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ELASTIC DEGRADATION AND DAMAGE: PLASTICITY-LIKE FORMULATION, STIFFNESS RECOVERY AND LOCALIZATION

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Abstract: Elastic-degrading and damage models have proliferated in the literature. Fundamental issues, however, still remain unsolved. Three specific aspects to which the author has contributed in recent years, are reviewed and discussed.

Resumen: Los modelos de degradación elástica y daño han proliferado en la literatura reciente. Algunos problemas fundamentales, sin embargo, todavía no han sido resueltos. En el artículo se repasan y discuten tres aspectos específicos en los que el autor ha contribuido en los últimos años.

1 Introduction

Models for Elastic Degradation and Damage (EDD) based on loading surfaces have become popular and widely used for constitutive modeling of engineering materials such as concrete, rock, ceramics, etc. EDD, however, is considerably more complex than ElastoPlasticity (EP), with several challenging questions that still remain unresolved. A number of formulations have been proposed, most of them with their own terminology and notation. Some of them include complicated features such as stiffness recovery due to Microcrack Closure/Reopening (MCR), combination of degradation and plasticity, and have been used to solve boundary value problems involving localization, sometimes without a full understanding of fundamental aspects. In this paper, the author reviews advances and pending problems in three areas: unification and standarization of EDD formulations, formulation of MCR effects and localization analysis based on the acoustic tensor.

2 Plasticity-like Formulation of EDD Models Based on a Loading Surface

Effective development in a field requires first a common terminology and notation. Recent years have seen various attempts to provide such a theory for EDD models based on a loading surface. The literature, however, still shows large dispersion of notations and approaches. One factor is the role of thermodynamics in EDD models, notably more important than it is in EP. This has caused historically a tendency to describe EDD in a different, more abstract way, resorting to concepts which are not as familiar to engineers as strain, stress or stiffness. Pioneering work by Dougill (1976), Hueckel and Maier (1977), Ortiz (1985) and others [1, 2, 3], however, already suggested that EDD models could be indeed formulated in a similar way to well known EP, and this should facilitate standardization. Such a general formulation was proposed recently by Carol, Rizzi and Willam [4], and is summarized in the following.

2.1 Plasticity-like equations and tangent operator

The fundamental relation of an elastic-degrading material is the secant stress-strain equation

$$\sigma_{ij} = E_{ijkl}\epsilon_{kl} \quad \text{or} \quad \epsilon_{ij} = C_{ijkl}\sigma_{kl} \tag{1a,b}$$

E_{ijkl} and C_{ijkl} denote the components of the elastic secant stiffness and compliance tensors **E** and **C**, that remain constant during unloading/reloading, and that are symmetric to avoid spurious energy dissipation or generation under closed stress or strain paths [4].

A loading function $F(\boldsymbol{\sigma}, \mathbf{p})$ is used to define an elastic domain $F<0$ in which no further degradation takes place (the secant stiffness remains unchanged). At the loading surface $F=0$, further degradation may occur, which is accompanied by increments of *degrading strain*, ϵ^d_{ij}. The degrading strain is defined as the excess strain beyond the value that corresponds to the increment of stress according to the current secant stiffness (Fig. 2).

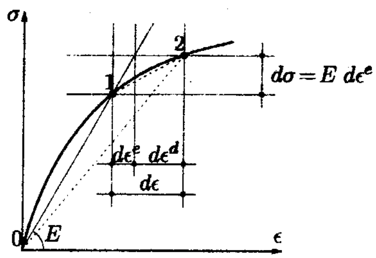

Fig. 2. Definition of the degrading strain rate

With these definitions, the following set of rate equations are considered

$$\dot{\sigma}_{ij} = E_{ijkl}(\dot{\epsilon}_{kl} - \dot{\epsilon}^d_{kl}) \tag{2}$$

$$\dot{\epsilon}^d_{kl} = \dot{\lambda} m_{kl} \qquad \left(\text{with } m_{ij} = \frac{\partial Q}{\partial \sigma_{ij}} \text{ optionally}\right) \tag{3a,b}$$

