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DOMAIN DECOMPOSITION FOR LINEAR EXTERIOR BOUNDARY VALUE PROBLEMS IN 2D ELASTICITY

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Abstract. In this paper we present new domain decomposition methods for solving linear exterior boundary value problems in elasticity. Our methods use a suitable Dirichlet to Neumann mapping which allows to transform the exterior problem into an equivalent boundary value problem in a bounded domain. Then, the use of Steklov–Poincaré operators and iterative solvers allows to obtain domain decomposition algorithms which can be naturally implemented on a parallel computing environment.

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1 INTRODUCTION

In this paper we present new domain decomposition methods for solving exterior boundary value problems in 2D elasticity. As a model problem we consider the exterior boundary value problem for the Lamé system. A rather complete survey on domain decomposition methods can be found in Quarteroni and Valli, (1999) and Smith et. al., (1996).

One of the main features of the domain decomposition methods that we present here is that they are based on the use of a suitable Dirichlet–to–Neumann mapping (DtN) to transform the exterior problem into an equivalent problem in a bounded domain. In what concerning the Laplace operator, the so–called uncoupling method can be used. This method allows to use a simple DtN mapping (see Gatica et. al, 1998) given just in terms of the hypersingular integral operator. However, in elasticity problems, the uncoupling method must be replaced for a more general class of DtN mappings (see Givoli, 1992; Givolli and Keller, 1989). Specifically, in this work we use the DtN mapping based on infinite Fourier series presented in Han and Wu, (1992). It is worth remarking that the idea of using Fourier series was first introduced in Feng, (1983).

The outline of the paper is as follows. In section 2, we present the method of Han and Wu for the numerical solution of the exterior boundary value problem for the Lamé system. Here, we present the DtN mapping, the variational formulation and also discuss about the error when considering a finite number of terms in the Fourier series. In section 3 we present the continuous Steklov–Poincaré operators related to our interface problem. In section 4 we show the finite dimensional approximations of the Steklov–Poicaré operators and finally, in section 5, we present some iteration–by–subdomain algorithms.

2 THE DTN MAPPING OF HAN AND WU

Let D be a bounded and simply connected domain in $\mathbf{R}^2$ with polygonal boundary Γ_D. Then, given $\mathbf{f} := (f_1, f_2) \in [L^2(\mathbf{R}^2 - \bar{D})]^2$ with compact support, $\mathbf{g}_0 \in [H^{1/2}(\Gamma_D)]^2$, the linear exterior boundary value problem reads: *Find* $\mathbf{u} := (u_1, u_2)$ *such that*

$$\begin{aligned} -\operatorname{div}\sigma(\mathbf{u}) &= \mathbf{f} \quad \text{in } \mathbf{R}^2 - \bar{D}\,, \\ \mathbf{u} &= \mathbf{g}_0 \quad \text{on } \Gamma_D\,, \\ \mathbf{u} &= \mathrm{O}(1)\,,\ \|x\| \to +\infty\,, \end{aligned} \tag{1}$$

where $\sigma(\mathbf{u}) := \lambda\,\mathbf{tr}\,e(\mathbf{u})\,\mathbf{I} + 2\,\mu\,e(\mathbf{u})$ is the stress tensor, $e(\mathbf{u})$ is the strain tensor given by $e_{ij}(\mathbf{u}) := \frac{1}{2}\{\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\}$, $i, j = 1, 2$, $\mathbf{tr}$ denotes the trace of a tensor, $\mathbf{I}$ denotes the identity tensor and λ, $\mu > 0$ are the Lamé constants.

Now, in order to transform (1), we introduce a circle Γ_N of radius R, centered at the origin, such that its interior region contains $\bar{D}$ and such that the support of $\mathbf{f}$ is contained in the annular region Ω bounded by Γ_D and Γ_N. Then, following Han and Wu, (1992), the exterior problem (1) can be transformed, equivalently, into the following boundary value problem in the bounded domain $\bar{\Omega}$: *Find* $\mathbf{u} \in [H^1(\Omega)]^2$ *such that*

$$\begin{aligned} -\operatorname{div}\sigma(\mathbf{u}) &= \mathbf{f} \quad \text{in } \Omega\,, \\ \mathbf{u} &= \mathbf{g}_0 \quad \text{on } \Gamma_D\,, \\ \sigma(\mathbf{u})\nu &= \tilde{\mathbf{T}}(\mathbf{u}) \quad \text{on } \Gamma_N\,, \end{aligned} \tag{2}$$

