ERROR ESTIMATION AND MESH ADAPTIVITY TECHNIQUES FOR STRUCTURAL AND FLUID MECHANICS PROBLEMS

Eugenio Oñate

Centro Internacional de Métodos Numéricos en Ingeniería
Universidad Politécnica de Cataluña
08034 Barcelona, España

SUMMARY

In this paper some adaptive mesh refinement (AMR) strategies for finite element analysis of structural and fluid flow problems are discussed. For structural problems two mesh optimality criteria based on the equal distribution of: (a) the *global error*, and (b) the *specific error* over the elements are studied. It has been found that the correct evaluation of the rate of convergence of the different error norms involved in the AMR procedure is essential to avoid oscillations in the refinement process. Extensions of these optimality criteria to fluid flow problems are presented The behaviour of the different AMR strategies proposed is compared in the analysis of some structural and fluid flow problems.

INTRODUCTION

The evaluation of discretization errors and the design of suitable meshes are nowadays two of the challenging issues in the analysis of structures and fluid flow problems using the finite element method (FEM).

The topic of error estimation and mesh adaptivity in the FEM is by no means new. Zienkiewicz and Zhu [1-5] have introduced a successfull adaptive mesh refinement (AMR) strategy for structural analysis using a simple estimate based on the difference of the discontinuous finite element stress (or strain) field with an "improved" smooth solution. This AMR strategy has been applied to the analysis of fluid flow problems [24]. Different authors [6–9], [19] have used similar AMR procedures for plate bending and shell analysis. For a comprehensive review of the topic of error estimation and adaptive mesh refinement see the reference list of Chapter 14 of [26].

In this paper we present an overview of different AMR strategies based on the Zienkiewicz and Zhu error estimator [1-6] for structural and fluid flow analysis. We will show how the AMR algorithm based on equal distribution of the error over all the elements requires a careful identification of the rate of convergence of the different error terms involved in the design of the new element size to avoid oscillations in the refinement process.

An interesting alternative AMR strategy can be based on the equal-distribution of the "specific error", i.e. the *error per unit area* (or volume) in all the elements in the mesh. We will see that this strategy allows to concentrate more and smaller elements in zones where stress concentrations (or discontinuities) occur, as it should be expected from the engineering point of view.

The layout of the paper is the following: In next section the basis of the error estimator and the AMR procedures proposed for structural analysis are detailed. Extension of the same concepts to deal with fluid flow problems are described next. Finally, two examples of application to the analysis of a plate and two problems of hypersonic compresible flow are presented showing the possibilities of the AMR strategies studied.

BASIC CONCEPTS OF ERROR ESTIMATION

In dealing with adaptive mesh refinement the following two concepts should be clearly defined:

(a) *Error estimator*. Since the "exact" solution is unknown, a method to approximately evaluate the error of the finite element solution should be defined.

(b) *Approximate correct solution*. A finite element solution is accepted as "correct" is the estimated error satisfies some prescribed *global* and *local* conditions.

Both concepts (a) and (b) are explained further in next sections.

Error estimator

One of the most popular error estimators for structural analysis problems is based on the error energy norm expressed as

$$\|e\|^2 = \int_\Omega [\sigma - \hat{\sigma}]^T \mathbf{D}^{-1} [\sigma - \hat{\sigma}] d\Omega \tag{1}$$

where σ are the exact stresses, $\hat{\sigma}$ are the stress values obtained from the finite element solution and D is the constitutive matrix [26].

Since the exact stresses are unknown they are approximated by

$$\sigma \simeq \sigma^* \tag{2}$$

where σ^* can be obtained by simple nodal averaring, least squares local and global smoothing, or other adequate projection methods [1], [3]. A simple approach is to use a global nodal smoothing with a lumped "mass" matrix giving the nodal smoothed values, σ_i^*, for each resultant stress component $\hat{\sigma}_i$ as

$$\sigma_i^* = \mathbf{M}_D^{-1} \int_\Omega \mathbf{N}_\sigma^T \hat{\sigma}_i d\Omega \tag{3}$$

where $\mathbf{M}_{D_{ii}} = \int_\Omega N_i d\Omega$ and $\mathbf{N}_\sigma$ are the chosen stress interpolating functions. Eq. (3) yields an accurate smoothed stress field for low order elements.

