} = \{\tau \in Y$ and $\tau(x) \in K$ p.p.$\}$ such that

$$\int_\Omega A\dot{\sigma}(\sigma - \tau)dx + \int_\Omega \bar{v} \text{ div } (\sigma - \tau)dx \leq \int_\Omega h(\sigma - \tau)dx \tag{6}$$

for all τ such that $\tau(t) \in \mathbf{K}_F = \{\tau \in \mathbf{K}/\tau.\eta = F^d\}$

$$\text{div } \sigma + \overline{f} = \dot{\bar{v}} \tag{7}$$

The boundary conditions

$$\sigma.\eta = F^d \quad \text{if } x \in \partial_F\Omega \quad t \in [0,T] \tag{8}$$

$$\bar{v} = 0 \quad \text{if } x \in \partial_v\Omega \quad t \in [0,T] \tag{9}$$

The initial conditions

$$\bar{v}(0) = 0, \quad \sigma(0) = \sigma_0 \qquad \text{if } x \in \Omega \tag{10}$$

IMPLICIT SCHEME

We use a finite difference discretisation in time.
We divide $[0,T]$ in N intervals $[t_n, t_{n+1}]$ with $\Delta t = \frac{T}{N}$ $u^n(x) = v^*(v, t_n)$ and $\overline{f}^n(x) = \overline{f}(x, t_n)$, $h^n(x) = h(x, t_n)$.
We consider $\sigma^0 = \sigma_0, \bar{v}^0 = 0$ and $(\sigma^{n+1}, \bar{v}^{n+1})$ is defined by induction as the solution of

$$\int_\Omega A\left(\frac{\sigma^{n+1} - \sigma^n}{\Delta t}\right).(\sigma^{n+1} - \tau)dx + \int_\Omega \bar{v}^{n+1} \text{ div } (\sigma^{n+1} - \tau)dx \leq\leq \int_\Omega h^{n+1}(\sigma^{n+1} - \tau)dx \tag{11}$$

for all $\tau \in \mathbf{K}_F$
and

$$\int_\Omega \left(\frac{\bar{v}^{n+1} - \bar{v}^n}{\Delta t}\right)\omega \, dx = \int_\Omega (\overline{f}^{n+1} + \text{ div } \sigma^{n+1})\omega \, dx \tag{12}$$

for all $\omega \in L^2(\Omega)^3$
with the boundary condition on σ

$$\sigma^{n+1}.\eta = F^d \quad \text{if } x \in \partial_F\Omega \tag{13}$$

and

$$\sigma^{n+1} \in \mathbf{K} \tag{14}$$

It is possible to see, using the theory of convex analysis (see for instance [3] and [4]) that there is a unique solution for the implicit scheme.
We shall give some definitions
For a sequence $(\chi_n)_{0 \leq n \leq N}$ in a vectorial space we define

$$\chi_N(t) = (\chi_n - \chi_{n-1})\frac{t - t_{n-1}}{\Delta t} + \chi_{n-1} \qquad \text{if } t \in [t_{n-1}, t_n]$$

$$\chi_N^*(t) = \chi_n \qquad \text{if } t \in [t_{n-1}, t_n]$$

We shall see what happens with the sequences $\sigma_N, \sigma_N^*, \bar{v}_N$ and $\bar{v}_N^*$ when $N \to +\infty$

A PRIORI ESTIMATES I

- Because the tensor A is positive defined we obtain

$$A(\sigma^{n+1} - \sigma^n)\sigma^{n+1} \geq \frac{1}{2}(A\sigma^{n+1}\sigma^{n+1} - A\sigma^n\sigma^n) \tag{15}$$

Therefore

$$\sum_{n=0}^{m}\int_\Omega A(\sigma^{n+1}-\sigma^n)\sigma^{n+1}dx \geq \frac{1}{2}\int_\Omega A\sigma^{m+1}\sigma^{m+1} - \frac{1}{2}\int_\Omega A\sigma^0\sigma^0 \tag{16}$$

• We also have

$$\sum_{i=0}^{m}\int_\Omega (\bar{v}^{n+1}-\bar{v}^n)\bar{v}^{n+1}\,dx \geq \frac{1}{2}\int_\Omega |\bar{v}^{m+1}|^2 dx \tag{17}$$

