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ADAPTIVITY FOR THE CONTROL VOLUME FINITE ELEMENT METHOD IN CONVECTION-DIFFUSION PROBLEMS

C. Padra† and **A. Larreteguy**‡

† División Mecánica Computacional, DIA
‡ División Termohidráulica, DIN
Centro Atómico Bariloche, CNEA
8400 San Carlos de Bariloche - Argentina

Abstract

In Computational Fluid Dynamics it is usual to find the problem of increasing the accuracy of a solution without adding unnecessary degrees of freedom. It is therefore necessary to update the mesh so as to ensure that it becomes fine enough in the critical region while remaining reasonably coarse in the rest of the domain. Local a posteriori error estimators are the adequate tool for identifying automatically these critical regions. They should use only given data and the numerical solution itself.

In this work the Control Volume Finite Element Method (CVFEM) for the Convection-Diffusion equation is considered. This is a nonconforming method in the sense that the interpolant space for the solution is not a subset of H^1. Despite of this fact, many years of numerical experiences have established the excellent behaviour of this method in non-selfadjoint problems.

In the conforming case several approaches have been introduced for selfadjoint problems by using the residual equations. In order to extend these techniques to the case we are dealing with, we have considered the treatment of the consistency terms arising in the error equation and the convective term wich is the no-selfadjoint part of the problem. Although some a posteriori error estimators for this problem have already been presented in the literature, most of them lack rigorous mathematical proof.

We present an error estimator that is a global upper bound of the true error under some hypotheses [11]. It has been included in a CVFEM code of our own. This code has been coupled together with an automatic mesh generator in order to obtain an adaptive loop. Evidence of the adequate behaviour of the adaptive procedure is given through numerical experimentation in well-known benchmark problems.

1 Statement of the problem

Consider a polygonal bounded region Ω with boundary $\delta\Omega$ in R^2. The problem to be solved can be put in the following form:

Find u that satisfies

$$\begin{cases} -\nabla \cdot (k\nabla u - \vec{b}u) & = & f in\ \Omega & \\ u & = & g_1 & \text{in } \Gamma_D \\ k\frac{\partial u}{\partial n} & = & g_2 & \text{in } \Gamma_N \end{cases} \tag{1}$$

We will assume that: a) the difusion coefficient k is bounded above and below by positive constants; and b) the convecting field $\vec{b}$ is incompressible ($\nabla \cdot \vec{b} = 0$). It is also assumed that:

- $\delta\Omega = \Gamma_D \cup \Gamma_N$
- $\Gamma_D \cap \Gamma_N = \emptyset$
- $\Gamma_N \subset \left\{x \in \delta\Omega \ : \ \vec{b} \cdot \vec{n} \geq 0\right\}$ $\quad \vec{n}$ outer normal

We will denote $L^2(R)$ and $H^1(R)$ the usual Sobolev spaces, equipped with the norms:

$$\| u \|_{L^2} = \left(\int_R u^2\right)^{1/2}$$

$$\| u \|_{H^1} = \left(\| u \|_{L^2(\Omega)} + \| \nabla u \|_{L^2(R)}\right)^{1/2}$$

Assuming $g \in H^{1/2}(\Gamma_D)$ we define:

$$H^1_E = \left\{v \in H^1_\Omega \ : \ v = g_1 \ in \ \Gamma_D\right\}$$

$$H^1_{E_0} = \left\{v \in H^1_\Omega \ : \ v = 0 \ \ in \ \Gamma_D\right\}$$

The weak formulation of the problem (1) reads:

$$find \ \ u \in H^1_E \ \ such \ that \ \ B(u,v) = F(v) \ \ \forall\, v \in H^1_{E_0} \tag{2}$$

where

$$B(u,v) \equiv (k\nabla u, \nabla v) - (\vec{b}u, \nabla v) + \int_{\Gamma_N} \vec{b} \cdot \vec{n}\ uv$$

$$F(v) \equiv (f,v) + \int_{\Gamma_N} g_2 v$$

The notation $(\cdot,\cdot)$ stands for the internal product of L^2 over Ω. The bilinear form $B(\cdot,\cdot)$ is *continuous* over $H^1_{E_0} \times H^1_{E_0}$ and *coercive*.