The strain energy of the exact solution is estimated as

$$\|U\|^2 \simeq \int_\Omega \left\{ \sigma^{*T} \mathbf{D}^{-1} \sigma^* \right\} d\Omega \tag{4}$$

Both $\|e\|$ and $\|U\|$ can be evaluated at the element level so that

$$\|e\|^2 = \sum_{i=1}^{n} \|e\|_i^2 \quad , \quad \|U\|^2 = \sum_{i=1}^{n} \|U\|_i^2 \tag{5}$$

where n is the total number of elements in the mesh.

Definition of correct solution

It is usually accepted that a solution is correct if the two following conditions are satisfied:

(a) The global error is less than a percentage value of the total strain energy, i.e.

$$\|e\| \leq \eta \|U\| \tag{6}$$

where η is the user's specified value of the permissible relative global error.

Eq.(6) allows to define a global error parameter, ξ_g, as

$$\xi_g = \frac{\|e\|}{\eta \|U\|} \tag{7}$$

Clearly the values $\xi_g \leq 1$ denote satisfaction of the global error criterium, whereas $\xi_g > 1$ indicates that further refinement is necessary.

(b) The distribution of the elements in the mesh satisfies a "mesh optimality criterium". This can be expressed as

$$\|e\|_i = \|e\|_{r_i} \tag{8}$$

where $\|e\|_i$ is the actual error norm in each element i and $\|e\|_{r_i}$ is the " required" error norm in the element, defined accordingly to the mesh optimality criterium chosen.

From eq.(8) we can define a local error parameter $\bar{\xi}_i$, for each element i as

$$\bar{\xi}_i = \frac{\|e\|_i}{\|e\|_{r_i}} \tag{9}$$

Note that a value of $\bar{\xi}_i = 1$ defines an "optimal" element size, whereas $\bar{\xi}_i > 1$ and $\bar{\xi}_i < 1$ indicate that the size of element i needs refinement and de-refinement, respectively.

We can define now a single *element refinement parameter*, combining the satisfaction of the above global and local conditions as

$$\xi_i = \bar{\xi}_i \xi_g = \frac{\|e\| \ \|e\|_i}{\eta \|U\| \ \|e\|_{r_i}} \tag{10}$$

Use of this parameter is discussed in next section.

The definition of the required error in each element, $\|e\|_{r_i}$, is a key issue and it strongly affects the distribution of the element sizes. This definition can be based on different mesh optimality criteria and some of these are discussed in next section, together with the general AMR strategy.

MESH OPTIMALITY CRITERIA AND AMR PROCEDURES

Mesh optimality criterium based on the equal-distribution of the global error

A popular mesh optimality criterium for structural analysis is based on the so called equal-distribution of the error , i.e. a mesh is defined as optimal if the global error is equally distributed over the elements. On the basis of this assumption we can define the required error for each element as the ratio between the global error and the total number of elements in the mesh, i.e.

$$\|e\|_{r_i} = \frac{\|e\|}{\sqrt{n}} \tag{11}$$

Combining (9) and (11) yields the expression of the local error parameter $\bar{\xi}_i$ as

$$\bar{\xi}_i = \frac{\|e\|_i}{\|e\|n^{-1/2}} \tag{12}$$

The element refinement parameter (see eq.(10)) is now obtained as

$$\xi_i = \bar{\xi}_i \xi_g = \frac{\|e\|_i}{\eta \|U\| n^{-1/2}} \tag{13}$$

The parameter ξ_i can be readily interpreted as the ratio between the element error and the distributed value of the permissible error over the mesh. However, the form $\xi_i = \bar{\xi}_i \xi_g$ allows to derive the correct AMR strategy. Thus by noting that the convergence rates of the element and global error norms are [18]

$$\begin{aligned} \|e\|_i &\simeq O(h^m)\Omega_i^{1/2} \simeq O(h_i^{m+\frac{d}{2}}) \\ \|e\| &\simeq O(h^m) \end{aligned} \tag{14}$$

where m is the degree of the shape function polynomials ($m = 1$ for linear elements, $m = 2$ for quadratic elements, etc.), and d is the number of dimensions of the problem ($d = 1, 2, 3$ for 1D , 2D and 3D problems, repectively) we can deduce that the new element size $\bar{h}_i$ can be obtained in terms of the existing size h_i using the expression