Using (15), (16), (17) and after summation of the successive inequations, (11) and (12) with $\tau \in \mathbf{K}_F$ and $w = \bar{v}^{n+1}$ we obtain

$$\frac{1}{2}\int_\Omega A\sigma^{m+1}\sigma^{m+1}dx + \frac{1}{2}\int_\Omega |\bar{v}^{m+1}|^2\,dx \leq \sum_{n=0}^{m}\int_\Omega \Delta t(\bar{f}^{n+1} + \text{ div }\tau)\bar{v}^{n+1}dx$$
$$+\sum_{n=0}^{m}\int_\Omega \Delta t h^{n+1}(\sigma^{n+1}-\tau)dx + \int_\Omega A(\sigma^{m+1}-\sigma^0)\tau\,dx + \frac{1}{2}\int_\Omega A\sigma_0\sigma_0\,dx \tag{18}$$

From the assumptions we have done in chapter 1, we have in particular $\bar{f} \in L^\infty(0,T;L^2(\Omega)^3)$ and $h \in L^\infty(0,T;L^2(\Omega)^3)$, using the coercivity of A we get from (18) the existence of C_1 and C_2, positive constants such that

$$\int_\Omega |\sigma^{m+1}|^2 dx + \int_\Omega |v^{m+1}|^2 dx \leq C_1 + C_2\Delta t\sum_{n=0}^{m}(\int_\Omega |\sigma^n|^2 dx + \int_\Omega |\bar{v}^n|^2 dx) \tag{19}$$

Therefore we can deduce from a discret version of the Gronwall Lemma that

$$\int_\Omega |\sigma^{m+1}|^2 dx + \int_\Omega |\bar{v}^{m+1}|^2 dx \leq C$$

Then

$$\begin{cases} \sigma_N \text{ and } \sigma_N^t & \text{are bounded in } L^\infty(0,T;\mathbf{L}^2(\Omega)) \\ & \text{independently of } N \\ \bar{v}_N \text{ and } \bar{v}_N^t & \text{are bounded in } L^\infty(0,T;L^2(\Omega)^3) \\ & \text{independently of } N \end{cases} \tag{20}$$

A PRIORI ESTIMATES II

We take the addition between the (11) inequation written at $t = t_n$ with $\tau = \sigma^n$ as test function and written at $t = t_{n-1}$ with $\tau = \sigma^{n+1}$ as test function and we obtain

$$\int_\Omega A\left(\frac{\sigma^{n+1}-2\sigma^n+\sigma^{n-1}}{\Delta t}\right)(\sigma^{n+1}-\sigma^n)dx + \int_\Omega (\bar{v}^{n+1}-\bar{v}^n)\text{ div }(\sigma^{n+1}-\sigma^n)dx$$
$$\leq \int_\Omega (h^{n+1}-h^n)(\sigma^{n+1}-\sigma^n)dx \tag{21}$$

Now we can take the difference between the (12) equation written at $t = t_{n+1}$ and $t = t_n$, with $w = \bar{v}^{n+1} - \bar{v}^n$ as test function and we obtain

$$\int_\Omega \text{div }\sigma^n(\bar{v}^{n+1}-\bar{v}^n)dx + \int_\Omega (\bar{f}^{n+1}-\bar{f}^n)(\bar{v}^{n+1}-\bar{v}^n)dx =$$
$$= \int_\Omega \frac{\bar{v}^{n+1}-2\bar{v}^n+\bar{v}^{n-1}}{\Delta t}(\bar{v}^{n+1}-\bar{v}^n)dx \tag{22}$$

With an easy computation and using that A is positive defined we get the following inequalities

$$A(\sigma^{n+1}-2\sigma^n+\sigma^{n-1})(\sigma^{n+1}-\sigma^n) \geq \frac{1}{2}A(\sigma^{n+1}-\sigma^n)(\sigma^{n+1}-\sigma^n) -$$
$$-\frac{1}{2}A(\sigma^n-\sigma^{n-1})(\sigma^n-\sigma^{n-1}) \tag{23}$$

and

$$(\overline{v}^{n+1} - 2\overline{v}^{n} + \overline{v}^{n-1})(\overline{v}^{n+1} - \overline{v}^{n}) \geq \frac{1}{2}|\overline{v}^{n+1} - \overline{v}^{n}|^2 - -\frac{1}{2}|\overline{v}^{n} - \overline{v}^{n-1}|^2 \qquad (24)$$