The existence and uniqueness of u that satisfies (2) for $f \in H^1(\Omega)$ follows from the Lax-Milgram theorem [3].

2 Discretization

The CVFEM method proposed by Baliga and Patankar [4,5] for the convection-diffusion equation is applied.

Consider a regular triangulation τ for the domain Ω [3]. Let N_i $(1 \leq i \leq n)$ be the vertices of the triangulation. Each vertex N_i is associated with a region S_i consisting of the union of all triangles $T \in \tau$ having N_i as a vertex.

From τ, a dual mesh $\mathcal{B}$ is constructed. The elements of the dual mesh will be called *control volumes*. The control volumes Ω_i are constructed joining the centroid of each triangle with the midpoints of its sides, thus spliting each triangle into three subregions. Each vertex N_i is associated with the control volume $\Omega_i \in \mathcal{B}$, $\Omega_i \subseteq S_i$, defined as the union of all the subregions converging to N_i.

The approximation space is defined as $V_h = \{v : v\mid_T \in C_T \forall T \in \tau$ and v is continuous at the nodes $\}$ where $C_T = \{\phi \in C(T) : \phi$ is exponential in the direction of the local mean velocity $\vec{b}_T$; linear in the normal direction and such that $-\nabla \cdot (k\nabla\phi - \vec{b}\phi) = 0 in\ T\}$

The CVFEM approximation is:
Find $u_h \in V_h$ such that

$$\int_{\delta\Omega_i} \left(\vec{b}u_h - k\nabla u_h\right) \cdot \vec{n}\, d\Gamma = \int_{\Omega_i} f\, d\Omega \qquad (3)$$

$$u_h(N_i) = g_1(N_i)\ \forall N_i \in \Gamma_D \qquad (4)$$

3 Error estimators

In this section we introduce an estimator that bounds the energy norm of the error.

Consider a family $\{\tau_j\}$ of regular triangulations in Ω [3] with it's associated numerical solutions u_j and exact errors $e_j = u - u_j$.

Let E_I be the set of all internal sides. For every internal side l we choose an arbitrary normal direction $\vec{n}$. For a side lying on the boundary $\vec{n}$ is taken to be the outer normal.

For the sake of simplicity we assume that k, $\vec{b}$, f and g_2 are piecewise constant and that g_1 is piecewise linear.

For a side l, $J_{l,n}$ and $J_{l,t}$ are defined as

$$J_{l,n} = \begin{cases} \left[\!\left[k\frac{\delta u_j}{\delta n} - \vec{b}\cdot\vec{n}u_j\right]\!\right]_l & \text{if } l \in E_I \\ 0 & \text{if } l \in \Gamma_D \\ 2\left(g_2 - k\frac{\partial u_j}{\partial t}\right)_l & \text{if } l \in \Gamma_N \end{cases}$$

$$J_{l,t} = \begin{cases} \left[\!\left[k\frac{\delta u_j}{\delta n}\right]\!\right]_l & \text{if } l \in E_I \\ 2\left(\frac{\partial g_1}{\partial t} - k\frac{\partial u_j}{\partial t}\right)_l & \text{if } l \in \Gamma_D \\ 0 & \text{if } l \in \Gamma_N \end{cases}$$

where $0 < k_0 \leq k(x,y)\ \forall\,(x,y) \in \Omega$ and $\left[\!\left[\cdot\right]\!\right]_l$ denotes the *jump* through the side l.

Let the norm $\mid\cdot\mid$ be defined as follows

$$\mid w \mid \equiv \| \nabla_j w \|_{L^2}$$

where $\nabla_j w$ be the L^2 vector defined by

$$\nabla_j w\mid_T \equiv \nabla(w\mid_T) \quad \forall w \in H^1(T),\ \forall T \in \tau_j$$

Theorem: the following inequality holds

$$\mid e_j \mid \leq C\left(\sum_T \eta_T^2\right)^{1/2} + \frac{\max b}{k_0} \| e_j \|$$

The proof of this theorem can be seen in [11].