$$\boxed{\bar{h}_i = \frac{h_i}{\xi}} \tag{15}$$

with

$$\boxed{\xi = \bar{\xi}_i^{\frac{2}{2m+d}} \xi_g^{1/m}} \tag{16}$$

Note that eq.(16) differs from the usual form [1–9], [11]

$$\xi = (\xi_i)^{1/m} = (\bar{\xi}_i \cdot \xi_g)^{1/m} \tag{17}$$

The expression of ξ given by (16) takes into account the different convergence rates of the element and global error norms. The authors have found that the use of (18) leads to a non-consistent mesh refinement which shows by an oscillating re and de-refinement of some mesh zones. These problems are overcome if expression (16) is used (see first example).

Some authors [9], [11] get round this problem by introducing a relaxation factor c such that $\bar{\xi}_i = c\bar{\xi}_i\xi_g$, or by defining and "ad hoc" value of exponent m in (17) [1].

A mesh optimality criterium based on the equal-distribution of the specific error

A clear alternative to the criterium of equal distribution of the error over all the elements is to assume that a mesh is optimal if the error per unit area (or volume) is the same over the whole mesh. It is clear then that

$$\frac{\|e\|_i}{\Omega_i^{1/2}} = \frac{\|e\|}{\Omega^{1/2}} = \alpha \tag{18}$$

where α is the required specific error tolerance ($\alpha \leq \frac{\eta\|U\|}{\Omega^{1/2}}$). Obviously in (18) Ω_i and Ω denote the element and total area (or volume) respectively.

The element error parameter $\bar{\xi}_i$ is defined now as the ratio between the element and global specific errors, i.e.

$$\bar{\xi}_i = \frac{\|e\|_i}{\Omega_i^{1/2}}\left[\frac{\|e\|}{\Omega^{1/2}}\right]^{-1} = \frac{\|e\|_i}{\|e\|}\left(\frac{\Omega}{\Omega_i}\right)^{1/2} \tag{19}$$

The element refinement parameter is obtained from (9), (10) and (19) as

$$\xi_i = \bar{\xi}_i\xi_g = \frac{\|e\|_i}{\eta\|U\|}\left(\frac{\Omega}{\Omega_i}\right)^{1/2} \tag{20}$$

The way we have defined the element error strongly affects its convergence rate, given now by

$$\frac{\|e\|_i}{\Omega_i^{1/2}} \leq O(h_i^m) \tag{21}$$

The new element size is therefore obtained from (15) with the expression of ξ given by

$$\xi = \left(\bar{\xi}_i\xi_g\right)^{1/m} = (\xi_i)^{1/m} \tag{22}$$

Note that this expression coincides with (17) and it has been (apparently) wrongly used by different authors in the context of the optimality criterium based on the equi-distribution of the global error, studied in previous section [1–9], [11]. The results obtained here clarify the correct form to be used for each mesh optimality criterium chosen.

To our knowledge the criterium of equal-distribution of the specific error was originaly introduced by Bugeda [20] and it has successfully been used in the context of optimum structural design problems by Bugeda and Oliver [21].

ERROR ESTIMATOR AND AMR STRATEGIES FOR FLUID FLOW PROBLEMS

For creeping incompressible flow problems the following error norm based on the analogy of Stokes flow with elasticity has been successfully used [24]

$$\|e\|^2 = \int_\Omega ([\dot{\epsilon} - \hat{\dot{\epsilon}}]^T[\sigma' - \hat{\sigma}'] - [p - \hat{p}][\dot{\epsilon}_{ii} - \hat{\dot{\epsilon}}_{ii}])d\Omega \tag{23}$$

where $\dot{\epsilon}$, σ' and p are exact values of stresses rates, deviatoric strains and pressure and $\hat{(\cdot)}$ stands for computed values.

Since $\dot{\epsilon}$, σ' and p are not known, an approximation of higher order accuracy than that given by the finite element method is used as

$$\|e\|^2 = \int_\Omega ([\dot{\epsilon}^* - \hat{\dot{\epsilon}}]^T[\sigma'^* - \hat{\sigma}'] - [p^* - \hat{p}][\dot{\epsilon}^*_{ii} - \hat{\dot{\epsilon}}_{ii}])d\Omega \tag{24}$$

Values of $\dot{\epsilon}^*$ and σ'^* can be obtained by projecting the discontinuos numerical solution into a continuos basis. The simplest option is to use nodal averaging of discontinuos element values. However, more sophisticated local and global smoothing techniques can be used as discussed in a previous section (see eq.(3)). For values of p^* identical procedures can be used. However if p is continuous it is usual to neglected its contribution in the error norm [24].