where $\tilde{\mathbf{T}} := (\tilde{T}_1, \tilde{T}_2) : [H^{1/2}(\Gamma_N)]^2 \to [H^{-1/2}(\Gamma_N)]^2$ is the DtN mapping defined by

$$\begin{aligned} \tilde{T}_1(\mathbf{v}) &:= \frac{2+2k}{1+2k}\,\frac{\mu}{\pi R}\sum_{n=1}^{\infty}\int_0^{2\pi}\frac{\partial^2 v_1}{\partial\phi^2}(R,\phi)\frac{\cos n(\theta-\phi)}{n}\,d\phi \\ &\quad -\frac{2k}{1+2k}\,\frac{\mu}{\pi R}\sum_{n=1}^{\infty}\int_0^{2\pi}\frac{\partial^2 v_2}{\partial\phi^2}(R,\phi)\frac{\sin n(\theta-\phi)}{n}\,d\phi\,, \end{aligned}$$

$$\begin{aligned} \tilde{T}_2(\mathbf{v}) &:= \frac{2+2k}{1+2k}\,\frac{\mu}{\pi R}\sum_{n=1}^{\infty}\int_0^{2\pi}\frac{\partial^2 v_2}{\partial\phi^2}(R,\phi)\frac{\cos n(\theta-\phi)}{n}\,d\phi \\ &\quad +\frac{2k}{1+2k}\,\frac{\mu}{\pi R}\sum_{n=1}^{\infty}\int_0^{2\pi}\frac{\partial^2 v_1}{\partial\phi^2}(R,\phi)\frac{\sin n(\theta-\phi)}{n}\,d\phi \end{aligned}$$

for all $x \in \Gamma_N$, $\mathbf{v} := (v_1, v_2)$, $v_i = v_i(r,\theta)$, $i = 1,2$, $x = r\cos\theta$, $y = r\sin\theta$, $k := \frac{\mu}{\lambda+\mu}$ and ν denotes the outward normal to Γ_N.

It is important to remark that the mapping $\tilde{\mathbf{T}}$ gives the exact boundary condition for the exterior problem and therefore, in practice, by taking just a finite number of terms on the Fourier series introduces and error that must be considered. Also, we note that $\tilde{\mathbf{T}}$ generates a boundary condition of nonlocal character.

In what follows, we consider approximations of the operator $\tilde{\mathbf{T}}$. This is, we define the sequence $\tilde{\mathbf{T}}^N := (\tilde{T}_1^N, \tilde{T}_2^N) : [H^{1/2}(\Gamma_N)]^2 \to [H^{-1/2}(\Gamma_N)]^2$, $N \in \mathbf{N} \cup \{0\}$, by

$$\begin{aligned} \tilde{T}_1^N(\mathbf{v}) &:= \frac{2+2k}{1+2k}\,\frac{\mu}{\pi R}\sum_{n=1}^{N}\int_0^{2\pi}\frac{\partial^2 v_1}{\partial\phi^2}(R,\phi)\frac{\cos n(\theta-\phi)}{n}\,d\phi \\ &\quad -\frac{2k}{1+2k}\,\frac{\mu}{\pi R}\sum_{n=1}^{N}\int_0^{2\pi}\frac{\partial^2 v_2}{\partial\phi^2}(R,\phi)\frac{\sin n(\theta-\phi)}{n}\,d\phi\,, \end{aligned}$$

$$\begin{aligned} \tilde{T}_2^N(\mathbf{v}) &:= \frac{2+2k}{1+2k}\,\frac{\mu}{\pi R}\sum_{n=1}^{N}\int_0^{2\pi}\frac{\partial^2 v_2}{\partial\phi^2}(R,\phi)\frac{\cos n(\theta-\phi)}{n}\,d\phi \\ &\quad +\frac{2k}{1+2k}\,\frac{\mu}{\pi R}\sum_{n=1}^{N}\int_0^{2\pi}\frac{\partial^2 v_1}{\partial\phi^2}(R,\phi)\frac{\sin n(\theta-\phi)}{n}\,d\phi\,. \end{aligned}$$