Therefore, from (21) and (22) we obtain

$$\frac{1}{2}\int_\Omega A\left(\frac{\sigma^{n+1} - \sigma^{n}}{\Delta t}\right)\left(\frac{\sigma^{n+1} - \sigma^{n}}{\Delta t}\right)dx - \frac{1}{2}\int_\Omega A\left(\frac{\sigma^{n} - \sigma^{n-1}}{\Delta t}\right)\left(\frac{\sigma^{n} - \sigma^{n-1}}{\delta t}\right)dx +$$
$$+ \frac{1}{2}\int_\Omega \left|\frac{\overline{v}^{n+1} - \overline{v}^{n}}{\Delta t}\right|^2 - \frac{1}{2}\int_\Omega \left|\frac{\overline{v}^{n} - \overline{v}^{n-1}}{\Delta t}\right|^2 dx \leq \int_\Omega \left(\frac{h^{n+1} - h^{n}}{\Delta t}\right)(\sigma^{n+1} - \sigma^{n})dx$$
$$+ \int_\Omega \left(\frac{\overline{f}^{n+1} - \overline{f}^{n}}{\Delta t}\right)(\overline{v}^{n+1} - \overline{v}^{n})dx \qquad (25)$$

From the assumptions in Chapter 1, we know that $h \in W^{1,\infty}(0,T;L^2(\Omega)^3)$ and $\overline{f} \in W^{1,\infty}(0,T;L^2(\Omega)^3)$ then $\delta h = \dfrac{h^{n+1} - h^{n}}{\Delta t}$ and $\delta \overline{f} = \dfrac{\overline{f}^{n+1} - \overline{f}^{n}}{\Delta t}$ are bounded in $L^2(\Omega)^3$. Therefore after summation over n in (25) and using A positive defined we get

$$\int_\Omega \left|\frac{\sigma^{m+1} - \sigma^{m}}{\Delta t}\right|^2 dx + \int_\Omega \left|\frac{\overline{v}^{m+1} - \overline{v}^{m}}{\Delta t}\right|^2 dx \leq\leq C_1 + C_2 \sum_{n=0}^{m} \left[\int_\Omega \left|\frac{\sigma^{n+1} - \sigma^{n}}{\Delta t}\right|^2 dx +\right.$$
$$\left. + \int_\Omega \left|\frac{\overline{v}^{n+1} - \overline{v}^{n}}{\Delta t}\right|^2 dx\right] + C_3\left[\int_\Omega \left|\frac{\sigma^{1} - \sigma^{0}}{\Delta t}\right|^2 dx + \int_\Omega \left|\frac{\overline{v}^{1}}{\Delta t}\right|^2 dx\right] \qquad (26)$$

with C_1, C_2, C_3 positive constants.
We take the equations (11) and (12) at $t = t_0$, $\omega = \overline{v}^1$ and $\tau = \sigma^0$ as test functions and we obtain taking into account that $\overline{v}^0 = 0$

$$\int_\Omega A\left(\frac{\sigma^{1} - \sigma^{0}}{\Delta t}\right)\left(\frac{\sigma^{1} - \sigma^{0}}{\Delta t}\right)dx + \int_\Omega \left|\frac{\overline{v}^{1}}{\Delta t}\right|^2 dx \leq \int_\Omega h^1\left(\frac{\sigma^{1} - \sigma^{0}}{\Delta t}\right)dx$$
$$+ \int_\Omega \left(\frac{\overline{v}^{1} - \overline{v}^{0}}{\Delta t}\right)(\operatorname{div} \sigma^0 + \overline{f}^1)dx$$