Remark: in order to have non-dimensional constants we introduce another norm, equivalent to the L^2 norm:

$$|||\, v \,||| \equiv \left(|\, \Omega \,|^{-1} \int_\Omega v^2 \right)^{1/2}$$

With this norm, calling $Pe = \frac{\max b |\Omega|^{1/2}}{k_0}$, we have

$$|\, e_j \,|^2 \leq C \left(\sum_T \eta_T^2 \right)^{1/2} + Pe \,|||\, e_j \,|||$$

The result depicted above is valid, with some changes in the proofing procedure, for the conforming elements introduced by O'Riordan and Stynes [1]. The term corresponding to the tangential jump is obviously null and must not appear in the estimator. This method is, however, not suitable for adaptivity because of its structured nature.

For the method of O'Riordan and Stynes it can be proved [1] that the order of the L^2 norm of the error is higher than the order of the corresponding H^1. Based on numerical experiences, we believe that the same is true in our case. Nevertheless, with additional hipothesis, we can prove that:

Corolary: if $\exists \;\lambda < 1$ (λ may depend from h) such that

$$Pe \,|||\, e_j \,||| \leq \lambda \,|\, e_j \,|$$

then

$$|\, e_j \,| \leq \frac{C}{1-\lambda} \left(\sum_T \eta_T \right)^{1/2}$$

Remark: if $|||\, e_j \,|||$ is of higher order than $|\, e_j \,|$ then $\lambda = \lambda(h)$ and $\lambda(h) \to 0$ when $h \to 0$.

Finally, we are going to take the following expression as the *global error estimator* ε:

$$\varepsilon \equiv \left(\sum_{T\in\tau} \eta_T^2 \right)^{1/2} \qquad (5)$$

4 Numerical Results

We will consider in this section numerical results obtained with an adaptive algorithm based on the proposed estimator.

These convection-diffusion problems are solved initially on a uniform mesh τ_0. The mesh τ_{j+1} at the $j+1$ adaptivity step, is obtained from the τ_j by refining the elements T of τ_j such that:

$$\eta_T \geq (TOL) * \eta_{max}$$

where $\eta_{max} = \max_{T\in\tau_j} \eta_T$ and $0 \leq TOL \leq 1$ is a used defined tolerance.

The selected triangles are splitted into 4, and the refinement is propagated to the neighbouring elements according to the algorithm of Rivara [6]. These procedure guarantees that, for every refinement step j, the minimum angle of τ_j is not smaller than one half the minimum angle of τ_0.

The automatic refinement process was implemented as a loop of three programs: PUTCON, TREX2D and ENREDO. PUTCON is a small user-defined program for adding boundary conditions to a given mesh. TREX2D is an in-house Fortran code [10] for solving two-dimensional convection-diffusion problems via the CVFEM method as proposed by Baliga and Patankar [4,5]. It was modified in order to evaluate the a-posteriori error estimations and output a list of commands for automatically driving the grid generator ENREDO (Venere and Dari [9]).

4.1 Problem 1: Convective Transport of a Step Change

This is a well known benchmark for convection dominated flows (see for example [7]). Consider a square domain with Dirichlet conditions over the whole boundary. The temperature (u) is set to 1 at the left and top sides, and to 0 at the right and bottom sides. It is convected by a uniform velocity field $\vec{v} = (2, 1)$. The diffusion coefficient is negligible.

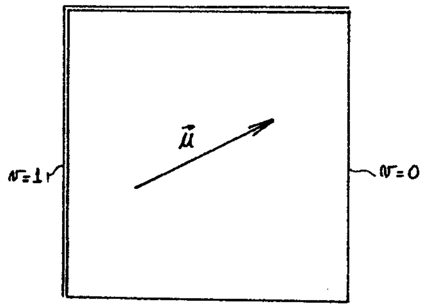

Figure 1: Problem 1: description

The initial uniform grid τ_0 has 32 elements ($NEL = 32$) and 25 nodes ($NOD = 25$). Two cases were considered, using two different values of the tolerance TOL: a-TOL = 0.3 and b-TOL = 0.7. Some representative grids and results obtained during the adaptive process are shown in figs. 3 and 4. Internal and external layers can be seen. They require refinement in order to be properly simulated.

The plot of the global error estimator ε versus NEL can be seen in fig. 2. Information about the evolution of the adaptive process can be found in table 1.

