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Model-Free Reliability Estimation by means of Chernov Bound

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Sumário: Apresenta-se uma abordagem para avaliar um estimador de confiabilidade a partir de um pequeno número de amostras. A abordagem pode ser utilizada com amostras reais de uma variável de controle ou com resultados de uma dada função de estado limite obtidos por meio de simulações de Monte Carlo. A simplicidade do método sugere sua aplicabilidade à normas de projeto que devam considerar aspectos probabilísticos.

Abstract: An approach to evaluate reliability estimates from a small number of samples is presented. The approach can be used either with actual samples of a reliability index or with the outcomes of a limit state function obtained through Monte Carlo Simulation. The simplicity of the method suggests its application to account for probabilistic issues in design codes.

1 Introduction

The theory described in this paper is useful for solving engineering problems of prescribed reliability. This means that it is aimed not at the *exact* calculation of the reliability — or optionally of the proneness to failure — but at verifying whether or not an acceptable reliability has been achieved by a given design. The evaluation of the reliability of complex systems, or in fact of any system characterized by very low proneness to failure, usually requires significant computational efforts. State of the art in the area were presented by Schuëller and Stix (1987) and Ayyub and McCuen (1995).

Almost all methods flexible enough to be applicable to arbitrary systems, i.e. systems involving non-linear failure criteria and arbitrarily distributed variables, are based on Monte Carlo simulation, or on variations thereof aimed at achieving maximum computational efficiency. Although the preferred approach is to determine the proneness to failure by simply counting the number of simulations of a failed state, related to the total number of simulations, the possibility of fitting a probability distribution to the limit state function remains as an appealing alternative, which does not require having simulated observations of a failed state. One of the arguments against the last scheme is that it passes through the selection of a mathematical model for the probability distribution. This is a difficult and always sensitive issue, bound to have a perceptible influence on the final result. Herein a method is proposed that enables the direct evaluation of an upper bound for the proneness to failure, on the basis of the availability of any number of observations or realizations of the limit state function.

Note that in reliability assessments of engineering systems this is often all that is needed: the analyst is usually interested in verifying whether a given reliability threshold has been satisfied, not in an *exact* reliability measure. Moreover, it has been argued by the authors (Riera et al., 1995) that, on account of phenomenological uncertainties, the assessment of total proneness to failure of engineering systems smaller than about 1×10^{-7} is, more often than not, irrelevant in a decision making process, for instance, in deciding whether a system is safe enough.

2 Problem definition and basic assumptions

In most practical cases, failure of engineering systems may be defined in numerical terms. A single control variate can be chosen, or a mathematical function of a set of control variates can be adopted to quantitatively represent the state of failure, F. This representation can always be rearranged in order to fit the form:

$$F : \{X | X \leq 0\} \tag{1}$$

such that the proneness to failure results defined by:

$$P_F = \text{Prob}\{X \leq 0\} \tag{2}$$

where the continuous variate X bears a physical uncertainty, represented by its unknown probability density function $p_X(x)$ or its cumulative distribution function $P_X(x)$. Consequently:

$$P_F = \int_{-\infty}^{0} p_X(x)dx = P_X(0) \tag{3}$$

The reader acquainted with structural reliability analysis may call the variate X *the safety margin* (the difference between capacity and demand) or even in a more mathematical vein ***the outcome of a limit state function***. Indeed, a much broader meaning can be assigned to X, for it can be any measured or calculated control quantity related to a failure criteria.

From the considerations above, it can be readly concluded that the basic problem of reliability estimation is the solution of equation 2, which is possible only if $p_X(x)$ exists; the existence of this ***continuous*** probability density is the first basic assumption of this work. The second basic assumption is that, although $p_X(x)$ is unknown, a set of samples from X is available.

The purpose of this paper is to apply the concept of Chernov Bound to provide a conservative but sometimes useful estimation of P_f, which at the same time is simple enough to be adopted as a codified procedure. The main feature of the approach is the independence of an arbitrary choice for $p_X(x)$, which was found to have an undesirably strong effect on the final results.