$$\dot{F} = n_{ij}\dot{\sigma}_{ij} - H\dot{\lambda} = 0 \quad \text{with } n_{ij} = \left.\frac{\partial F}{\partial \sigma_{ij}}\right]_{\lambda=ct} \text{ and } H = -\left.\frac{\partial F}{\partial \lambda}\right]_{\sigma=ct} \tag{4a,b,c}$$

m defines the directions of the flow rule for degrading strains (3a), that can be derived as the gradient of the potential Q (3b). Eq. (4a) represents the consistency condition, and H has the usual meaning of a hardening/softening modulus. The previous equations can be combined in the traditional way, yielding the following expressions for the degradation multiplier and the tangent stiffness:

$$\dot{\lambda} = \frac{n_{cd}E_{cdkl}\dot{\epsilon}_{kl}}{H + n_{pq}E_{pqrs}m_{rs}} \quad ; \quad E^{tan}_{ijkl} = E_{ijkl} - \frac{E_{ijab}m_{ab}n_{cd}E_{cdkl}}{H + n_{pq}E_{pqrs}m_{rs}} \tag{5a,b}$$

Eqs. (2–5) are analogous to those in classical plasticity except for the secant stiffness instead of the initial stiffness, and the degrading strain which takes the place of the plastic strain. F and $\mathbf{m}$ are assumed to be defined in such a way that the denominator $H+n_{pq}E_{pqrs}m_{rs}$ remains always positive. The model is called associated *in the stress space* when $\mathbf{m}$ is proportional to $\mathbf{n}$ and consequently the tangent stiffness exhibits major symmetry. If $\mathbf{m}$ is derived from a potential Q, associativity can be alternatively stated as $Q=F$.

2.2 Degradation rule for compliance

In contrast to plasticity, however, the previous Eqs. (2) to (4) (and the additional definitions inherent to H) are not sufficient to define the evolution of an elastic-degrading model, since no evolution law has been specified for the (variable) secant stiffness. In order to do that, (1) can be differentiated and compared to (2), and relation $\dot{E}_{ijkl}=-E_{ijpq}\dot{C}_{pqrs}E_{rskl}$ (obtainded from differentiation of $\mathbf{E}:\mathbf{C}=\mathbf{I}$) substituted, leading to

$$E_{ijkl}\dot{\epsilon}^{d}_{kl} = -\dot{E}_{ijkl}\epsilon_{kl} \quad \text{or} \quad \dot{\epsilon}^{d}_{pq} = \dot{C}_{pqrs}\sigma_{rs} \tag{6a,b}$$

which provides the relation between secant compliance and degrading strain rates. When the first is known, the second follows (but not the opposite). A "generalized flow rule" or *degradation rule* for the secant compliance can be defined and related to the flow rule for the degrading strains [4]

$$\dot{C}_{ijkl} = \dot{\lambda}M_{ijkl} \quad \text{and} \quad m_{ij} = M_{ijkl}\sigma_{kl} \tag{7a,b}$$

$\dot{\lambda}$ specifies the magnitude and $\mathbf{M}$ the direction of the rate of change of $\mathbf{C}$, and (7b) follows from replacing (3a) and (7a) into (6b). This growth equation (in essence equivalent to Eq. 2.7 in [2] and to Eq. 3.36 in [3]), indicates that once the degradation rule has been established, the corresponding flow rule for degrading strains follows automatically. The requirement that $\mathbf{E}$ and $\mathbf{C}$ must remain always symmetric requires that $\mathbf{M}$ is also symmetric. Similar to $\mathbf{m}$, $\mathbf{M}$ can be also derived from a degradation potential Q', although this requires recourse to some concepts of thermodynamics as explained in [4]. With the definition of the degradation rule, the elastic-degrading formulation is closed. The final set of equations strictly necessary to integrate the model for a prescribed strain history reduces to (5a), (7a) and (1) (plus the definitions inherent to H). In addition, equation (5b) with $\mathbf{m}$ from (7b) is also necessary to implement the tangent stiffness for incremental-iterative solution procedures, or for localization analysis, as developed in sect. 4.