The porcentage error in the mesh is now defined as

$$\eta = \frac{\|e\|}{\|U\|} \simeq \frac{\|e\|}{\|\hat{U}\|} \times 100 \tag{25}$$

$$\text{with} \qquad \|\hat{U}\|^2 = \int_\Omega (\hat{\dot{\epsilon}}^T \hat{\sigma}' - \hat{p}\hat{\dot{\epsilon}}_{ii})d\Omega \tag{26}$$

The AMR strategy can now be based on the same global and local error criteria studied in a previous section for structural problems and it involves the following steps:

1) Compute global and local error parameters ξ_g and $\bar{\xi}_i$ (viz. eqs.(7) and (9)). Computation of $\bar{\xi}_g$ can be based on the criteria of equal distribution of the global error (eq.(12)) or of the specific error (eq.(19)).

2) Compute new elements sises by eq.(15) with the element refinement parameter ξ as given by eqs.(16) or (22), accordingly to mesh optimality criterium chosen.

For high speed flow situations the same AMR strategy can be used. However the flow-elasticity analogy does not hold and (23) will yield only an approximation of the actual error measure, howeveer useful for practical applications.

A common alternative for one-dimensional compressible flow problems analysed with linear elements, is to assume the error for each problem variable v as given by

$$\| e_v \|_i^2 \simeq C(h_i)^2 \left| \frac{d^2 v}{dx^2} \right|_i \tag{27}$$

where $|\cdot|_i$ means an average value over the ith element. The condition of uniform distribution of the error implies now simply

$$(h_i)^2 \left| \frac{d^2 v}{dx^2} \right|_i = k \text{ (constant)} \tag{28}$$

Above concepts can be extended to 2D/3D flow situations to derive the following AMR strategy:

1) At each element center the following matrix is computed

$$M_{ij} = \frac{\partial^2 v}{\partial x_i \partial x_j} \qquad i,j = \begin{matrix} 1,2 \text{ for 2D problems} \\ 1,2,3 \text{ for 3D problems} \end{matrix} \tag{29}$$

where v is the variable which error is to be computed. Tipically $v = \rho$ (density) or $v = M$ (mach number) are chosen.

2) The eigenvalues of **M** are computed (i.e. λ^1, λ^2 for 2D problems). From eq.(28) it can be written

$$[h_i^1]^2 \lambda_i^1 = [h_i^2]^2 \lambda_i^2 = (h^{min})^2 \lambda^{max} \tag{29}$$

where h^{min} and λ^{max} are the specified value of the minimum element size and the maximum eigenvalue computed in the mesh, respectively.

3) Eq.(29) yields the new element sizes as

$$h_i^j = h^{min} \sqrt{\frac{\lambda^{max}}{\lambda_i^j}} \qquad j = 1,2 \text{ for 2D} \tag{30}$$

Usually $h_i^1 = h_i^2$ is taken, thus implying equal size elements. However the possibility of stretching the elements (i.e. $h_i^1 \neq h_i^2$) has also been successfully exploited in practice [22,23].

The mesh is redefined using the new element sizes given by (30). This can imply either refinement or enlargement of some element zones. The definition of the new mesh can be based on the enrichment of the previous one, subdividing or eliminating elements, or in the complete regeneration of a new mesh [22,23,25]. For the second option an efficient mesh generator for triangular or quadrilateral elements of different orders has to be used. This mesh generator should allow to combine structured with unstructured meshes in the same domain. An example of application is the modelling of the boundary layer region with an structured mesh whereas an unstructured mesh can be used for the rest of the flow domain. The unstructured mesh generator used in all the examples presented in next section is based on the advancing front technique [20,22,23].

It is interesting to note that this last AMR strategy is in fact equivalent to that based on the equal distribution of specific error studied in a previous section. This explains its better ability to capture the discontinuities induced by shocks in practical hypersonic flow computations [18], [22], [23], [25].