It. Step j	NOD	NEL	ε	NOD	NEL	ε
1	25	32	0.7376	25	32	0.7376
3	121	209	0.3053	75	124	0.3703
5	450	830	0.1396	144	248	0.2610
7	1638	3094	0.0676	224	395	0.2119
9				440	805	0.1436
11				653	1201	0.1167
13				898	1660	0.1004
15				1269	2357	0.0796
17				1704	3162	0.0662

Table 1: Problem 1 - evolution of the adaptive process

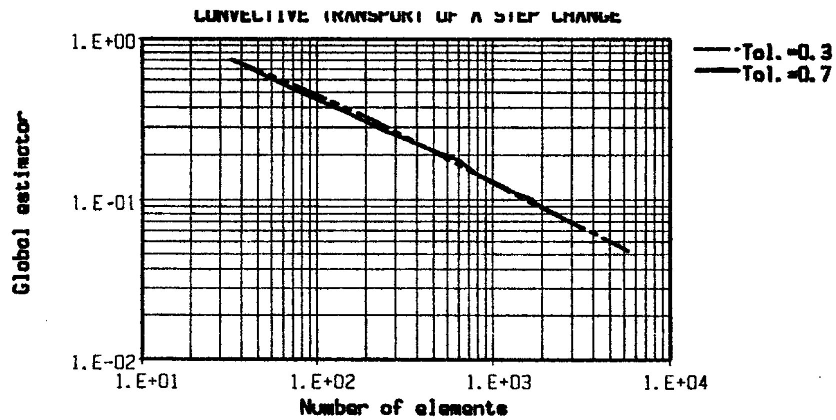

Figure 2: Problem 1: ε vs NEL

4.2 Problem 2: Step Change in a Recirculating Flow

This benchmark takes into account the case of convection subjected to a recirculating flow, and can be found for example in [8].

Consider a rectangular domain with the boundary conditions shown in fig 5. The temperature is convected by a velocity field that can be analitically expressed as $\vec{v} = (2y(1-x^2), -2x(1-y^2))$. The inlet boundary condition along $-1 \leq x \leq 0, y = 0$ is given by $u(x,0) = 1 + \tanh(20x+10)$. On the tangential boundaries the condition is $u = 1 - tanh(10)$. Natural boundary conditions are applied at the outlet. The diffusion coefficient is negligible. The initial uniform grid τ_0 has 64 elements and 50 nodes. As in the preceeding problem two cases were considered for the values of the tolerance TOL: a-$TOL = 0.3$ and b-$TOL = 0.7$. Some representative grids and results are shown in figs. 7 and 8.

The plot of the global error estimator ε versus NEL is shown in fig. 6 and the information about the adaptive process can be found in table 2.

It. Step j	NOD	NEL	ε	NOD	NEL	ε
1	45	64	0.2748	45	64	0.2748
3	300	561	0.1273	99	171	0.0896
5	646	1240	0.1120	164	297	0.0651
7	829	1599	0.0342	240	444	0.1410
9	1659	3249	0.0121	261	484	0.1276
11				380	718	0.0638
13				416	789	0.0478
15				540	1036	0.0330
17				677	1308	0.0246
19				776	1501	0.0254