2.3 Elastic-damage formulation

In elastic-degrading formulations the state of degradation is characterized by the secant compliance (or stiffness) tensor itself, with 21 independent components, and the corresponding evolution laws must also involve 21 components (those of the tensor M_{ijkl}). Alternatively, it is reasonable to assume a reduced set of variables which still fully characterize the state of degradation or *damage* in the material, for which simple evolution laws can be postulated. These are the *damage variables*, $\mathcal{D}_*$, the number and nature of which (scalar, vectorial or tensorial) does not need to be specified for the development of the general theory (the subindex $*$ represents the desired number of indices). According to that concept, one can write

$$C_{ijkl} = C_{ijkl}(C^0_{pqrs}, \mathcal{D}_*) \quad \text{and} \quad \dot{C}_{ijkl} = \frac{\partial C_{ijkl}}{\partial \mathcal{D}_*}\dot{\mathcal{D}}_* \tag{8a,b}$$

where C_{ijkl} are a set of known, continuous and differentiable functions and repetition of subscript $*$ implies summation over all the indices represented by the symbol. A flow rule for the damage variables $\mathcal{D}_*$, can be formulated, and its relation to the flow rule for compliance can be established [4] as

$$\dot{\mathcal{D}}_* = \dot{\lambda}\mathcal{M}_* \qquad \text{and} \qquad M_{ijkl} = \frac{\partial C_{ijkl}}{\partial \mathcal{D}_*}\mathcal{M}_* \tag{9a,b}$$

Similar to (3a) and (7a), $\dot{\lambda}$ specifies the intensity and $\mathcal{M}_*$ the direction of the increment of the damage variables in the *damage space*. The final equations for the evolution of elastic damage are the same as for the elastic degradation where M_{ijkl} is replaced by $\mathcal{M}_*$ according to (9b).

2.4 General formulation for associated $(1-D)$ *scalar damage models*

Using the unified theory described in previous sections it is possible to derive a general formulation for associated scalar damage models of the traditional $(1-D)$ type. In [4], this is done in terms of strain, and it is shown that it includes as particular cases a number of models of this type found in the literature [5, 6, 7]. Here, the derivation is presented as stress-based and with a different choice of the scalar multiplier λ, although the resulting formulation is fully equivalent. First, consider the traditional assumption for the secant stiffness, and the corresponding inverse equation in terms of compliance:

$$E_{ijkl} = (1-D)E^0_{ijkl} \quad ; \qquad C_{ijkl} = \frac{1}{(1-D)}C^0_{ijkl} \tag{10a,b}$$

The choice of damage variable and scalar multiplier, and the subsequent identification of degradation rules for damage, stiffness and stress, leads to:

$$\mathcal{D}_* = \text{scalar} = D \;\; ; \quad \dot{\lambda} = \frac{\dot{D}}{(1-D)} \;\; ; \quad \mathcal{M}_* = \text{scalar} = 1-D \tag{11a,b,c}$$

$$\frac{\partial C_{ijkl}}{\partial \mathcal{D}_*} = \frac{1}{(1-D)^2}C^0_{ijkl} = \frac{1}{(1-D)}C_{ijkl} \;\; ; \quad M_{ijkl} = C_{ijkl} \tag{11d,e}$$

$$m_{ij} = C_{ijlk}\sigma_{kl} = \epsilon_{kl} \tag{11f}$$

The following loading function based on the elastic energy is chosen:

$$F = w - r(D) \qquad \text{with} \qquad w = \frac{1}{2}\sigma_{ij}C_{ijkl}\sigma_{kl} = \frac{1}{2}\epsilon_{ij}E_{ijkl}\epsilon_{kl} \tag{12a,b}$$

This definition includes loading functions of the type $F = f(w, D) - r'(D)$ since for $F = 0$ one can isolate w and rewrite as above. As particular cases of $f(w, D)$, the cases with functions based on the stress-based or strain-based undamaged free energies $w^0 = \sigma_{ij}C^0_{ijkl}\sigma_{kl}/2 = (1-D)w$ or $\bar{w}^0 = \epsilon_{ij}E^0_{ijkl}\epsilon_{kl} = w/(1-D)$ are also included. From F, the gradient in stress space can be obtained

$$n_{ij} = \frac{\partial F}{\partial \sigma_{ij}} = C_{ijkl}\sigma_{kl} = \epsilon_{ij} \tag{13}$$

n_{ij} turns out equal to m_{ij} which means associativity in the stress space. From F one can also obtain the hardening modulus $H = -\partial F/\partial\lambda$ for constant stress, which, given (11b), is

$$H = -w + (1-D)\frac{\partial r}{\partial D} \tag{14}$$

Finally, with m_{ij}, n_{ij} and H, one can obtain the following expressions for the tangent stiffness:

$$E^{tan}_{ijkl} = (1-D)E^0_{ijkl} - \frac{1}{\bar{H}}\sigma_{ij}\sigma_{kl} \qquad \text{with} \qquad \bar{H} = w + (1-D)\frac{\partial r}{\partial D} \tag{15a,b}$$