3 Review of mathematical background

3.1 The moment generating function

The moment generating function, $G_X(s)$ of a random variate X is defined as the expected value of the function e^{sX}. Hence,

$$G_X(s) = \mathrm{E}\{e^{sX}\} = \int_{-\infty}^{+\infty} e^{sx} p_X(x)dx \tag{4}$$

The statistical moments of X can be obtained from $G_X(s)$ by observing that:

$$\frac{d^k G_X(s)}{ds^k} = \mathrm{E}\left\{X^k e^{sX}\right\} \tag{5}$$

and consequently:

$$M_{X,k} = \mathrm{E}\left\{X^k\right\} = \frac{d^k G_X(0)}{ds^k} \tag{6}$$

From the equations above, it can be concluded that the moment generating function can alternatively be defined by the moments of X, rather than by the probability density $p_X(x)$. This is possible by means of a MacLaurin expansion of $G_X(s)$ around the value $s = 0$:

$$G_X(s) = 1 + \sum_{k=1}^{\infty} \frac{s^k}{k!} M_{X,k} \tag{7}$$

The moment generating function plays a central role in the Chernov Bound definition. Equations 4 and 7 offer two alternative ways of estimating $G_X(s)$, as discussed in Section 5.

3.2 The Tchebychev's inequality

The Tchebychev's inequality states that, for any positive ϵ:

$$\mathrm{Prob}\left\{|X - \mu_X| \leq \epsilon\right\} \leq \left(\frac{\sigma_X}{\epsilon}\right)^2 \tag{8}$$

where μ_X and σ_X are the mean value and the standard deviation of X, respectively. By setting $\epsilon = \mu_X$ and defining $\delta_X = \sigma_X/\mu_X$, the coefficient of variation of X, eq. 8 provides a bound for the solution of eq. 2 as:

$$\mathrm{Prob}\left\{X \leq 0\right\} \leq \delta_X^2 \tag{9}$$

Unfortunately, this bound can be regarded as too conservative for meaningful applications, as exemplified in Section 4.

3.3 The Chernov Bound

The Chernov bound (Papoulis, 1984) can be seen as an improvement of the bound resulting from Tchebychev's inequality. But while the latter makes use only of the two first statistical moments of a random variate and leads to excessively conservative results, the former may consider as many moments as available — or even all of them at once, depending on the chosen estimator for the moment generating function — and yields values that can be truly useful for engineering purposes.

The derivation of Chernov bound starts by considering that, for any real valued α:

$$\mathrm{Prob}\left\{Y \geq \alpha\right\} = \int_{\alpha}^{+\infty} f_Y(y)\, dy \tag{10}$$

If now α is positive and large enough to satisfy the condition:

$$-\int_{-\infty}^{0} y f_Y(y)\, dy \leq \int_{0}^{\alpha} y f_Y(y)\, dy \tag{11}$$

then the following inequalities hold:

$$\mathrm{E}\{Y\} = \int_{-\infty}^{+\infty} y f_Y(y)\, dy \geq \int_{\alpha}^{+\infty} y f_Y(y)\, dy \geq \alpha \int_{\alpha}^{+\infty} f_Y(y)\, dy \tag{12}$$

which combined with eq. 10 leads to the general inequality:

$$\mathrm{Prob}\{Y \geq \alpha\} \leq \frac{\mathrm{E}\{Y\}}{\alpha} \tag{13}$$

By replacing $Y = e^{sX}$ and $\alpha = e^{s\beta}$ in eq. 13 it results:

$$\mathrm{Prob}\left\{e^{sX} \geq e^{s\beta}\right\} \leq e^{-s\beta} G_X(s) \tag{14}$$