EXAMPLES

Analysis of a plate bending problem

The first example is the analysis of a simply supported square plate under uniformly distributed loading. Figure 1 shows the geometry of the plate, material properties and the initial mesh of 68 six nodes triangular Reissner-Mindlin plate elements based on an assumed shear strain formulation [14–16]. The plate edges are *soft* simply supported ($w = 0$) to ensure the development of a boundary layer due to the zero values of the twist moments along the supported sides. The values of m and d in eq.(16) are $m = 1$ and $d = 2$ in this case.

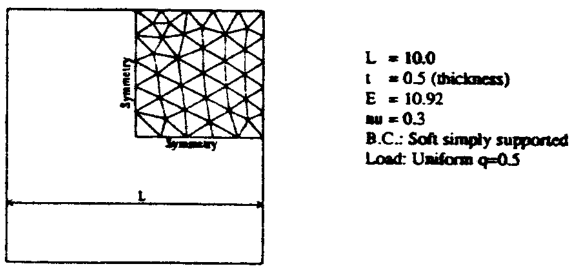

Figure 1. Symmetric quadrant of an uniformly loaded plate with 'soft' simple supports. Initial mesh of 68 triangular plate elements [14,15].

Figure 2 shows the sequency of refined meshes obtained with the three AMR strategies studied in the first part of the paper. A value of the permissible global error η =5% has been chosen in this case. First column (strategy A) shows the results obtained using the criterium of equal-distribution of global error over all the elements and the (wrong) value of ξ defining the new element sizes, as given by eq.(17). Note the oscillations in the AMR process clearly shown by the alternative re and de-refinements of the same mesh zones.

Results labelled as strategy B in Figure 2 have been obtained with the same mesh optimality criterium, but using now the correct expression for ξ as given by eq.(16). Note that the refinement oscillations dissapear and the AMR process converges in a consistent manner.

Finally results for strategy C have been obtained with the mesh optimality criterium based on the equal distribution of the specific error, with the element size parameter ξ as given by eq.(22). It can be clearly seen that: (a) The AMR process converges without oscillations, and (b) This AMR strategy concentrates more and smaller elements in the vicinity of the supported edge (where the error is greater due to the boundary layer effect), whereas in the center of the plate bigger elements

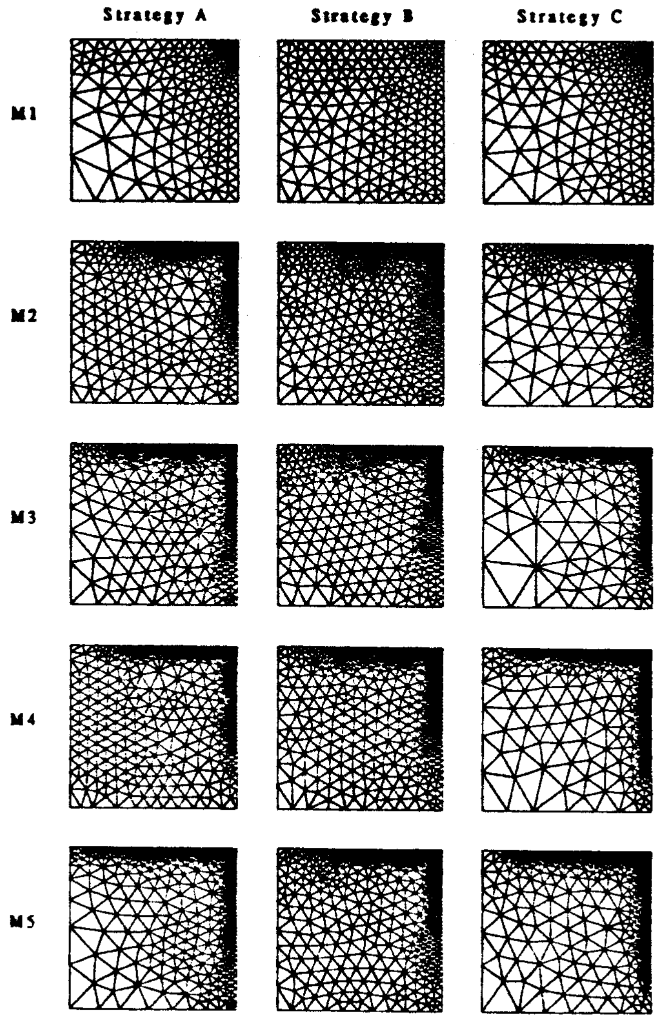

Figure 2. Symmetric quadrant of an uniformly loaded plate with "soft" simple supports. Sequence of meshes obtained with AMR strategies based on: (A) Equal distribution of global error and inconsistent definition of parameter ξ (via eq.(17)); (B) Idem with ξ consistently defined by eq.(16) and (C) Equal distribution of specific error.

than in previous cases are allowed. The prize to be paid is the increase in the total number of elements with respect to strategies A and B for the same global accuracy.