3 Spurious Dissipation in Microcrack Closure/Reopening Models

Stiffness degradation is normally attributed to opening of microcracks subject to tension. This poses the additional problem of modeling stiffness recovery when stresses are reversed and microcracks close. Normally, MCR effects are introduced by resorting to spectral decomposition of strain (or stress) into positive and negative parts $\epsilon = \epsilon^+ + \epsilon^-$, and by defining the 4th order projection operators $\epsilon^+ = \mathbf{P}^+_\epsilon \epsilon$ and $\epsilon^- = \mathbf{P}^-_\epsilon \epsilon$, that are functions of strain (or stress) state. Then, the ***active stiffness*** is defined as a combination of initial and secant stiffness, e.g. $\mathbf{E}^{ac} = \mathbf{E}^0 - \mathbf{P}^+_\epsilon(\mathbf{E}^0 - \mathbf{E})\mathbf{P}^+_\epsilon$ [3, 7, 5]. The formulation of MCR effects, however, should satisfy certain fundamental principles. Recovery of stiffness does not mean that the degradation process which led to microcracking is reversed; it is only a transient closure, and reopening of the same microcracks should occur upon new tensile conditions without additional energy dissipation. Assuming that no further degradation occurs during opening and closing of existing microcracks, this can be stated as that the MCR model should conserve energy along the line of hyperelastic solids. In Carol and Willam (1994 and 1996), [8, 9] various types of models are examined with regard to this postulate. The concept of ***spurious dissipation*** (that should be zero if the formulation is conservative) is introduced. General isotropic as well as some specific anisotropic secant stiffnesses are examined with different projection operators. Results show that all MCR models are energy conservative when the secant stiffness is isotropic, but ***all of them exhibit significant spurious dissipation*** if anisotropic secant stiffness is subjected to stress/strain histories which involve rotation of principle axes. The conclusion seems general for all MCR models based on positive/negative tensorial decomposition of strain. The only type of formulation that seems capable of representing these effects with anisotropic degradation and no spurious dissipation, would be along the line of the microplane model [10, 11], with opening/closure conditions decided independently for each orientation in space, and where the active stiffness is obtained by integration over the appropriate subdomain of the solid angle [8]. Beyond energy conservation, additional requirements have been proposed concerning convexity of the corresponding energy potential [12]. A fully consistent formulation of MCR effects valid for the recovery of both isotropic and anisotropic degradation remains, therefore, an open subject for which further research is required.

4 Localization of Elastic-Degrading Models

The availability of general expressions for the tangential operator $\mathbf{E}^{tan}$ of EDD models [4], immediately suggests the possibility of a bifurcation study based on eigenvalue analysis of $\mathbf{E}^{tan}$ itself and of the associated localization tensor $\mathbf{Q}^{tan}$, in analogy to elastoplastic bifurcation [13, 14]. Such an analysis provides information about the failure modes, which are specially important for their implementation and use in the context of FE computations. Preliminary results were presented by Neilsen and Schreier (1992) [6]. In 1995, Rizzi, Carol and Willam [15, 16] used the plasticity analogy of $\mathbf{E}^{tan}$ to extend the closed-form solutions proposed for EP [17, 18], to scalar damage models subjected to various loading scenarios in plane strain, plane stress and 3D. In those references, the solutions were presented in terms of strains. Here, equivalent solutions are presented in terms of stress, which provides simpler expressions and other interesting advantages.