But for $s \leq 0$ and $X \geq 0$ it is true that:

$$\mathrm{Prob}\left\{e^{sX} \geq e^{s\beta}\right\} = \mathrm{Prob}\{X \leq \beta\}, \quad s \leq 0 \tag{15}$$

and the so-called Chernov Bound is finally obtained as:

$$\mathrm{Prob}\{X \leq \beta\} \leq e^{-s\beta} G_X(s), \quad s \leq 0 \tag{16}$$

Considering now the inicial purpose of solving equation 2, the application of Chernov bound for $\beta = 0$ and non-positive values of s leads to:

$$\mathrm{Prob}\{X \leq 0\} \leq G_X(s), \quad s \leq 0 \tag{17}$$

where the smallest value of the moment generating function will provide the less conservative bound. The task of finding the minimum of $G_X(s)$ will be examined in the next sections.

Eq. 17 can be better understood with the aid of Fig. 1. It is seen that, if $s \leq 0$, then $e^{sx} \geq 1$ for $x \leq 0$. Consequently:

$$\mathrm{Prob}\{X \leq 0\} = \int_{-\infty}^{0} p_X(x)\, dx \leq \int_{-\infty}^{0} e^{sx} p_X(x)\, dx \leq \int_{-\infty}^{\infty} e^{sx} p_X(x)\, dx = G_X(s) \tag{18}$$

Hence, the surplus of Chernov bound is:

$$G_X(s) - P_F = \int_{-\infty}^{0} \left(e^{sx} - 1\right) p_X(x)\, dx + \int_{0}^{\infty} e^{sx} p_X(x)\, dx, \quad s \leq 0 \tag{19}$$

In this equation, the value of s can be tuned in order to minimize the right-hand side, in which the first term increases for decreasing s, while the second term decreases for decreasing s. This implies the existence of an optimum value that must be calculated according to the specific function $p_X(x)$.

4 Application to variates with given distribution

4.1 Application to a normally distributed variate

In order to allow a direct comparison of equations 9 with 17, and of both with an exact solution of equation 2, a normal (Gaussian) random variate X is used in the following. In this case, the probability density function is given by:

$$p_X(x) = \frac{1}{\sigma_X \sqrt{2\pi}} \exp\left[-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right] \tag{20}$$

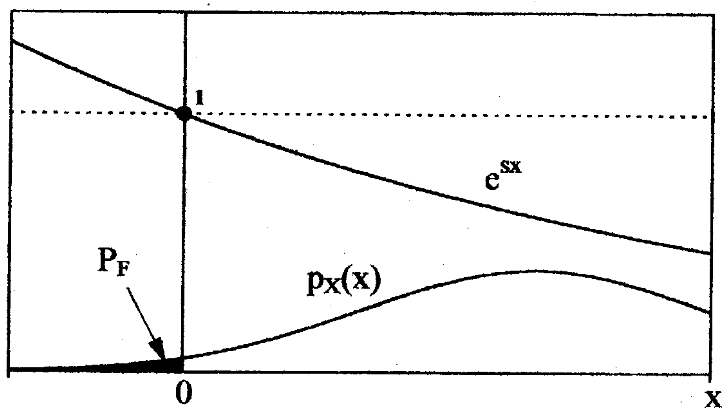

Figure 1: Graphical scheme for illustrating the Chernov Bound.

and the moment generating function has the closed form derived directly from equation 4 as:

$$G_X(s) = \exp\left(\mu_X s + \frac{\sigma_X^2 s^2}{2}\right) \tag{21}$$

It can be easily shown that this function has a minimum at $s = -\mu_X/\sigma_X^2$, which leads to:

$$[G_X(s)]_{\min} = \exp\left(-\frac{\mu_X^2}{2\sigma_X^2}\right) \tag{22}$$

For example, let $\mu_X = \beta$ and $\sigma_X = 1$. A comparison of results is given in Table 1, where $\Phi(x)$ is the standard Gaussian cumulative distribution function. Table 1 highlights the advantage of Chernov over Tchebychev bound. It is important to emphasize at this point that, for the initial purpose of reliability estimation, the difference of approximately one order of magnitude between the Chernov bound and the exact solution of equation 1 may be acceptable as acceptable in many practical applications.