This sequence leads to define, for $N \in \mathbf{N} \cup \{0\}$, the following boundary value problems: *Find* $\mathbf{u}_N \in [H^1(\Omega)]^2$ *such that*

$$\begin{aligned} -\operatorname{div}\sigma(\mathbf{u}_N) &= \mathbf{f} \quad \text{in } \Omega\,, \\ \mathbf{u}_N &= \mathbf{g}_0 \quad \text{on } \Gamma_D\,, \\ \sigma(\mathbf{u}_N)\nu &= \tilde{\mathbf{T}}^N(\mathbf{u}_N) \quad \text{on } \Gamma_N\,. \end{aligned} \tag{3}$$

On the other hand, by using integration by parts, we obtain the following variational formulation of (2): *Find* $\mathbf{u} \in [H^1(\Omega)]^2$ *such that* $\mathbf{u} = \mathbf{g}_0$ *on* Γ_D *and*

$$\tilde{A}(\mathbf{u},\mathbf{v}) + B(\mathbf{u},\mathbf{v}) = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,dx \quad \forall\,\mathbf{v} := (v_1,v_2) \in [H^1_{\Gamma_D}(\Omega)]^2\,, \tag{4}$$

where

$$\tilde{A}(\mathbf{u},\mathbf{v}) := \int_\Omega \sigma(\mathbf{u}) : e(\mathbf{v})\, dx\,,$$

$$\sigma(\mathbf{u}) : e(\mathbf{v}) := \sum_{i,j=1}^{2} \sigma_{ij}(\mathbf{u}) e_{ij}(\mathbf{v})$$

and

$$\begin{aligned} B(\mathbf{u},\mathbf{v}) \; &:= \; \frac{2+2k}{1+2k}\,\frac{\mu}{\pi}\sum_{n=1}^{\infty}\int_0^{2\pi}\int_0^{2\pi}\left\{\frac{\partial u_1}{\partial\phi}(R,\phi)\frac{\partial v_1}{\partial\theta}(R,\theta)+\frac{\partial u_2}{\partial\phi}(R,\phi)\frac{\partial v_2}{\partial\theta}(R,\theta)\right\} \\ &\quad \times\frac{\cos n(\theta-\phi)}{n}\,d\phi\,d\theta \\ &+ \; \frac{2k}{1+2k}\,\frac{\mu}{\pi}\sum_{n=1}^{\infty}\int_0^{2\pi}\int_0^{2\pi}\left\{\frac{\partial u_1}{\partial\phi}(R,\phi)\frac{\partial v_2}{\partial\theta}(R,\theta)-\frac{\partial u_2}{\partial\phi}(R,\phi)\frac{\partial v_1}{\partial\theta}(R,\theta)\right\} \\ &\quad \times\frac{\sin n(\theta-\phi)}{n}\,d\phi\,d\theta\,. \end{aligned} \tag{5}$$

Analogously, by using integration by parts, we obtain the following variational formulation of (3): *Find* $\mathbf{u}_N \in [H^1(\Omega)]^2$ *such that* $\mathbf{u}_N = \mathbf{g}_0$ *on* Γ_D *and*

$$\tilde{A}(\mathbf{u}_N,\mathbf{v}) + B_N(\mathbf{u}_N,\mathbf{v}) = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,dx \quad \forall\,\mathbf{v}\in[H^1_{\Gamma_D}(\Omega)]^2\,, \tag{6}$$

where

$$\begin{aligned} B_N(\mathbf{u},\mathbf{v}) \; &:= \; \frac{2+2k}{1+2k}\frac{\mu}{\pi}\sum_{n=1}^{N}\int_0^{2\pi}\int_0^{2\pi}\left\{\frac{\partial u_1}{\partial\phi}(R,\phi)\frac{\partial v_1}{\partial\theta}(R,\theta)+\frac{\partial u_2}{\partial\phi}(R,\phi)\frac{\partial v_2}{\partial\theta}(R,\theta)\right\} \\ &\quad \times\frac{\cos n(\theta-\phi)}{n}\,d\phi\,d\theta \\ &+ \; \frac{2k}{1+2k}\,\frac{\mu}{\pi}\sum_{n=1}^{N}\int_0^{2\pi}\int_0^{2\pi}\left\{\frac{\partial u_1}{\partial\phi}(R,\phi)\frac{\partial v_2}{\partial\theta}(R,\theta)-\frac{\partial u_2}{\partial\phi}(R,\phi)\frac{\partial v_1}{\partial\theta}(R,\theta)\right\} \\ &\quad \times\frac{\sin n(\theta-\phi)}{n}\,d\phi\,d\theta\,. \end{aligned} \tag{7}$$

The following result is necessary to prove the uniqueness of solution for both variational formulations (4) and (6).

