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3D Solid Incompressible Viscoelastic Finite Element in Large Strains for the Cornea. Refractive Surgery Application.

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Abstract

Several models[1,2,3,4] have been developed in order to reproduce the corneal behavior in ophthalmological procedures as tonometry, radial keratotomy and photokeratectomy with Excimer Laser. It has been found that the viscoelastic effect of a biological soft tissue, as the cornea, is negligible in tonometry[5]. Nevertheless, clinical studies on humans showed refractive changes with time in radial keratotomy. Wound healing is responsible for the long time effect (measured in years). On the other hand, the short-time effect (hours or days) has not been clarified.

In this work a 3D viscoelastic finite element model is developed taking into account incompressibility and large strains. An internal variable is introduced by means of a multiplicative decomposition of the deformation gradient. The final goal of the study is to determinate the importance of the viscoelastic effect in radial keratotomy.

1 Introduction

In order to simulate the viscoelastic effect in the cornea after a refractive procedure, as the radial keratotomy, we developed a three-dimensional incompressible viscoelastic finite element in large strains.

We follow the Le Tallec's approach [6,7] where a differential form is chosen which introduces an internal variable through a multiplicative decomposition of the deformation gradient. This model is chosen so that it was thermodinamically consistent, preserving incompressibility, and easy to solve as is the initial hyperelastic model.

The viscoelastic incompressibility constraints the admissible solution of the internal variable (five unknown instead of six unknown of the deformation tensor). This reduces the computational cost instead of a weak form of it.

The final mixed problem in three variables displacements-pressure-internal variable is reduced to an standard mixed problem displacements-pressure by eliminating the internal variable.

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2 Corneal Rheology

The cornea is a soft tissue composite layered material made up of collagen fibrils and ground substance. X-ray diffraction measurements made on wet rat tail tendon (which is composed primarily of collagen type I and ground substance) showed that 55% of the volume of the fiber is occupied by collagen molecules and the remaining 45% by water. Such information is available for enucleated human corneal stroma, but not for stroma in its normal hydrated state.
It is assumed that collagen fibers Poisson 'ratio is 0.40 and ground substance's Poisson 'ratio is 0.49. This result could change in the case of incisions made in the cornea where the material could swell.
Hanna et al.[2] used a finite element model to simulate the corneal lamella. They calculate the macroscopic constitutive properties from the experimental microscopic ones. They found that the fibrils filled approximately 35% of the total volume. They assumed that corneal shrinkage induced by fixation was 20% identical in both collagen and ground substance. They found similar values for radial Poisson's ratio (0.47), and moderate compressibility in the plane of the cornea (0.26). They found elastic modulus values of 11 MPa and 39 KPa for corneal plane and radial plane, respectively.
Kobayashi et al.[3] determined the viscoelastic response of intact cornea subjected to physiological intraocular pressure determined from local deformations.The tests results of five paired eyes showed that in prepressurized enucleated eyes, the viscoelastic response was insignificant while significant viscoelastic response existed in the nonpressurized eyes.
Seiler et al. [10] studied the viscoelastic properties of human corneal strips with and without Bowman's layer by relaxation measurements to determine the relative contribution of Bowman's layer to the biomechanical properties of the cornea. At a strain of 2%, the stress was measured to be (5.06±2.01)KPa and (4.72±1.30)KPa with and without Bowman's layer, respectively. They found fast relaxing time of 10.58 s and slow relaxing time of 269 for intact cornea.
Seiler et al. obtained the shear compliance spectra by applying dynamic mechanical shear compliance spectroscopy to different corneal tissues as a function of corneal hydration and temperature in the frequency range from 0.1 mHz to 100 Hz.
In this work we will try to simulate the viscoelastic behavior of the cornea described by this authors. We will also simulate the viscoelastic effect in the Refractive Surgery, in particular with arcuate incisions for the correction of astigmatism.

2.1 Viscoelastic constitutive law

2.1.1 Small Strain

When a viscoelastic material is subjected to a step loading, there exist both an instantaneous and a long term equilibrium response.The standard linear solid or Kelvin model is composed of combinations of linear springs with spring constant μ and dashpots with coefficient of viscosity ν . It is able to model the two responses in a simple way.
If we add the force exerted in the elastic branch with the force in the viscoelastic branch we have

$$f = \mu_0 e + \mu e_e$$

The force in the viscoelastic branch can be written as

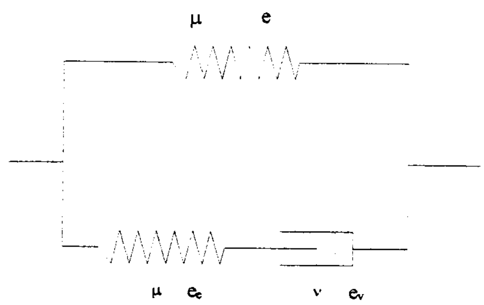


Figure 1: The standard linear solid or Kelvin model.