4.2 Application to a uniformly distributed variate

Another instructive comparison can be performed by using a uniformly distributed random variate, such that:

$$p_X(x) = 1, \quad -a \leq x \leq 1 - a \tag{23}$$

where a may be a small positive real number that represents a shift to the left in the distribution. In this case the moment generating function can also be analitically derived as:

$$G_X(s) = \frac{1}{s}\left\{\exp\left[(1-a)s\right] - \exp\left(-as\right)\right\} \tag{24}$$

For small values of a, this function has a minimum approximately located at $s = -1/a$, which after replacing gives:

$$[G_X(s)]_{\min} = ae\left[1 - \exp\left(-1/a\right)\right] \tag{25}$$

Table 1: Comparison among Chernov and Tchebychev bounds with exact solutions for Prob $\{X \leq 0\}$, in the case where X is a Gaussian variate with $\mu_X = \beta$ and $\sigma_X = 1$.

β	Exact $\Phi(-\beta)$	Chernov $\exp(-\beta^2/2)$	Tchebychev $1/\beta^2$
3.090	1×10^{-3}	8.4×10^{-3}	105×10^{-3}
3.719	1×10^{-4}	9.9×10^{-4}	723×10^{-4}
4.265	1×10^{-5}	11.2×10^{-5}	5497×10^{-5}
4.753	1×10^{-6}	12.4×10^{-6}	44265×10^{-6}

Table 2: Comparison among Chernov and Tchebychev bounds with exact solutions for Prob $\{X \leq 0\}$, in the case where X is a variate uniformly distributed in the intervall $[-a, 1-a]$.

Exact a	Chernov $\approx ae$	Tchebychev $\left[\sqrt{12}(0.5-a)\right]^{-2}$
1×10^{-3}	2.7×10^{-3}	335×10^{-3}
1×10^{-4}	2.7×10^{-4}	3335×10^{-4}
1×10^{-5}	2.7×10^{-5}	33335×10^{-5}
1×10^{-6}	2.7×10^{-6}	333335×10^{-6}

A comparison of the bound provided by eq. 25 with Tchebychev bound is presented in Table 2. Which illustrates the clear superiority of Chernov bound. Note that here the error is not increasing for smaller failure probabilities. Furthermore, for $a \to 0$ the bound will also have zero as a limit.

5 Estimating the moment generating function

5.1 The direct method

It must be clear that the neat advantage of the Chernov bound stems out from our knowledge of the underlying distribution type, which is very rarely known in real engineering problems. However, it is possible to estimate $G_X(s)$ from a limited number of samples, as will be discussed in the following. Given now n samples $\{X_1, X_2, \ldots, X_n\}$ of a random variable X with probability density $p_X(x)$, an estimator for the expected value of the moment generating function $G_X(s)$ may be straightforwardly defined as:

$$\hat{G}_X(s) = \frac{1}{n} \sum_{i=1}^{n} \exp(sX_i) \tag{26}$$

In order to find a minimum for eq. 26, its first derivative is calculated and set equal to zero:

$$\sum_{i=1}^{n} X_i \exp(sX_i) = 0 \tag{27}$$

It is clearly seen that, if only positive samples of X are available, which is the most likely case in common practice, then there will be no value of s satisfying eq. 27. This means that:

$$\lim_{s \to -\infty} \hat{G}_X(s) = 0 \tag{28}$$

Therefore the choice of s leading to a safe bound must be carried out with the aid of an auxiliary criterion.

One of such criterion may be provided by the definition of an acceptable statistical error in the estimator $\hat{G}_X(s)$. It can be shown that this error increases with $|s|$, in the same way as the statistical error in the estimation of moments $M_{X,k}$ increases with the order k (see eq. 6).