Lemma 1. The bilinear forms B and B_N, $N\in\mathbf{N}$, are symmetric and continuous on $[H^1(\Omega)]^2\times[H^1(\Omega)]^2$. Moreover, $B(\mathbf{v},\mathbf{v})\geq 0$ for all $\mathbf{v}\in[H^1_{\Gamma_D}(\Omega)]^2$ and $B_N(\mathbf{v},\mathbf{v})\geq 0$, $N\in\mathbf{N}$, for all $\mathbf{v}\in[H^1_{\Gamma_D}(\Omega)]^2$.

Proof. See Lemma 2 in Han and Wu, (1992).

Now, we note that $\tilde{A}$ is a continuous bilinear form on $[H^1(\Omega)]^2\times[H^1(\Omega)]^2$. Moreover, as a consecuence of Korn's inequality, it follows that $\tilde{A}$ is coercive on $[H^1_{\Gamma_D}(\Omega)]^2$. Then, by using Lemma 1, we deduce that both variational formulations (4) and (6) satisfy the hypothesis of the Lax–Milgram Lemma.

The following result allows to get control when using a finite number of terms in the the Fourier series that defines $\tilde{\mathbf{T}}$.

Theorem 1. There exists $C > 0$, independent of N, such that for all $l \geq 2$,

$$|B(\mathbf{u},\mathbf{v}) - B_N(\mathbf{u},\mathbf{v})| \leq \frac{C}{N^{l-1}} \|\mathbf{u}\|_{[H^{l-1/2}(\Gamma_N)]^2} \|\mathbf{v}\|_{[H^{1/2}(\Gamma_N)]^2}$$
$$\forall\, \mathbf{v} \in [H^1_{\Gamma_D}(\Omega)]^2$$

and

$$\|\mathbf{u} - \mathbf{u}_N\|_{[H^1(\Omega)]^2} \leq \frac{C}{\beta_0 N^{l-1}} \|\mathbf{u}\|_{[H^{l-1/2}(\Gamma_N)]^2}\,,$$

where β_0 is the coerciveness constant of $\tilde{A}$.

Proof. See Lemma 3 and Theorem 1 in Han and Wu, (1992).

Theorem 1 shows that, for solving (4), is enough to consider the problem (6) since the error $\|\mathbf{u} - \mathbf{u}_N\|_{[H^1(\Omega)]^2}$ is controlled by the number of terms in the Fourier series. In what follows, we develop a domain decomposition method for the exterior problem (1) by considering its approximation by (6).

3 STEKLOV–POINCARÉ OPERATORS

Let Γ be a polygonal and closed curve contained in Ω, such that its interior region contains $\bar{D}$ and such that it splits Ω into two subdomains, Ω_1 and Ω_2. This is, Ω_1 is the annular region bounded by Γ_D and Γ, Ω_2 by Γ and Γ_N and $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$. Also, we denote by ν_i, $i = 1, 2$, the outward normal to $\partial\Omega_i$ on Γ. With this, and given $\lambda \in [H^{1/2}(\Gamma)]^2$, we define the following boundary value problems:

- *Find* $\mathbf{u}_{N,1}(\lambda) \in [H^1(\Omega_1)]^2$ *such that*

$$\begin{aligned} -\operatorname{div}\sigma(\mathbf{u}_{N,1}(\lambda)) &= \mathbf{f} \quad \text{in } \Omega_1\,, \\ \mathbf{u}_{N,1}(\lambda) &= \mathbf{g}_0 \quad \text{on } \Gamma_D\,, \\ \mathbf{u}_{N,1}(\lambda) &= \lambda \quad \text{on } \Gamma\,. \end{aligned} \tag{8}$$

