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ON THE NUMERICAL SOLUTION OF A VARIATIONAL INEQUALITY

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Abstract

In this work we analize some special features of the numerical solution of a variational inequality which arises from an optimization problem with monotone controls. We give estimates of its convergence rate and we show an example which shows the optimality of these estimates.

Resumen

En este trabajo se analizan algunas particularidades de la solución numérica de una inecuación variacional proveniente de un problema de optimización con controles monótonos. Se dan estimaciones de la velocidad de convergencia y ejemplos donde se muestra que estas estimaciones son críticas.

1 Introduction

We consider the following differential equation

$$\begin{cases} y'' + (y - F + F'')^- = 0\,, \quad \text{in } (0,T)\,, \\ y(0) = 0\,, \\ y(T) = F(T)\,, \end{cases} \tag{1}$$

where $F \in H^3[0,T]$ and $x^- = -\min(x,0)$.

This equation stems (by using the methodology of variational inequalities in Sobolev spaces) from the optimization problem with monotone controls presented in [2]. The solution of (1) has the following minimality property:

Theorem 1.1 *y is the minimum element of the set*

$$U = \{u \in H^1[0,T] \,:\, u(0) \geq 0,\ u(T) \geq F(T),\ u'' \leq 0,\ u'' - u \leq F'' - F\}\,, \tag{2}$$

where u'' is understood as the second derivative in a weak sense, i.e. (2) means

$$\langle u', v' \rangle \geq 0\,, \quad \forall v \in H_0^1[0,T]\,,\ v \geq 0\,, \tag{3}$$

$$\langle u', v' \rangle \geq \langle -F'' + F - u, v \rangle\,, \quad \forall v \in H_0^1[0,T]\,,\ v \geq 0 \tag{4}$$

and

$$\langle u, v \rangle = \int_0^T u(t)\, v(t)\, dt\,. \tag{5}$$

The set U is called the set of supersolutions. By virtue of Theorem 1.1 (the proof of this theorem is included in [2]), the problem (1) can be reformulated in the following sense:

$$\boxed{P_C: \text{ Find } y \text{ the minimum element of the set } U.}$$

A numerical procedure to solve numerically this problem was presented in [1]. Here we continue with that development: specifically we present a sharper estimate of the convergence rate and an example which shows that this estimate is optimal.

2 The discretized problem

2.1 Finite difference approximation

Let Ω^k be the following set of points in $[0,T]$

$$\Omega^k = \{t_i \;:\; t_i = i\;k;\; i = 0, ..., n\}\,, \qquad k = \frac{T}{n}\,.$$

We define the following external approximation of the space $H^2[0,T]$

$$V_k = \left\{ f : [0,T] \to \Re \;\; / \;\; f(t) = \sum_{i=0}^{n} \alpha_i \Psi_i(t), \;\; \alpha_i \in \Re \right\},$$

where $\Psi_i : \Re \to \Re$ are defined by

$$\Psi_i(t) = \frac{1}{k}\left(1 - \left|\frac{t - t_i}{k}\right|\right)^+ ,$$

and $x^+ = \max(x, 0)$. We also define

$$V_{0,k}^+ = \{f \in V_k \;:\; f(0) = f(T) = 0,\; f(t_j) \geq 0\,,\; \forall j = 1, ..., n-1\}\,.$$

2.2 The approximated problem

The discretization of the restrictions that define the set U determines the subset $U_k \subset V_k$, which has the following form

$$U_k = \{u_k \in V_k \;:\; u_k(0) \geq 0\,,\; u_k(T) \geq F(T)\,, \text{ and } u_k \text{ verifies (6)}\}$$

$$\begin{cases} \langle u_k', v_k' \rangle \geq 0\,, & \forall v_k \in V_{0,k}^+\,, \\ \langle u_k', v_k' \rangle \geq -\langle -F'' + F - u_k, v_k \rangle\,, & \forall v_k \in V_{0,k}^+\,. \end{cases} \tag{6}$$