A summary of the main results obtained for this example is given in Table I. Columns 1, 2 and 3 show the number of elements, the total strain energy and the global error parameter ξ_g for each mesh. Columns 3, 4 and 5, 6 show the average value of the local error parameter $(\bar{\xi}_i^2)_a$ and its mean deviation $(\bar{\xi}_i^2)_{\sigma^2}$ over each mesh for the two optimality criteria studied, respectively. From the numbers shown in the table we deduce:

(a) Strategy A (equal distribution of global error and non consistent definition of ξ) converges to the global permisible error chosen. On the order hand, the remeshing inconsistencies previously mentioned yield oscillations in the mean deviation of the local error parameter (column 5).

(b) Strategy B (Equal distribution of global error and the right definition of ξ) also converges to the global permissible error and it shows a consistent distribution of the local error (see column 5). Note a small oscillation in the mean deviation of $\bar{\xi}_i$ for the final remeshing stages once the "optimal mesh" has been reached.

(c) Strategy C (Equal distribution of specific error) converges to the global error showing no oscillations in the distribution of element sizes. However, note that the number of elements involved in the final meshes is much bigger than that obtained with strategies A and B. This is due to the finer discretization due to the zones with high stress gradients.

Also note that by comparing columns 5 and 7 some conclussions between the different "philosophies" of the AMR strategies based on the equal distribution of the *global* and the *specific* errors can be drawn. For instance, we can see that the simultaneous satisfaction of both optimality criteria is not possible, as expected. Further instructive conclusions can be extracted from these results and they will be reported in a separate publication [18].

Mach 5 inviscid flow over a compression corner

Figure 3 shows the geometry of the problem and the initial mesh of 250 three node triangular elements with equal interpolation for all variables. The analysis has been performed solving the compressible Euler equations using a Taylor-Galerkin approach [21], [23], [25] with the AMR strategy presented in last part of previous section. Figure 3 also shows the sequence of adaptive remeshings and the results for the pressure contours obtained with each mesh. The improvement in the definition of the shock is obvious.

Hypersonic inviscid flow over a double ellipse

The final example corresponds to the laminar inviscid non reactive flow past a double ellipse for $M_\infty = 8.5$ and $\alpha = 30°$ [25]. Again numerical results have been obtained by solving the compressible Euler equations with a Taylor-Galerkin approach and 3 node equal interpolation triangles. Figure 4 shows the finite element meshes obtained after three consecutive remeshings using the same AMR strategy as in previous example. Figure 4 also shows the Mach number and pressure contours, and the plots of the pressure coefficient and the Mach number variation along the stagnation streamline for the finer mesh are.

CONCLUDING REMARKS

In this paper two different mesh optimality criteria based on the equal distribution of the *global* and *specific* error in a finite element mesh have been studied in the context of structural and fluid problems. We have seen that the correct evaluation of the rate of convergence of the different error norms involved in the AMR strategy is essential to avoid oscillations in the refinement process. Also, the mesh optimality criteria based on *global* and *specific* error distribution are conceptually very different. Thus, whereas the former leads to meshes with smaller number of elements, the second captures better the effect of high gradients. Further research should allow to balance the possibilites of these two criteria for use in practical structural and fluid flow applications.