Now, in order to estimate the variance of the estimator provided by eq. 26, the expected value and the variance of $\hat{G}_X(s)$ should be calculated as:

$$\mathrm{E}\left\{\hat{G}_X(s)\right\} = \frac{1}{n}\sum_{i=1}^{n} \mathrm{E}\{\exp(sX_i)\} \tag{29}$$

$$\mathrm{V}\left\{\hat{G}_X(s)\right\} = \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{V}\{\exp(sX_i)\} \tag{30}$$

where for any i:

$$\mathrm{E}\{\exp(sX_i)\} = \mathrm{E}\{\exp(sX)\} \tag{31}$$

$$\mathrm{V}\{\exp(sX_i)\} = \mathrm{E}\{\exp(2sX)\} - \mathrm{E}^2\{\exp(sX)\} \tag{32}$$

and hence:

$$\mathrm{E}\left\{\hat{G}_X(s)\right\} = G_X(s) \tag{33}$$

$$\mathrm{V}\left\{\hat{G}_X(s)\right\} = \frac{1}{n}\left[G_X(2s) - G_X^2(s)\right] \tag{34}$$

The coefficient of variation δ_G of the estimator $\hat{G}_X(s)$ is defined as:

$$\delta_G(s) = \frac{\mathrm{V}^{1/2}\left\{\hat{G}_X(s)\right\}}{\mathrm{E}\left\{\hat{G}_X(s)\right\}} \tag{35}$$

Combining equations 33, 34 and 35 and replacing the moment generating function by its estimator finally results in:

$$n\hat{\delta}_G^2(s) = \frac{\hat{G}_X(2s)}{\hat{G}_X^2(s)} - 1 \tag{36}$$

Eq. 36 can be used to choose the appropriate value of s in two different ways: (1) For a given number n of available samples, s is chosen in order to respect an acceptable coefficient of variation $\delta_G(s)$. (2) For a specified coefficient of variation $\delta_G(s)$, the number n of samples to be taken is chosen in order to reach the smallest possible value of $G_X(s)$.

It must be aknowledged that the estimator $\hat{\delta}_G$ has a statistical error even larger that $\hat{G}_X(s)$, for it makes use of values of $2s$. This fact, in view of eq. 28, may lead to difficulties in finding a truly reliable bound. The problem is overcome by combining this method of estimation with a second approach, presented in the following.

5.2 The moments method

The moment generating function can be alternativelly estimated through its McLaurin expansion, eq. 7. Each moment $M_{X,k}$ is estimated through eq. 7 as:

$$\hat{M}_{X,k}(m) = \frac{1}{n}\sum_{i=1}^{n} X_i^k \tag{37}$$

$$\hat{G}_X(s,m) = \frac{1}{n}\sum_{i=1}^{n}\sum_{k=0}^{m} \frac{s^k}{k!} X_i^k \tag{38}$$

where only a finite number m of moments, i.e. terms in the expansion can be evaluated. It is known that, if all samples X_i are positive and if the number m of considered moments is even, then eq. 38 has a minimum for $s < 0$. This minimum decreases as m increases, leading to the same problem already faced in the direct estimation of $G_X(s)$.

A criterium for specifying m may be proposed on the basis of the statistical error of $\hat{M}_{X,k}(m)$. Hence, the expected value and the variance are, respectivelly:

$$\mathrm{E}\left\{\hat{M}_{X,k}(m)\right\} = M_{X,k} \tag{39}$$

$$\mathrm{V}\left\{\hat{M}_{X,k}(m)\right\} = \frac{1}{n}\sum_{k=0}^{m}\left(M_{X,2k} - M_{X,k}^2\right) \tag{40}$$

and the coefficient of variation results:

$$\delta_M(m) = \frac{\mathrm{V}^{1/2}\left\{\hat{M}_{X,k}(m)\right\}}{\mathrm{E}\left\{\hat{M}_{X,k}(m)\right\}} \tag{41}$$

Intensive numerical experimentation has shown that the best results are achieved under the condition:

$$\delta_M(m_{0.95}) = 0.95, \qquad m_{0.95} < n \tag{42}$$

where m is taken as the largest even integer smaller than $m_{0.95}$. A mathematical justification of this criterium is still missing.