Then we can state the discretized problem as:

$$\boxed{\text{P}_k\text{: Find } y_k \text{ the minimum element of } U_k\,.}$$

3 The reduced problem

In order to grasp the essential features of the approximation error of our procedure, we are going to analyze, in a first place, a simplified problem. It consists in looking for $u \in H^2[0,T]$ such that

$$\begin{cases} u'' + (\varphi)^- = 0\,, \quad \text{in } (0,T)\,, \\ u(0) = 0\,, \\ u(T) = b\,, \end{cases} \tag{7}$$

where $\varphi \in H^1[0,T]$. It is easy to prove that u is the minimum element of the set S

$$S = \left\{ s \in H^1[0,T] \ : \ s(0) \geq 0,\ s(T) \geq b,\ s'' \leq 0,\ s'' \leq \varphi \right\},$$

where s'' is understood in the weak sense (3)-(5).

3.1 The discrete solution

In a similar way as to what was done in Section 2.2, we define the discrete solution in the following form:

$$\boxed{\text{P}_k\text{: Find } u_k \text{ the minimum element of } S_k\,,}$$

where

$$S_k = \{ s_k \in V_k \ : \ s_k(0) \geq 0,\ s_k(T) \geq b,\ \text{ and } s_k \text{ verifies (8)} \}$$

$$\begin{cases} \langle s_k', v_k' \rangle \geq 0\,, & \forall v_k \in V_{0,k}^+\,, \\ \langle s_k', v_k' \rangle \geq -\langle \varphi_k, v_k \rangle\,, & \forall v_k \in V_{0,k}^+\,. \end{cases} \tag{8}$$

Let

$$\Psi(t) = \frac{1}{k}\left(1 - \left|\frac{t}{k}\right|\right)^+, \tag{9}$$

we define

$$\overline{v}(t) = (v * \Psi)(t) = \int_{-k}^{k} v(t-s)\,\Psi(s)\,ds. \tag{10}$$

Remark 3.1 *It is easy to prove that $\overline{v}(t_i) = \langle v, \Psi_i \rangle$. Definition (10) can be also used for elements of $(H_0^1(0,T))'$; in the particular case of $v_k \in V_k$, we get $\overline{v_k''}(t_i) = -\langle v_k', \Psi_i' \rangle$, $\forall i = 1, \ldots, n-1$.*

Lemma 3.1 *u_k verifies the following difference equation*

$$\begin{cases} \overline{u_k''}(t_j) = -\left(\overline{\varphi}(t_j)\right)^-\,, \quad \forall j = 1, \ldots, n-1\,, \\ u_k(0) = 0\,, \\ u_k(T) = b\,. \end{cases} \tag{11}$$

Proof. From the definition of u_k, it is verified

$$\langle u_k', v_k' \rangle \geq 0\,, \qquad v_k \in V_{0,k}^+\,.$$

In consecuence, we have

$$\langle u_k', \Psi_j' \rangle \geq 0\,, \qquad \forall j = 1, ..., n-1$$

and from the definition of convolution, it results

$$\overline{u''}(t_j) \leq 0\,, \qquad \forall j = 1, ..., n-1\,. \tag{12}$$

In the same way

$$\langle u_k', v_k' \rangle + \langle \varphi, v_k \rangle \geq 0, \qquad v_k \in V_{0,k}^+\,.$$

This expression is equivalent to

$$\overline{u_k''}(t_j) + \overline{\varphi}(t_j) \leq 0\,, \quad j = 1, ..., n-1\,. \tag{13}$$

By (12) and (13) we have

$$\overline{u_k''}(t_j) \leq -(\overline{\varphi}(t_j))^-\,, \qquad j = 1, ..., n-1\,. \tag{14}$$

Let us denote z_k the solution of (11). Therefore

$$\overline{u_k''}(t_j) \leq \overline{z_k''}(t_j)\,,$$

and by using the Discrete Maximum Principle (see [3]), we get

$$u_k(t_j) \geq z_k(t_j)\,.$$

It is easy to check that $z_k \in S_k$; in consequence z_k is the minimum element of S_k (i.e. $z_k = u_k$).