	n° elements	$\|U\|$	ξ_g	Equal distribution of global error		Equal distribution of specific error	
				$(\xi_i^e)_a$	$(\xi_i^e)_{\sigma^2}$	$(\xi_i^e)_a$	$(\xi_i^e)_{\sigma^2}$
	STRATEGY A						
M0	68	28.632598	2.7074	1.000	3.272	1.064	3.740
M1	358	29.454666	1.5824	1.000	3.018	2.176	18.052
M2	820	29.803484	0.9516	1.000	1.552	3.492	32.316
M3	939	29.759661	0.9216	1.000	1.430	3.559	28.100
M4	966	29.861299	0.8514	1.000	1.166	4.023	37.199
M5	863	29.697644	0.9680	1.000	2.425	3.513	22.261
M6	1040	29.884956	0.8550	1.000	1.554	3.653	38.984
	STRATEGY B						
M0	68	28.632598	2.7074	1.000	3.272	1.064	3.740
M1	415	29.635824	1.5622	1.000	4.304	1.773	18.635
M2	923	29.839586	0.9878	1.000	2.672	2.762	30.087
M3	1007	29.836037	0.9274	1.000	1.489	3.163	29.755
M4	1038	29.847635	0.8818	1.000	1.262	3.400	33.205
M5	1024	29.843843	0.8702	1.000	1.319	3.490	35.147
M6	981	29.834833	0.8532	1.000	1.390	3.783	39.833
	STRATEGY C						
M0	68	28.632598	2.7074	1.000	3.272	1.064	3.740
M1	369	29.485505	1.5276	1.000	2.752	2.277	19.408
M2	1617	29.662097	0.8594	1.000	7.674	2.137	7.863
M3	3079	29.486021	0.9380	1.000	43.741	1.037	0.662
M4	3941	29.633166	0.9258	1.000	45.376	0.877	0.310
M5	4243	29.740783	0.6772	1.000	24.215	1.519	0.829

TABLE I. Symmetric quadrants of an uniformly loaded plate with "soft" simple supports. Some statistical results of the AMR processes.

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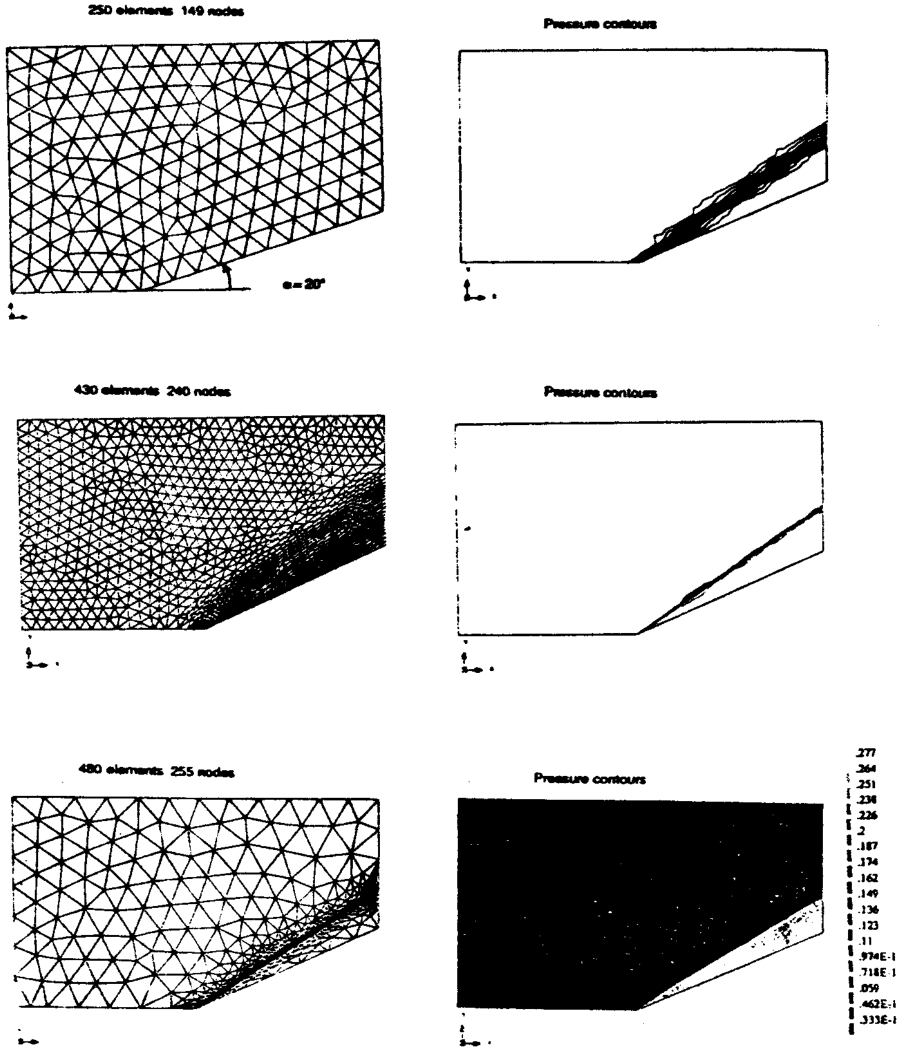

Figure 3. Mach 5 flow over a compression corner. Sequence of refined meshes of linear triangular elements and pressure contours.