- *Find* $\mathbf{u}_{N,2}(\lambda) \in [H^1(\Omega_2)]^2$ *such that*

$$\begin{aligned} -\operatorname{div}\sigma(\mathbf{u}_{N,2}(\lambda)) &= \mathbf{f} \quad \text{in } \Omega_2\,, \\ \mathbf{u}_{N,2}(\lambda) &= \lambda \quad \text{on } \Gamma\,, \\ \sigma(\mathbf{u}_{N,2}(\lambda))\nu &= \tilde{\mathbf{T}}^N(\mathbf{u}_{N,2}(\lambda)) \quad \text{on } \Gamma_N\,. \end{aligned} \tag{9}$$

On the other hand, the continuity of the normal stresses on Γ leads to the following interface problem: *Find* $\bar{\lambda} \in [H^{1/2}(\Gamma)]^2$ *such that*

$$[\sigma(\mathbf{u}_{N,1}(\bar{\lambda}))\nu_1 + \sigma(\mathbf{u}_{N,2}(\bar{\lambda}))\nu_2, \mu] = 0 \quad \forall\, \mu \in [H^{1/2}(\Gamma)]^2\,, \tag{10}$$

where $[\cdot,\cdot]$ denotes the duality pairing between $[H^{-1/2}(\Gamma)]^2$ and $[H^{1/2}(\Gamma)]^2$ defined by means of the inner product on $[L^2(\Gamma)]^2$.

Now, we define linear extension operators $\mathbf{R}_i$, $i = 1, 2$, as follows

- $\mathbf{R}_1 : [H^{1/2}(\Gamma)]^2 \ni \lambda \to \mathbf{R}_1\lambda \in [H^1(\Omega_1)]^2$ as the solution of

$$\begin{aligned} -\operatorname{div}\sigma(\mathbf{R}_1\lambda) &= 0 \quad \text{in } \Omega_1\,, \\ \mathbf{R}_1\lambda &= 0 \quad \text{on } \Gamma_D\,, \\ \mathbf{R}_1\lambda &= \lambda \quad \text{on } \Gamma\,. \end{aligned} \tag{11}$$

- $\mathbf{R}_2 : [H^{1/2}(\Gamma)]^2 \ni \lambda \to \mathbf{R}_2\lambda \in [H^1(\Omega_2)]^2$ as the solution of

$$\begin{aligned} -\operatorname{div}\sigma(\mathbf{R}_2\lambda) &= 0 \quad \text{in } \Omega_2\,, \\ \mathbf{R}_2\lambda &= \lambda \quad \text{on } \Gamma\,, \\ \sigma(\mathbf{R}_2\lambda)\nu &= \tilde{\mathbf{T}}^N(\mathbf{R}_2\lambda) \quad \text{on } \Gamma_N\,. \end{aligned} \tag{12}$$

It is worth remarking that for any $\lambda \in [H^{1/2}(\Gamma)]^2$, $\mathbf{R}_1\lambda$ and $\mathbf{R}_2\lambda$ are, respectively, the harmonic extensions of λ to Ω_1 and Ω_2. Moreover, we define $\tilde{\mathbf{w}}_1 \in [H^1(\Omega_1)]^2$ and $\mathbf{w}_2 \in [H^1(\Omega_2)]^2$ as the unique weak solutions of the following boundary value problems:

- *Find* $\tilde{\mathbf{w}}_1 \in [H^1(\Omega_1)]^2$ *such that*

$$\begin{aligned} -\operatorname{div}\sigma(\tilde{\mathbf{w}}_1) &= \mathbf{f} \quad \text{in } \Omega_1\,, \\ \tilde{\mathbf{w}}_1 &= \mathbf{g}_0 \quad \text{on } \Gamma_D\,, \\ \tilde{\mathbf{w}}_1 &= 0 \quad \text{on } \Gamma\,. \end{aligned} \tag{13}$$

- *Find* $\mathbf{w}_2 \in [H^1(\Omega_2)]^2$ *such that*

$$\begin{aligned} -\operatorname{div}\sigma(\mathbf{w}_2) &= \mathbf{f} \quad \text{in } \Omega_2\,, \\ \mathbf{w}_2 &= 0 \quad \text{on } \Gamma\,, \\ \sigma(\mathbf{w}_2)\nu &= \tilde{\mathbf{T}}^N(\mathbf{w}_2) \quad \text{on } \Gamma_N\,. \end{aligned} \tag{14}$$