□

Remark 3.2 *Let us define $u_I \in V_k$: $u_I(t_j) = u(t_j)\,, \forall j = 0, ..., n$. Then, u_I is the solution of*

$$\begin{cases} \overline{u_I''}(t_j) = \overline{-\varphi^-}(t_j), \quad \forall j = 1, ..., n-1\,, \\ u_I(0) = 0\,, \\ u_I(T) = b\,. \end{cases} \tag{15}$$

The function $\min(x, 0)$ *is concave, then*

$$\overline{-\varphi^-}(t_j) \leq -(\overline{\varphi}(t_j))^-, \;\; \forall j = 1, ..., n-1\,.$$

Therefore

$$\overline{u_I''}(t_j) \leq \overline{u_k''}(t_j)\,,$$

and by using the Discrete Maximum Principle (see [3]), we get

$$u_I(t_j) \geq u_k(t_j)\,.$$

So, we have obtained one of the estimates, to find the other we use the auxiliary result given by the Lemma (3.2).

Note 3.1 *We will denote,* $\forall w \in V_k$

$$\|w\|_{l^\infty(0,T)} = \max_{i=1,\ldots,n-1} |w(t_i)| \,, \tag{16}$$

$$\|w\|_{l^1(0,T)} = k \sum_{i=1}^{n-1} |w(t_i)| \,, \tag{17}$$

$$\|w\|_{l^2(0,T)} = \left(\sum_{i=1}^{n-1} k \, |w(t_i)|^2 \right)^{\frac{1}{2}} . \tag{18}$$

Lemma 3.2 *Let* v_k *verify the following difference equation*

$$\begin{cases} \overline{v_k''}(t_j) = w = \{w_j\} \,, & \forall j = 1, \ldots, n-1 \,, \\ v_k(0) = v_k(T) = 0, & \end{cases} \tag{19}$$

then, there are positive constants $c, \bar{c}$ *(independent of* n*) such that*

$$v_k(t_j) \le c \, \|w\|_{l^1(0,T)} \le \bar{c} \, \|w\|_{l^2(0,T)} \,. \tag{20}$$

Proof. Let γ_i $(i = 1, \ldots, n-1)$ be the triangular function defined by:

$$\begin{cases} \gamma_i(0) = 0 \,, \\ \gamma_i(t_i) = -\dfrac{i(n-i)\, k^2}{n} \,, \\ \gamma_i(T) = 0 \,, \end{cases} \tag{21}$$

it is easy to prove that

$$\overline{\gamma_i''}(t_j) = 1 \quad \text{and} \quad \|\gamma_i\|_{l^\infty(0,T)} \le \frac{T}{4} k \,, \quad \forall i = 1, \ldots, n-1 \,. \tag{22}$$

Then, to a generic w, we have that the solution of (19) is given by

$$v(t) = \sum_{i=1}^{n} w_i \, \gamma_i(t) \,. \tag{23}$$

Finally, (20) appears as a result of (17), (18) and (22).

□

We will use the previous Lemma to estimate the error of approximation. In consequence we need to obtain a bound of the differences:

$$\overline{\varphi^-}(t_j) - (\overline{\varphi}(t_j))^- \,, \ \forall j = 1, \ldots, n-1 \,.$$

Lemma 3.3 $\forall \varphi \in H^1(0,T)$ *the following inequality holds*

$$\left\| (\overline{\varphi})^- - \overline{\varphi^-} \right\|_{l^2} \le 2 \, \|\varphi'\|_{L^2(0,T)} \, k \,. \tag{24}$$