Therefore using A positive defined, $h \in L^\infty(0,T;L^2(\Omega)^3)$ and $\overline{f} \in L^\infty(0,T;L^2(\Omega)^3)$ we have

$$\int_\Omega \left|\frac{\sigma^{1} - \sigma^{0}}{\Delta t}\right|^2 dx + \int_\Omega \left|\frac{\overline{v}^{1}}{\Delta t}\right|^2 dx \leq C_4 \qquad (27)$$

with C_4 a positive constant.
Using (27) we can apply the discret version of the Gronwall's lemma on (26) and we deduce that there exists a positive constant C_5 such that

$$\left\|\frac{\sigma^{m+1} - \sigma^{m}}{\Delta t}\right\|_{L^2(\Omega)} + \left\|\frac{\overline{v}^{m+1} - \overline{v}^{m}}{\Delta t}\right\|_{L^2(\Omega)^3} \leq C_5 \qquad (28)$$

with $m = 0, \ldots, N-1$.
From (12) and (28) it follows that

$$\|\operatorname{div} \sigma^m\|_{L^2(\Omega)^3} \leq C_5$$

Therefore

$$\begin{cases} \dot{\sigma}_N & \text{is a bounded sequence in } L^\infty(0,T;L^2(\Omega)) \\ \dot{\overline{v}}_N & \text{is a bounded sequence in } L^\infty(0,T;L^2(\Omega)^3) \\ \operatorname{div} \sigma_N \text{ and } \operatorname{div} \sigma_N^{c} & \text{are bounded sequences} \\ & \text{in } L^\infty(0,T;L^2(\Omega)^3) \end{cases} \qquad (29)$$

PASSING TO THE LIMIT $N \to +\infty$

From (20) and (29), there exist $\sigma \in L^\infty(0,T;Y) \cap W^{1,\infty}(0,T;\mathbf{L}^2(\Omega))$, $\sigma^e \in L^\infty(0,T;Y)$, $\bar{v} \in W^{1,\infty}(0,T;L^2(\Omega)^3)$ and $\bar{v}^e \in L^\infty(0,T;L^2(\Omega)^3)$ and subsequences of $(\sigma_N,\bar{v}_N)$, $(\sigma_N^d,\bar{v}_N^d)$ such that

$$\left\{\begin{array}{lll} \sigma_N \to \sigma & \text{in } L^\infty(0,T;\mathbf{L}^2(\Omega)) \text{ weak } * \\ \dot\sigma_N \to \dot\sigma & \text{in } L^\infty(0,T;\mathbf{L}^2(\Omega)) \text{ weak } * \\ \sigma_N^e \to \sigma^e & \text{in } L^\infty(0,T;\mathbf{L}^2(\Omega)) \text{ weak } * \\ \operatorname{div} \sigma_N \to \operatorname{div} \sigma & \text{in } L^\infty(0,T;L^2(\Omega)^3) \text{ weak } * \\ \operatorname{div} \sigma_N^e \to \operatorname{div} \sigma^e & \text{in } L^\infty(0,T;L^2(\Omega)^3) \text{ weak } * \\ \bar{v}_N \to \bar{v} & \text{in } L^\infty(0,T;L^2(\Omega)^3) \text{ weak } * \\ \bar{v}_N^e \to \bar{v}^e & \text{in } L^\infty(0,T;L^2(\Omega)^3) \text{ weak } * \\ \dot{\bar{v}}_N \to \dot{\bar{v}} & \text{in } L^\infty(0,T;L^2(\Omega)^3) \text{ weak } * \end{array}\right. \tag{30}$$

From the definitions of $\sigma_N, \sigma_N^e, \bar{v}_N, \bar{v}_N^e$, (20) and (28) it is easy to see that

$$\left\{\begin{array}{l} \lim_{N\to+\infty}(\sigma_N - \sigma_N^e) = 0 \quad \text{in } L^\infty(0,T;Y) \\ \lim_{N\to+\infty}(\bar{v}_N - \bar{v}_N^e) = 0 \quad \text{in } L^\infty(0,T;L^2(\Omega)^3) \end{array}\right. \tag{31}$$

Therefore we can deduce

$$\left\{\begin{array}{l} \sigma = \sigma^e \\ \bar{v} = \bar{v}^e \end{array}\right. \tag{32}$$