$$v\ \dot{e_v} = \mu e_e$$

We solve this linear rate equation obtaining the elongation response function (creep):

$$e(t) = e_e + e_v = \frac{f}{\mu_0}\left[1 + \left(\frac{\mu_0}{\mu + \mu_0} - 1\right) \mathrm{e}^{-t/\tau}\right]$$

where $E_0 = \mu + \mu_0$ measures the instantaneous elastic stiffness, $E_\infty = \mu_0$ measures the long term elastic stiffness, and $\tau = (\nu/\mu)\,[1 + (\mu/\mu_0)]$ is a characteristic relaxation time which indicates how long it takes for the material to reach its long term equilibrium response.

2.1.2 Finite Strains. Generalized Linear Model.

We generalize the above simple model to three-dimensional situations involving isochoric large deformations.
In the finite strains case, the right Cauchy-Green tensors C, C_e, and C_v measure the total deformation, the elastic part and the viscous part of the viscous branch, respectively. Variable C_v is an internal viscoelastic variable.
In analogy with the small strains case we assume a free energy potential of the form:

$$\Psi(C, C_v) = \Psi_0(C) + \Psi_e(C_e)$$

where Ψ_0 measures the stored energy of the elastic branch (long term behavior) and Ψ_e measures the stored energy of the viscous branch which, disappears in relaxation.
The intrinsic dissipation in the dashpot must satisfy the Clausius-Duhem inequality (2^0 law of thermodynamics),

$$\Phi(\dot{C_v}) : \dot{C_v} \geq 0$$

We choose,

$$\Phi(\dot{C_v}) = -\nu\ \dot{C_v^{-1}}$$

where ν is a symmetric definite-positive tensorial viscosity.
If the model is linear,

$$\Psi_e(C_e) = \mathrm{tr}(C_e K C_e)$$

assuming compressibility, we have as in small strains the linear rate equation:

$$\nu \dot{C}_v + K(C_v - C) = 0$$

$$C_v(t) = \frac{K}{\nu}\int_0^{-\infty} e^{-sK/\nu} C(t-s)ds$$

2.1.3 Finite Strains. Multiplicative decomposition of the deformation

In the finite strains case the additive decomposition of the deformation is equivalent to a multiplicative decomposition of the stretch $\lambda = \lambda^e \lambda^v$.
If we decompose $F = RU$ and $R = I$ then $U = U_e U_v = U_v U_e$. But if U_e and U_v are not coaxial then $U_e U_v \neq U_v U_e$ and this is not physically true in crystals. So the addtive decomposition of the deformation in finite strains is not physically true.
A better approach is taken into account. In crystals we were allowed to formally decompose the deformation at microscopic level into

$$\varphi = \varphi_e \circ \varphi_v$$

Therefore the deformation gradient and right Cauchy-Green strain tensor are

$$F = F_e F_v$$
$$C = F_v^T C_e F_v$$
$$J = J_e J_v$$

where $J = \det(F) > 0$. This idea is taken from plasticity where $F = F_e F_p$ [16].
So, by applying the first principle of thermodynamics in an isothermic process we have

$$work - free\ energy = dissipated\ heat$$

Then, in a reference configuration we have

$$\frac{1}{2} S : dC - d\Psi = \Phi(\dot{C}_v) : dC_v$$

where S is the symmetric Piola-Kirchhoff stress tensor. By introducing the isotropic components, the viscoelastic constitutive laws are

$$S = 2\frac{\partial \Psi(C, C_v)}{\partial C} - pC^{-1}$$
$$\det(C) = 1$$

$$\Phi(\dot{C}_v) = -\frac{\partial \Psi(C, C_v)}{\partial C_v} + qC_v^{-1}$$
$$\det(C_v) = 1$$

The first equation is a standard hyperelastic constitutive law with C_v as a constitutive parameter. The third equation is a first order differential equation in time where the variable C_v introduces the time dependence in the model. The other two equations are the incompressibility relations. If the material were elastically compressible the second relation would be dropped.