Table 2: Problem 2 - evolution of the adaptive process

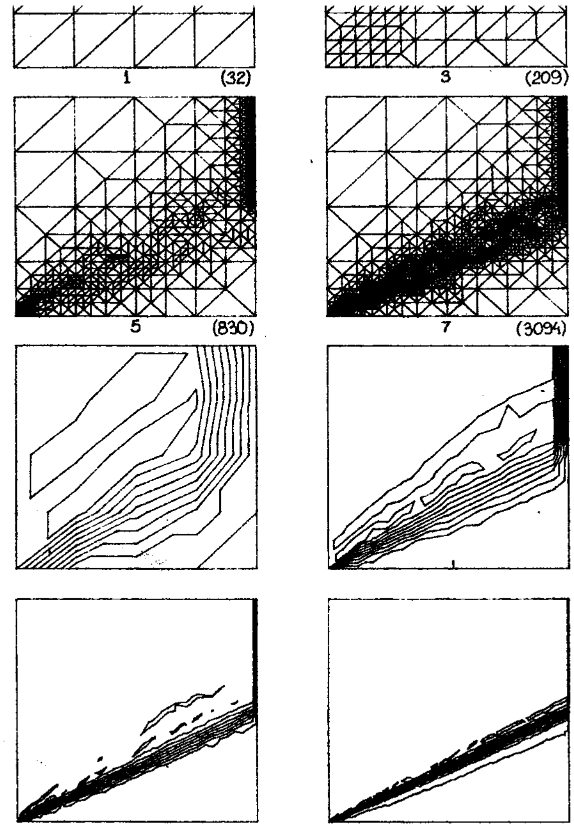

Figure 3: Problem 1: Case a - $TOL = 0.3$ - Some grids and results

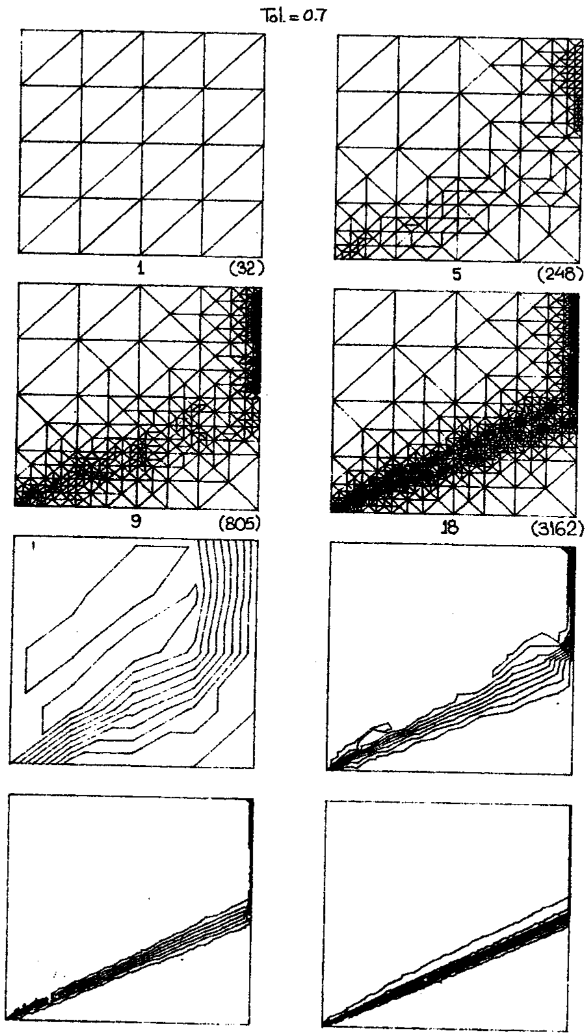

Figure 4: Problem 1: Case b - $TOL = 0.7$ - Some grids and results

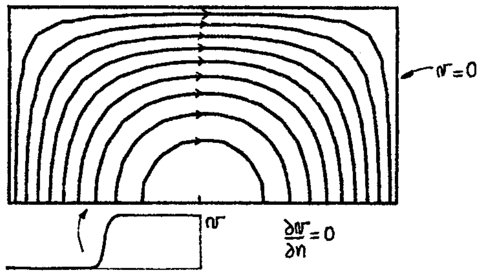

Figure 5: Problem 2: description

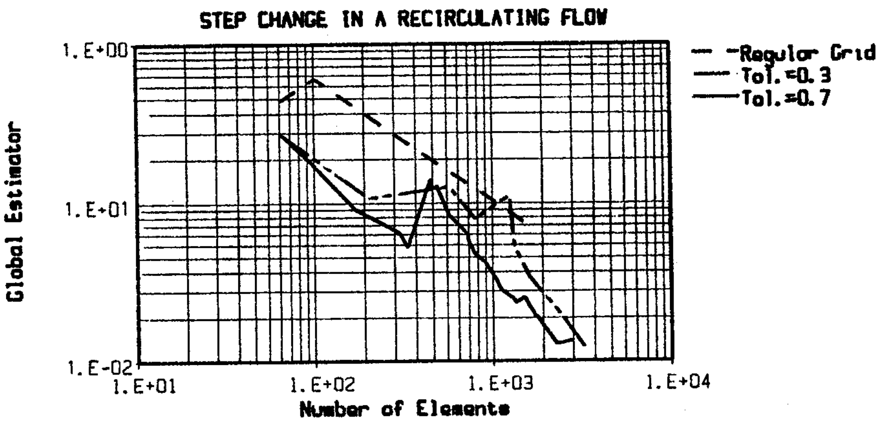

Figure 6: Problem 1: ε vs NEL

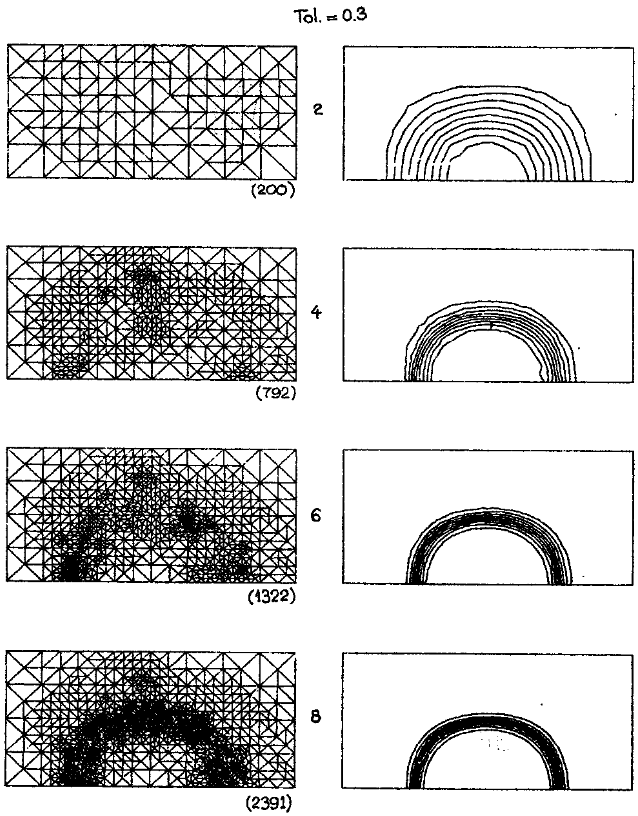

Figure 7: Problem 2: Case a - $TOL = 0.3$ - Some grids and results

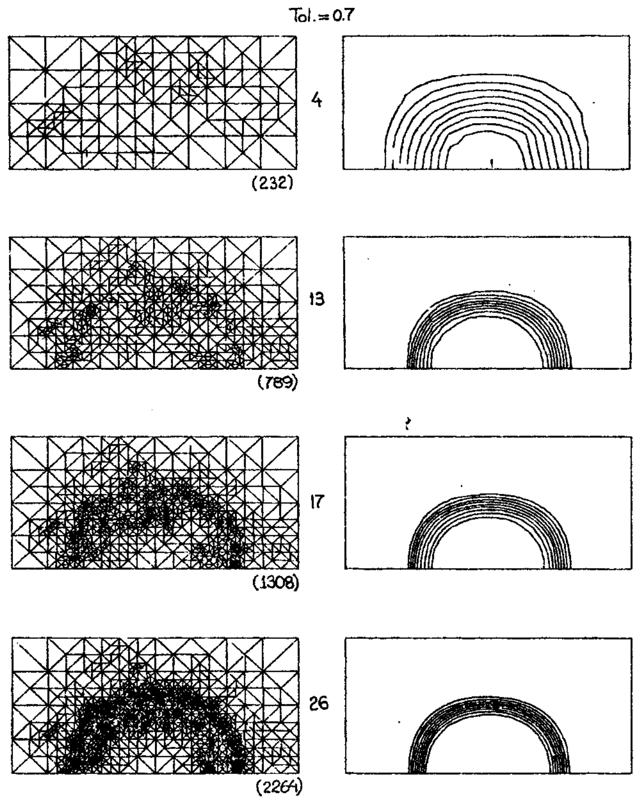

Figure 8: Problem 2: Case b - $TOL = 0.7$ - Some grids and results

5 Conclusions

We have introduced a technique for constructing a posteriori error estimators for the CVFEM method applied to convection dominated elliptic problems in two dimensions.

Our numerical computations show the good behaviour of the proposed estimator used as an error indicator within an adaptive loop. The secuence of grids obtained in the numerical experiments suggest that the application of techniques such stretching and/or derefinement could considerably improve the adaptive process.

The results of this work can be easily extended to other Petrov-Galerkin methods for which the residual vanishes inside each element.

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