The application $T_1 : W^{1,2}(0,T;\mathbf{L}^2(\Omega)) \to \mathbf{L}^2(\Omega)$, $T_1(\tau) = \tau(0)$ is continuous and convex. Then by a lower semicontinuity argument

$$\|\sigma(0) - \sigma_0\| \leq \liminf_{N\to+\infty} \|\sigma_N(0) - \sigma_0\|_{\mathbf{L}^2(\Omega)} = 0$$

Therefore we deduce

$$\sigma(0) = \sigma_0 \tag{33}$$

with the same technique we obtain

$$\bar{v}(0) = 0 \tag{34}$$

It remains to show that $\sigma \in \mathbf{K}_F$

- The application $T_2 : L^2(0,T;\mathbf{L}^2(\Omega)) \to \mathbb{R}$, $T_2(\tau) = \|\tau - P_K\tau\|_{L^2(0,T;\mathbf{L}^2(\Omega))}$ is convex and continue, with K convex of plasticity. Then by a lower semicontinuity argument

$$\|\sigma - P_K\sigma\|_{L^2(0,T;\mathbf{L}^2(\Omega))} \leq \liminf_{N\to\infty} \|\sigma_N - P_K\sigma_N\|_{L^2(0,T;\mathbf{L}^2(\Omega))} = 0$$

Therefore

$$\sigma(x,t) \in K \text{ p.p.} \tag{35}$$

- For all $\omega \in C^\infty([0,T]\times\Omega,\mathbb{R}^3)$ with $\omega = 0$ if $x \in \partial_v\Omega$ we have from the Green formula

$$\int_0^T \int_\Omega \operatorname{div} \sigma_N.\omega \, dx\, dt = \int_0^T \int_{\partial_F\Omega} F^d(x)\omega \, dx\, dt - \int_0^T \int_\Omega \sigma_N.\varepsilon(\omega) dx\, dt$$

Passing to the limit we obtain

$$\sigma\eta = F^d \text{ p.p. in } \partial_F\Omega \tag{36}$$

In order to obtain the movement equation we remark the following

If

$$\chi \in W^{1,\infty}(0,T;L^2(\Omega)^m)$$

then

$$\chi_N^\epsilon \to \chi \text{ in } L^\infty(0,T;L^2(\Omega)^m) \tag{37}$$

with $m \geq 1$.

Movement Equation

By the definition of σ_N^ϵ, v_N^ϵ, σ_N and v_N we can deduce

$$\int_0^T \int_\Omega (\text{ div } \sigma_N^\epsilon + \overline{f}_N - \dot{v}_N).\omega_N^\epsilon \, dx \, dt = 0$$

with $\omega \in C_c^\infty(0,T;L^2(\Omega)^3)$

From (37) we have in particular

$$\begin{cases} \omega_N^\epsilon \to \omega & \text{in } L^2(0,T;L^2(\Omega)^3) \\ \overline{f}_N \to \overline{f} & \text{in } L^2(0,T;L^2(\Omega)^3) \end{cases} \tag{38}$$

Using (30) and (32) we can pass to the limit and by using a density argument we deduce

$$\text{div } \sigma + \overline{f} = \dot{v} \tag{38}$$

The Constitutive Law

We will see that it holds in two steps

FIRST STEP

We consider $\tau \in \mathbf{K}_F$ with the following assumptions

$$\begin{cases} \tau \in W^{1,\infty}(0,T;\mathbf{L}^2(\Omega)) \\ \text{div } \tau \in W^{1,\infty}(0,T;L^2(\Omega)^3) \end{cases} \tag{39}$$

From (11) we get

$$\sum_{n=0}^{m} \left[\int_\Omega A \frac{\sigma^{n+1} - \sigma^n}{\Delta t} (\sigma^{n+1} - \tau(t_{n+1}))dx + \int \overline{v}^{n+1} \text{ div } (\sigma^{n+1} - \tau(t_{n+1}))dx \right] \leq$$
$$\leq \sum_{n=0}^{m} \int_\Omega h^{n+1} (\sigma^{n+1} - \tau(t_{n+1}))dx \tag{40}$$

and we get from (12)

$$\sum_{n=0}^{m} \int_\Omega \text{ div } \sigma^{n+1} \overline{v}^{n+1} \, dx = \sum_{n=0}^{m} \int_\Omega \left(\frac{\overline{v}^{n+1} - \overline{v}^n}{\Delta t} - \overline{f}(t_{n+1}) \right) \overline{v}^{n+1} \, dx \tag{41}$$