4.1 Solution for $\bar{H}^{\text{crit}}(\hat{\mathbf{N}})$ in 3D

The general solution proposed by Ottossen an Runesson [17] for the critical value of the hardening modulus required to obtain singularity of Q_{jk} for a plane of given orientation $\hat{\mathbf{N}}$, can be written as:

$$\bar{H}^{\text{crit}} = -\hat{N}_i \bar{n}_{ij} Q^{-1}_{jk} \bar{m}_{kl} \hat{N}_l \tag{16}$$

where Q^{-1}_{jk} are the components of the inverse matrix of Q_{jk}, and

$$\bar{n}_{ij} = E_{ijkl} n_{kl} \,, \qquad \bar{m}_{ij} = -E_{ijkl} m_{kl} \qquad \text{and} \qquad Q_{jk} = \hat{N}_i E_{ijkl} \hat{N}_l \tag{17a,b,c}$$

Due to the pefect analogy of the expresion of $\mathbf{E}^{\tan}$, these equations are valid also for elastic degrading materials provided E_{ijkl} represents the secant stiffness instead of the initial.

In the case of traditional scalar damage as described in Sect. 2.4, $n_{ij} = m_{ij} = \epsilon_{ij}$ and $E_{ijkl} = (1-D)E^0_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, where $\lambda = (1-D)\lambda^0$ and $\mu = (1-D)\mu^0$. By substituting those expressions, one can obtain

$$\bar{H}^{\text{crit}}(\hat{\mathbf{N}}) = \frac{\sigma_T^2}{\mu} + \frac{\sigma_N^2}{\lambda + 2\mu} \tag{18}$$

where σ_N and σ_T are the intensities of the normal and shear components of the stress (traction) on the plane of orientation $\hat{\mathbf{N}}$. Eq. (18) is equivalent to its counterpart in terms of strains, Eq. 39 in [16]; one can be converted into the other by using (1a,b) and (10a). Smilarly to that case, the localization condition in terms of stresses (18) can also be represented graphically as an ellipse in the Mohr axes σ_N, σ_T. Here, however, the representation is simpler since the ellipse has a fixed center at the origin, while in terms of strains the center moved along the horizontal axis depending on ϵ_V. The major and minor axes of the ellipse are given by $\sqrt{(\lambda+2\mu)\bar{H}}$ and $\sqrt{\mu\bar{H}}$ respectively.

4.2 Maximization with $\hat{\mathbf{N}}$

The onset of discontinuous bifurcation at material level will be obtained by maximizing $\bar{H}^{\text{crit}}$ with $\hat{\mathbf{N}}$, which will give $\bar{H}^{\text{db}}$ and $\hat{\mathbf{N}}^{\text{db}}$. Graphically, that is equivalent to establishing the tangency condition between the ellipse (that shrinks as loading progresses because $\bar{H}$ and the secant moduli decrease) and the largest Mohr circle of stresses (that expands with increasing load). Geometrical considerations lead to the following three cases of tangency depending on the ratio between $\sigma_1+\sigma_3$ and $\sigma_1-\sigma_3$:

$$\begin{aligned}
&(a) \qquad \frac{1}{1-2\nu}(\sigma_1-\sigma_3) \le \sigma_1+\sigma_3 \\
&(b) \qquad -\frac{1}{1-2\nu}(\sigma_1-\sigma_3) < \sigma_1+\sigma_3 < \frac{1}{1-2\nu}(\sigma_1-\sigma_3) \\
&(c) \qquad \sigma_1+\sigma_3 \le -\frac{1}{1-2\nu}(\sigma_1-\sigma_3)
\end{aligned} \tag{19a,b,c}$$

Case (a) corresponds to tangency on the right hand end of the ellipse where the radius of the Mohr circle is smaller than the radius of curvature of the ellipse at that end. In this case, the first plane to satisfy $\det\mathbf{Q} = 0$ coincides with the plane of major principal stress. The corresponding strain-based hardening modulus is obtained by enforcing that $\sigma_N = \sigma_1$. Describing the localization direction in terms of the angle θ^{db} between $\hat{\mathbf{N}}$ and the major principal stress, in this case we have

$$\theta^{\text{db}} = 0 \qquad\qquad \bar{H}^{\text{db}} = \frac{\sigma_1^2}{\lambda+2\mu} \tag{20a,b}$$

Case (C) is the same on the left side with $\sigma_N = \sigma_3$, yielding

$$\theta^{\mathrm{db}} = 90^\circ \,, \qquad \bar{H}^{\mathrm{db}} = \frac{\sigma_3^2}{\lambda+2\mu} \tag{21a,b}$$