Proof. We will use the following partition of $[0,T] = I^0 \cup I^- \cup I^+$, where

$$I^+ = \{t \in [0,T] \ : \ \varphi(t+s) > 0, \ \forall s \in [-k,k]\},$$

$$I^- = \{t \in [0,T] \ : \ \varphi(t+s) < 0, \ \forall s \in [-k,k]\},$$

$$I^0 = \{t \in [0,T] \ : \ \exists\, s \in [-k,k], \ \varphi(t+s) = 0\}.$$

For all $t_j \in I^+ \cup I^-$, we have

$$\overline{\varphi^-}(t_j) = (\overline{\varphi})^-(t_j).$$

Then, it remains to estimate the value of $\left(\overline{\varphi^-} - (\overline{\varphi})^-\right)$, $\forall t_j \in I^0$. Taking in mind that $\forall t \in [0,T]$

$$-\varphi^-(t) \leq \varphi(t),$$

we have

$$\overline{\varphi^-}(t_j) \geq -\overline{\varphi}(t_j),$$

therefore

$$\overline{\varphi^-}(t_j) \geq (\overline{\varphi}(t_j))^-. \tag{25}$$

For the other inequality, we have by (25)

$$0 \leq \overline{\varphi^-}(t_j) - (\overline{\varphi}(t_j))^- \leq \overline{\varphi^-}(t_j).$$

Also, $\forall t \in [0,T]$ it is valid that

$$\varphi^-(t) \leq |\varphi(t)|,$$

and then

$$\overline{\varphi^-}(t_j) \leq \overline{|\varphi|}(t_j).$$

Let $t_j \in I^0$, then $\exists \zeta \in (-k,k)$ such that $\varphi(t_j - \zeta) = 0$. Then, $\forall t \in (t_{j-1}, t_{j+1})$ we have

$$|\varphi(t)| \leq \int_{t_j-\zeta}^{t} |\varphi'(r)|\, dr \leq \int_{t_{j-1}}^{t_{j+1}} |\varphi'(r)|\, dr \leq \sqrt{2\,k}\ \|\varphi'\|_{L^2(t_{j-1},t_{j+1})};$$

therefore

$$\overline{|\varphi|}(t_j) \leq \sqrt{2\,k}\ \|\varphi'\|_{L^2(t_{j-1},t_{j+1})},$$

$$\sum_{j=1}^{n-1} k\,(\overline{|\varphi|}(t_j))^2 \leq \sum_{j=1}^{n-1} 2\,k^2\, \|\varphi'\|^2_{L^2(t_{j-1},t_{j+1})} \leq 4\,k^2\, \|\varphi'\|^2_{L^2(0,T)}.$$

Then

$$\left\|\overline{|\varphi|}\right\|_{l^2(0,T)} \leq 2\,k\, \|\varphi'\|_{L^2(0,T)}$$

and in consequence

$$\overline{\varphi^-}(t) - (\overline{\varphi})^-(t) \leq 2\, \|\varphi'\|_{L^2(0,T)}\ k. \tag{26}$$

□

Theorem 3.1 *The following estimate for the error of approximation holds*

$$\|u_k - u\|_{l^\infty(0,T)} \leq C\ \|\varphi'\|_{L^2(0,T)}\ k. \tag{27}$$

Proof. Let us define $E_k(t_j) = u(t_j) - u_k(t_j)$. From (15) and (11) we have

$$\begin{cases} \overline{E_k''}(t_j) = \overline{-\varphi^-}(t_j) + (\overline{\varphi})^-(t_j) \quad \forall t_j\,,\ j = 1,..,n-1\,, \\ E_k(0) = E_k(T) = 0\,. \end{cases} \tag{28}$$

We apply now to this equation the results established in Lemmas 3.2 and 3.3. From both estimations we get (27).

□

4 Estimation optimality

We present in this Section an example which shows that for a given partition Ω_k of $[0,T]$ there exist an *ad-hoc* purposely devised data for which the estimate (27) is achieved. The example is defined in the following way

$$\varphi(t) = \frac{\sqrt{2T}}{n\pi}\sin\left(\frac{n\pi t}{T}\right).$$

It is easy to check that

$$\begin{cases} \|\varphi'\|_{L^2(0,T)} = 1\,, \\ \overline{\varphi}^- \equiv 0, \\ \overline{(\varphi)^-}(t_j) = \dfrac{\sqrt{2T}}{n\pi^2} \quad \forall t_j\ j = 1,...,n-1\,. \end{cases} \tag{29}$$