Combining (40) and (41) we obtain

$$\sum_{n=0}^{m} \int_\Omega A \frac{\sigma^{n+1} - \sigma^n}{\Delta t} \sigma^{n+1} \, dx + \sum_{n=0}^{m} \int_\Omega \left(\frac{\overline{v}^{n+1} - \overline{v}^n}{\Delta t} \right) \overline{v}^{n+1} \, dx \leq$$
$$\leq \sum_{n=0}^{m} \left[\int_\Omega A \frac{\sigma^{n+1} - \sigma^n}{\Delta t} \tau(t_{n+1}) \, dx + \int_\Omega \overline{v}^{n+1} (\text{ div } \tau(t_{n+1}) + \overline{f}(t_{n+1}))dx \right] +$$
$$+ \sum_{n=0}^{m} \int_\Omega h^{n+1} (\sigma^{n+1} - \tau(t_{n+1}))dx \tag{42}$$

From (16), (17) and the definitions of $\sigma_N^\epsilon, \tau_N^\epsilon, \overline{v}_N$ and $\overline{v}_N^\epsilon$ we get with $t = n\Delta t$

$$\frac{1}{2}\int_\Omega A\sigma_N(t_n)\sigma_N(t_n)dx + \frac{1}{2}\int_\Omega |\overline{v}_N(t_n)|^2 dx - \frac{1}{2}\int_\Omega A\sigma_0\sigma_0\, dx \leq$$
$$\leq \int_0^t \int_\Omega A\dot{\sigma}_N.\tau_N^\epsilon\, dx\, ds + \int_0^t \int_\Omega \overline{v}_N^\epsilon.(\text{ div } \tau_N^\epsilon + \overline{f}_N^\epsilon)dx\, ds + \int_0^t \int_\Omega h^\epsilon(\sigma_N^\epsilon - \tau_N^\epsilon)dx\, ds \tag{43}$$

We shall use the following

- The application

$T_3 : W^{1,2}([0,T], L^2(\Omega)) \to L^1(0,T)$

$$T_3(\xi) = \frac{1}{2}\int_\Omega A\xi(t)\xi(t)dx \text{ is continuous and convex}$$

Therefore

$$\liminf_{N\to+\infty} \frac{1}{2}\int_\Omega A\sigma_N(t)\sigma_N(t) \geq \frac{1}{2}\int_\Omega A\sigma(t)\sigma(t)\, dx \tag{44}$$

and with the same technique we get

$$\liminf_{N\to+\infty} \frac{1}{2}\int_\Omega |\overline{v}_N(t)|^2 dx \geq \frac{1}{2}\int_\Omega |\overline{v}_N(x,t)|^2 dx \tag{45}$$

- From (37) and (30) we get in particular

$$\begin{cases} \dot{\sigma}_N \to \dot{\sigma} & \text{in } L^2(0,t;\mathbf{L}^2(\Omega)) \text{ weak} \\ \tau_N^\epsilon \to \tau & \text{in } L^2(0,t;\mathbf{L}^2(\Omega)) \\ \overline{v}_N^\epsilon \to \overline{v} & \text{in } L^2(0,t;L^2(\Omega)^3) \text{ weak} \\ \text{div } \tau_N^\epsilon \to \text{ div } \tau & \text{in } L^2(0,t;L^2(\Omega)^3) \\ \overline{f}_N^\epsilon \to \overline{f} & \text{in } L^2(0,t;L^2(\Omega)^3) \\ h_N^\epsilon \to h & \text{in } L^2(0,t;L^2(\Omega)^3) \\ \sigma_N^\epsilon \to \sigma & \text{in } L^2(0,t;L^2(\Omega)^3) \text{ weak} \end{cases} \tag{46}$$