5.3 Example

Although application of eqs. 26 and 38 are straightforward, there are some important aspects of the estimation procedure that are now clarified by means of a brief numerical example. The estimators are applied to simulated samples of a Gaussian random variate X, with mean $\mu_X = 4.265$ and standard deviation $\sigma_X = 1$, leading to $\mathrm{Prob}\left\{X \leq 0\right\} = 1 \times 10^{-5}$. The theoretical bound is given by eq. 21, with a minimum $\left[G_X(s)\right]_{\min} = 1.1 \times 10^{-4}$, provided in Table 1.

It is observed that both estimators eqs. 26 and 38 are very sensitive to samples lying in the lower tail of $p_X(x)$, what can be understood by recalling Fig. 1. For this reason, if simulation is used to produce samples of X, the simulation technique must be accurate in generating extreme values. Hence, instead of using a random numbers generator with maximum period, like an IBM System / 360 (Rubinstein, 1981), a kind of ***numerical integration*** over the sample space should be carried

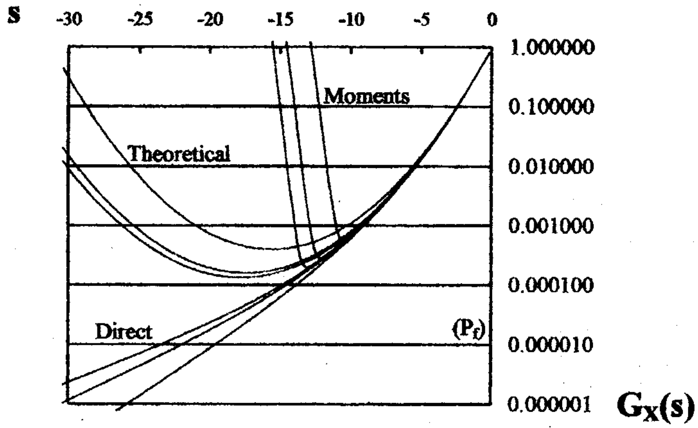

Figure 2: Comparison among theoretical and estimated moment generation function for the numerical example. Each set of three curves are obtained for 128, 512 and 1024 samples, respectively.

out. This requires the use of a generator with prescribed periodicity (Bourgund et. al. 1986), which provides random ordered samples in the form:

$$X_i = \mu_X + \sigma_X \Phi^{-1}\left[(i + 1/2)/(n+1\right], \quad i = 1, 2, \ldots, n \tag{43}$$

where n is the period and $\Phi^{-1}[\cdot]$ is the inverse standard Gaussian cumulative distribution. The results obtained for $n = 128$, 512 and 1024 are presented in Fig. 2.

Two important conclusions are drawn from this example: (1) If the variate X is a function $X = F(Y_1, Y_2, \ldots)$ of random variables Y_i with given distribution, the Chernov bound is better estimated by means of a fair numerical integration scheme aimed at a good representation of statistical moments and extreme values. Pure Monte Carlo simulation is not recommended, mainly in the case where only a small number of function calls is allowed. (2) If the samples X_i are obtained as measurements of a real quantity, the probability distribution may not be of any known type and the sensitivity presented by the Chernov bound to extreme values is consistent with the goal of the analysis.

6 Conclusions

The theoretical importance and potential practical usefulness of little known Chernov bound has been demonstrated in this paper. In practical reliability assessments through simulations or other numerical methods, the use of Chernov bound would free the analyst from often questionable assumptions concerning distribution models of the limit state function, or of the behavior of the latter at the tails.

A criterium is proposed for estimating the bound, which allows the verification of very low proneness to failure even in the case where a relatively small number of samples of the limit state function is available. It has been verified that optimum efficiency is achieved with the utilization of an specific numerical integration scheme, which is aimed at an accurate evaluation of statistical moments of the limit state function. This subject is presently under further investigations.

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