2.2 Equilibrium equations. Variational formulation

By writing the weak form of the equilibrium equations in a fixed reference configuration Ω, neglecting the body forces, considering $\bar{t}_p$ pressure external forces in $\partial\Omega$, and choosing an adequate dissipation form $(\Phi(\dot{C}_v) = -v\ \dot{\overline{C_v^{-1}}})$ we obtain the classical variational formulation together with the dissipative constitutive laws

$$\int_\Omega P : \text{GRAD}\ \eta\ dV = \int_{\partial\Omega} (\bar{t}_p \cdot \eta)\ dV$$

$$\int_\Omega q(\det F - 1) dV = 0$$

$$v\ \dot{\overline{C_v^{-1}}} - \frac{\partial \Psi(C, C_v)}{\partial C_v} + qC_v^{-1} = 0$$
$$\det(C_v) = 1$$

where $C_v(\cdot, t_0) = \textit{given\_value}$ with t_0 the initial time. We call the third equation as *dissipation equation.*

3 Approximation in space

3.1 Continuous problem

When the internal variable C_v is given the continuous problem reduces to a standard well-posed mixed problem where

$$\eta \in H = \{w \in W^{1,S}(\Omega; E), w|_{\Gamma_0} = 0\},$$

and

$$q \in P = L^{S_*}(\Omega; R),$$
$$\tfrac{1}{3S} + \tfrac{1}{S_*} = 1$$

The number $S \geq 1$ is such that the integrals in the weak form of the equilibrium equations make sense for any choice of u and η.For example $W^{1,S} = H^1$ and $L^{S*} = L^2$.
We will not use the mixed form of the incompressibility constraint $\det(C_v) = 1$. Instead of this we propose a constrained space of the internal variable such that

$$C_v \in A = \{A \in L^2(\Omega), \det(A) = 1\}$$

The incompressibility condition can be rewritten by developing the expression of the determinant in the third line:

$$a_{31} \operatorname{cof}_{31} A + a_{32} \operatorname{cof}_{32} A + a_{33} \operatorname{cof}_{33} A = 1$$

Since C_v is positive definite, the diagonal cofactors are different from zero. Then we can write

$$a_{33} = \frac{1 - a_{31} \operatorname{cof}_{31} A - a_{32} \operatorname{cof}_{32} A}{\operatorname{cof}_{33} A}$$

So in the dissipation equation we have five unknowns (the components of C_v, taking into account that is symmetric, except a_{33}, which is given by the above equation), and we have five equations since the six components of the dissipation equation are not independent because of the incompressibility condition. That is $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ where $a_6 = a_{33} = a_6(a_1, a_2, a_3, a_4, a_5)$.

4 Approximation in time

Since the main cost is associated with the solution of the first equation giving u as a function of the viscoelastic variable, either an explicit or an implicit scheme have a similar cost per time step.
We choose an implicit scheme because it is unconditionally stable. This is important in situations where the time scales are of different order of magnitude.
The Euler scheme is chosen; it is not second order accurate as the midpoint rule, but it requires less computer memory and has very nice stiff stability and long term convergence properties [6].
Let interval-time $\Delta t > 0$; for each iteration $n \geq 0$, we have to solve

$$\int_{\Omega_h} F_h^{n+1}\left(2\frac{\partial\Psi}{\partial C}(C_h^{n+1}, C_{vh}^{n+1}) - p_h^{n+1}(C^{n+1})^{-1}\right) : \eta_h \, dV = \int_{\delta\Omega} (\bar{t}_p \cdot \eta_h) \, dV$$

$$\int_{\Omega_h} q_h(\det F_h^{n+1} - 1) dV = 0$$

$$v\frac{(C_v^{n+1})^{-1} - (C_v^{n})^{-1}}{\Delta t} - \frac{\partial\Psi(C^{n+1}, C_v^{n+1})}{\partial C_v} - q(C_v^{n+1})^{-1} = 0$$

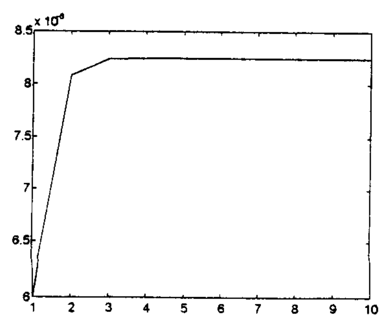
$$\det(C_v^{n+1}) = 1$$

with $C_v(\cdot, t_0) = C_v^0 = given\_value.$

5 Results

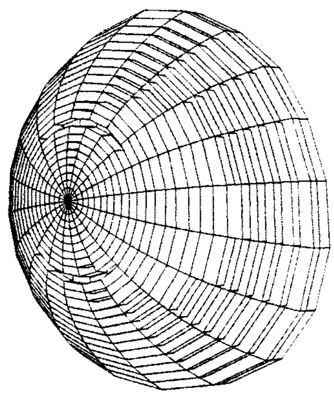
As a first test of this viscoelastic approach, we study the deformation of a rectangular strip in pure traction with

$$\begin{aligned} \Psi_e(C_e) &= \Psi_1(\operatorname{tr} C_e) + \Psi_2(\det C_e) \\ &= k_1 \operatorname{tr} C_e - k_1 \ln(\det C_e) \end{aligned}$$



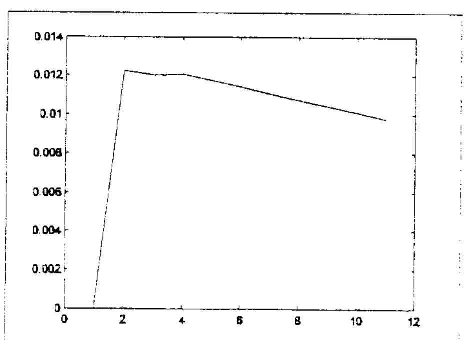
Time history of the displacements at the free edge of a strip.