In consequence, it is

$$\begin{cases} u_k \equiv 0\,, \\ u_I(t_j) = \dfrac{\sqrt{2T}}{2n\pi^2}(T - t_j)\,t_j\,, \end{cases} \tag{30}$$

then

$$\|u_I - u_k\|_{l^\infty(0,T)} = \frac{\sqrt{2T}}{2n\pi^2}T^2/4\,.$$

Finally, taking into account Remark 3.2 we get

$$\|u - u_k\|_{l^\infty(0,T)} = \frac{\sqrt{2T}}{2n\pi^2}T^2/4\,.$$

5 General case

Now we are going to consider the general case, i.e.

$$\begin{cases} y'' + (y - F + F'')^- = 0\,, \qquad \text{in } (0,T)\,, \\ y(0) = 0, \\ y(T) = F(T). \end{cases} \tag{31}$$

By definition the discrete solution verifies

$$\begin{cases} \overline{y_k''}(t_j) = -\left(\overline{(y_k - F + F'')(t_j)}\right)^- & j = 1, ..., n-1, \\ y_k(0) = 0, \\ y_k(T) = F(T). \end{cases} \tag{32}$$

To obtain the error estimation we use basically the estimate (27) and the continuity of an operator P associated to the solution of (32).

Theorem 5.1 *Let us denote by y the solution of (31) and y_k the solution of (32), we have*

$$\|y_k - y\|_{l^\infty(0,T)} \le M\, k.$$

Proof. Let $w_k \in V_k$ such that

$$\begin{cases} \overline{w_k''}(t_j) = -\left(\overline{(y - F + F'')(t_j)}\right)^- & j = 1, ..., n-1 \\ w_k(0) = 0, \\ w_k(T) = F(T). \end{cases} \tag{33}$$

As y is a fixed function, if we consider in (26) : $\varphi = y - F - F''$, we get the inequality

$$\|y - w_k\|_{l^\infty(0,T)} \le M\, k. \tag{34}$$

Let us define the operator $P : H^2(0,T) \to V_k$, such that $P(f) = z_k$, where

$$\begin{cases} \overline{z_k''}(t_j) = -\left(\overline{(f + z_k - F + F'')(t_j)}\right)^- & j = 1, ..., n-1 \\ z_k(0) = 0, \\ z_k(T) = F(T). \end{cases} \tag{35}$$

It is obvious that $w_k = P(y - w_k)$ and $y_k = P(0)$. P is a continuous operator and there exist a constant $\overline{M}$ such that

$$\|y_k - w_k\| = \|P(y - w_k) - P(0)\| \le \overline{M}\ \|y - w_k\|, \tag{36}$$

Finally, from (34) and (36)

$$\|y_k - y\| \le \|y_k - w_k\| + \|y - w_k\| \le M\left(\overline{M} + 1\right) k. \tag{37}$$

□

Remark 5.1 Estimation optimality
To show the optimality of the estimate (37) we can study the behavior of the solution corresponding to the following data F:

$$F(t) = \frac{\sqrt{2T}}{n^3\,\pi}\sin\left(\frac{n\,\pi\,t}{T}\right).$$

It is easy to check – mutatis mutandis – by using essentially the arguments used in Section 4 that

$$\|y - y_k\|_{l^\infty(0,T)} \ge C\frac{1}{n}.$$

Conclusions

We have obtained in this paper a critical estimate of the error associated to the numerical procedure presented in [1], for the computational solution of the optimal control problem analyzed theoretically in [2].
This estimate represents in a certain sense the worst case that can arise in this procedure, but really, for a particular data, the sequence of errors $E_k = y - y_k$ has a faster rate of convergence than that given by (37), i.e. we have

$$\lim_{k\to 0} \frac{\|y - y_k\|_{l^\infty(0,T)}}{k} = 0 .$$

The origin of this improvement is the fact that the estimate (27) can be modified – obtaining a tighter estimate – in the following form

$$\|u_k - u\|_{l^\infty(0,T)} \leq C \, \|\varphi'\|_{L^2(I^0)} \; k ,$$

and because $\|\varphi'\|_{L^2(I^0)} \to 0$, due to the regularity of φ.

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