Using (44), (45) and (46) we can pass to the limit $N \to +\infty$ and we obtain

$$\frac{1}{2}\int_\Omega A\sigma(t)\sigma(t)dx - \frac{1}{2}\int_\Omega A\sigma_0\sigma_0\, dx + \frac{1}{2}\int_\Omega |\overline{v}(t)|^2 dx \leq \int_0^t \int_\Omega A\dot{\sigma}\tau\, dx\, ds +$$
$$+ \int_0^t \int_\Omega h(\sigma - \tau)dx\, ds + \int_0^t \int_\Omega (\overline{f} + \text{ div } \tau)\overline{v}\, dx\, ds \tag{47}$$

Combining with the movement equation we conclude

$$\int_0^t \int_\Omega A\dot{\sigma}(\sigma - \tau)dx\, ds + \int_0^t \int_\Omega \overline{v}\text{ div } (\sigma - \tau)dx\, ds \leq \int_0^t \int_\Omega h(\sigma - \tau)dx\, ds \tag{48}$$

SECOND STEP

We consider $\tau \in L^\infty(0,T;Y)$ with $\tau(t) \in \mathbf{K}_F$

We define

$$\begin{cases} \tau_n(x,t) = \int_0^t \rho_n(s)\tau(x,t-s)ds \\ \text{with } \rho_n \text{ such that} \\ \rho_n \in C_0^\infty([0,t]) \\ \int_0^t \rho_n(s)ds = 1, \quad \rho_n \geq 0 \end{cases} \tag{49}$$

We will show that

$$\tau_n \to \tau \quad \text{in } L^2(0,t;\mathbf{L}^2(\Omega)) \tag{50}$$

with

$$\tau_n(t) \in \mathbf{K}_F$$

We shall use some results

- We define (see for instance [2])

$$J(y) = \inf \{\alpha > 0 / \frac{y}{\alpha} \in K\}$$

with $y \in \mathbf{R}^9_s$ and K convex of plasticity

J is a convex function and from the definition of J, $K = \{y \in \mathbf{R}^9_s / J(y) \leq 1\}$ because K is a close convex.

In order to show $\tau_n(x,t) \in K$ p.p. we write

$$J(\tau_n(x,t)) = J\left(\int_0^t \rho_n(s)\tau(x,t-s)ds\right)$$

Using J convex we get from (49)

$$J(\tau_n(x,t)) \leq \int_0^t \rho_n(s) J(\tau(x,t-s))ds$$

Therefore, taking into account that $\tau(x,t) \in K$ p.p. we obtain

$$J(\tau_n(x,t) \leq 1 \text{ p.p.}$$

From the definition we have directly

$$\tau_n.\eta = F^d$$

Then $\tau_n \in \mathbf{K}_F$.

In order to show the convergence of τ_n to τ, we define

$$G_n(x) = \|\tau_n(x) - \tau(x)\|_{L^2(0,T;\mathbf{R}^9_s)} \tag{51}$$

It is easy to see that

$$\begin{cases} G_n^2(x) \to 0 \text{ p.p. in } \Omega \\ G_n^2(x) \leq C \text{ p.p. in } \Omega \end{cases} \tag{52}$$

Therefore we can apply the Lebesgue dominated convergence theorem to obtain

$$\lim_{n\to+\infty} \int_\Omega G_n^2(x)dx = 0 \tag{53}$$

Then we have $\tau_n \to \tau$ in $L^2(0,T;\mathbf{L}^2(\Omega))$ and we have proved (50).

If we write (48) with η as test function, passing to the limit $n \to +\infty$ we get

$$\int_0^t \int_\Omega A\dot{\sigma}(\sigma-\tau)dx\,ds + \int_0^t \int_\Omega \bar{v}\,\mathrm{div}\,(\sigma-\tau)dx\,ds \leq \int_0^t \int_\Omega h(\sigma-\tau)dx\,ds \tag{54}$$

for all $\tau \in \mathbf{K}_F$

REMARK

In fact, we have proved (54) that is a weak version of (6). It is easy to see that we have a unique solution $(\sigma, \bar{v})$ such that (54), (7) and (8) are satisfied.

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