Case (b) is more complicated with tangency at intermediate points of the ellipse. Writing the corresponding equations and enforcing a single solution, one obtains:

$$\tan^2\theta^{\mathrm{db}} = \frac{1}{4}\frac{(\sigma_1-\sigma_3)^2 - (1-2\nu)^2(\sigma_1+\sigma_3)^2}{((1-\nu)\sigma_1 - \nu\sigma_3)^2} \qquad \bar{H}^{\mathrm{db}} = \frac{1}{\mu}\left(\frac{\sigma_1-\sigma_3}{2}\right)^2 + \frac{1}{\lambda+\mu}\left(\frac{\sigma_1+\sigma_3}{2}\right)^2 \tag{22a,b}$$

It is immediate to verify that, in the limit cases given by inequalities (19), these equations collapse into (20) and (21) The usual stress-based hardening softening modulus H is related to $\bar{H}$ in previous equations using $\bar{H} = H + n_{ij}E_{ijkl}m_{kl}$, which in this model can be rewritten as

$$H = \bar{H} - 2w \tag{23}$$

with w=current elastic energy (12b). Note that, while θ^{db} and $\bar{H}$ are independent of the intermediate principal stress σ_2, H in general is not, because of the term w.

4.3 Examples

The previous equations can be used to study specific loading situations. The first example is uniaxial tension, where $\sigma_1 > 0$ and $\sigma_2 = \sigma_3 = 0$. Since $0 < 1-2\nu < 1$, this corresponds always to case (b). Substituting the values of the stresses and introducing $w = \sigma_1^2/E$, Eqs. (22–23) yield

$$\tan^2\theta^{\mathrm{db}} = \frac{\nu}{1-\nu} \qquad \text{and} \qquad H^{\mathrm{db}} = -\frac{\nu^2}{E}\sigma_1^2 \tag{24a,b}$$

The angle reproduces exactly the formula obtained in [16], which exhibits a dependence on the Poisson ratio not observed in von Mises plasticity, with $\theta^{\mathrm{db}} = 0$ for $\nu = 0$, $\theta^{\mathrm{db}} = 33.21^\circ$ for $\nu = 0.3$ and $\theta^{\mathrm{db}} = 45^\circ$ for $\nu = 0.5$. On the other hand, H^{db} is always negative indicating that a certain amount of softening is always required to achieve localization.

The second example is uniaxial extension, where $\sigma_1 > 0$, $\sigma_3 = 0$ and $\epsilon_2 = 0$. This case would correspond to an apparent uniaxial tension state applied under plane-strain conditions, with out-of-plane intermediate stress σ_2. The definitions of the three cases and the localization angle do not depend on σ_2 and therefore they are the same as before. H however does depend on σ_2 through the elastic energy w, and this leads to the new value $H^{\mathrm{db}} = 0$, i.e. no softening required in this case.

The third example is pure shear, with $\sigma_1 > 0$, $\sigma_2 = 0$ and $\sigma_3 = -\sigma_1$. This also corresponds to case (b), and applying the various expressions we obtain the results $H^{\mathrm{db}} = 0$ and $\theta^{\mathrm{db}} = 45^\circ$. Additional results for scalar damage and plane stress can be found in [16].

5 Concluding remarks

The formulation of degradation and damage in a format which is similar to classical elastoplasticity offers several advantages. One is to standarize the notation and terminology, which is always necessary to advance in any field. Also, it becomes possible to use a number of existing developements such as some analytical solutions for the localization condition based on the acoustic tensor. Damage, however, is more complex than plasticity and poses a number of new difficulties not completely solved at present. Recovery of stiffness upon load reversal in a fully energy-consistent framework is one of them, since the formulations proposed to this end in the literature all seem to generate/disipate spurious energy if used in conjunction with anisotropic degradation. With regard to the localization study, existing solutions from plasticity can be used to obtain the critical hardening modulus for any given plane orientation. The subsequent maximization process is however considerably more complicated in the general case of anisotropic degradation. The elastoplastic solutions can only be applied directly to traditional scalar damage of the $(1-D)$ type, yielding some new dependencies such as the one on Poisson ratio. Recent studies extend these solutions to scalar damage combined with von Mises plasticity [19].

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