We observed a time history of the displacements observed at the free edge of the strip. The finite element mesh consists of 80 elements of order 1 with 180 nodes. We observe a transition in time from a stiff instantaneous response to a softer long term behavior. The adimensional numerical values were $k_1 = 1e5, \nu = 0.0, E = 1e6, f = 100, \eta = 5e4$.
The next calculation studies the history of the deformation of a human eye after surgery. The finite element mesh consists of 2 layers of 23 rings of 20 elements in each layer. The boundary is fixed. The posterior layer is loaded by a constant pressure equivalent to the intraocular pressure.
The refractive surgery consists of 2 arcuate incisions of 60 degrees of arc in a radious of 3 mm from the apex of the cornea.



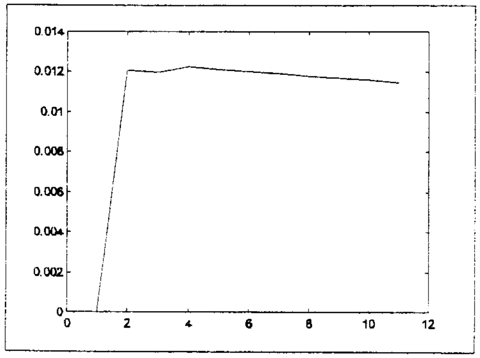
Corneal Mesh with 2 arcuate incisions.

In the next figure we show the time history of the displacement of the incision gape.



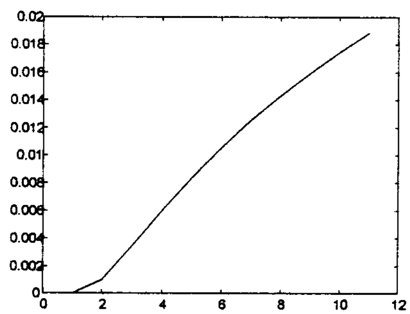
Time history of the displacement of the incision gape.

We show the time history of the edge displacement of the incision:



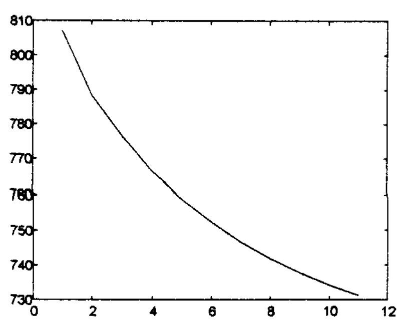
Time history of the edge displacement of the incision.

In this case we show the time history of the apex z-displacement.



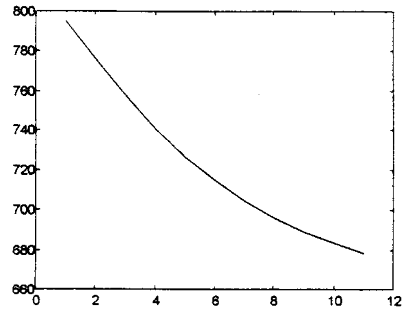
Time history of the apex z-displacement.

We observed that in any case the cornea increase the depth chamber leading to a myopic shift.
The keratometric curvature (max curvature vs min curvature at 3 mm from the apex) changed with the time. The time history of flattest axis (greater radious of curvature) is showed



The time history of flattest axis.

The time history of the steepest axis (lower radious of curvature) is showed



The time history of the steepest axis.

We observed a flatenning of the overall radious of curvature with the time as we expect from clinical results.

6 Conclusions

A viscoelastic finite element as proposed by Le Tallec has been implemented and applied to the solution of large strains displacements in the cornea after surgery.
The implementation was performed in an object oriented program for finite element analysis. Tasks as introduction of new constitutive laws have been greatly facilitated in this environment.
We observed the creep of the cornea in several ways as the displacement of the incision edge, the incision gape, the elevation of the apex, and the keratometric curvature.
The overall flattening in the steady case is in agreement with the fact that the deepest effect of the surgery is the instataneous response after surgery, and after some time (days, months...) the cornea tends to recover its old shape a little.
This is a preliminary result. More cases have to be simulated. An open question is about the viscoelastic parameters in refractive surgery.

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