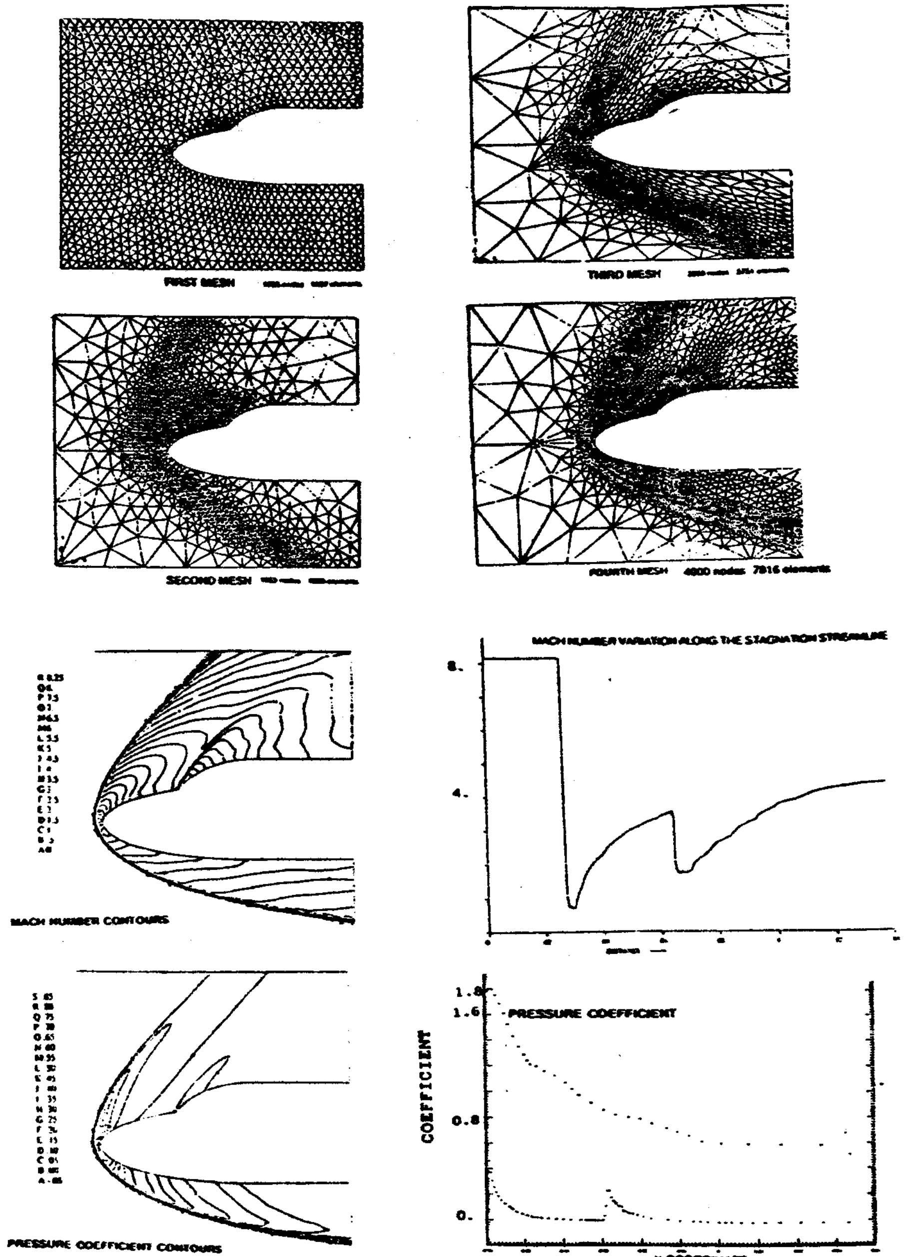

Figure 4 Hipersonic flow ($M_\infty = 8.5$, $\alpha = 30°$) over a double ellipse. Sequence of refined meshes. Mach number and pressure contours and plots of pressure coefficient and Mach number variation for the finer mesh.

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