Also, we denote by $\mathbf{g} \in [H^1(\Omega)]^2$ a smooth extension of $\mathbf{g}_0$ on Ω, such that $\mathbf{g} := 0$ in $\Omega_2 \cup \Gamma$, and define $\mathbf{w}_1 := \tilde{\mathbf{w}}_1 - \mathbf{g}$. Therefore, it follows that,

$$\mathbf{u}_{N,1}(\lambda) = \mathbf{w}_1 + \mathbf{g} + \mathbf{R}_1\lambda \tag{15}$$

and

$$\mathbf{u}_{N,2}(\lambda) = \mathbf{w}_2 + \mathbf{R}_2\lambda\,. \tag{16}$$

Now, we introduce continuous and symmetric bilinear forms $A_1 : [H^1(\Omega_1)]^2 \times [H^1(\Omega_1)]^2 \to \mathbf{R}$, $A_2 : [H^1(\Omega_2)]^2 \times [H^1(\Omega_2)]^2 \to \mathbf{R}$ and $A : [H^1(\Omega)]^2 \times [H^1(\Omega)]^2 \to \mathbf{R}$, given by

$$A_1(\mathbf{z}_1, \mathbf{v}_1) := \int_{\Omega_1} \sigma(\mathbf{z}_1) : e(\mathbf{v}_1)\, dx\,,$$

$$A_2(\mathbf{z}_2, \mathbf{v}_2) := \int_{\Omega_2} \sigma(\mathbf{z}_2) : e(\mathbf{v}_2)\, dx \,+\, B_N(\mathbf{z}_2, \mathbf{v}_2)$$

and

$$A(\mathbf{z}, \mathbf{v}) := \sum_{k=1}^{2} A_k(\mathbf{z}_k, \mathbf{v}_k) \quad \forall\, \mathbf{z}, \mathbf{v} \in [H^1(\Omega)]^2\,,$$

where $\mathbf{z}_k := \mathbf{z}|_{\Omega_k}$ y $\mathbf{v}_k := \mathbf{v}|_{\Omega_k}$. Then, the unique solution $\mathbf{u}_N \in [H^1(\Omega)]^2$ of (6) satisfies $\mathbf{u}_N = \mathbf{g}_0$ on Γ_D and

$$A(\mathbf{u}_N, \mathbf{v}) = F(\mathbf{v}) \quad \forall\, \mathbf{v} \in [H^1_{\Gamma_D}(\Omega)]^2\,,$$

where

$$F(\mathbf{v}) := \int_{\Omega} \mathbf{f} \cdot \mathbf{v}\, dx\,.$$

Note that for any $\lambda \in [H^{1/2}(\Gamma)]^2$, the harmonic extensions $\mathbf{R}_i$, $i = 1, 2$, satisfy

$$A_1(\mathbf{R}_1\lambda, \varphi) = 0 \quad \forall\, \varphi \in [H_0^1(\Omega_1)]^2 \tag{17}$$

and

$$A_2(\mathbf{R}_2\lambda, \varphi) = 0 \quad \forall\, \varphi \in [H_\Gamma^1(\Omega_2)]^2\,, \tag{18}$$

where $H_\Gamma^1(\Omega_2) := \{v \in H^1(\Omega_2) : v = 0 \text{ on } \Gamma\}$.

Lemma 2. The solution $\mathbf{u}_N$ of (6) is related with $\mathbf{w}_1$ and $\mathbf{w}_2$ trough the following identities:

$$\mathbf{w}_1 = (\mathbf{u}_N - \mathbf{g})|_{\Omega_1} - \mathbf{R}_1((\mathbf{u}_N - \mathbf{g})|_\Gamma)$$

and

$$\mathbf{w}_2 = \mathbf{u}_N|_{\Omega_2} - \mathbf{R}_2(\mathbf{u}_N|_\Gamma)\,.$$

Proof. It is analogous to the proof of Lemma 3 in Gatica et. al., (1998).

In what follows, we define the functions

$$\mathbf{w} := \begin{cases} \mathbf{w}_1\,, & \text{en } \Omega_1 \cup \Gamma, \\ \mathbf{w}_2\,, & \text{en } \Omega_2 \cup \Gamma, \end{cases}$$

and

$$\mathbf{R}\lambda := \begin{cases} \mathbf{R}_1\lambda\,, & \text{en } \Omega_1 \cup \Gamma, \\ \mathbf{R}_2\lambda\,, & \text{en } \Omega_2 \cup \Gamma, \end{cases}$$

for all $\lambda \in [H^{1/2}(\Gamma)]^2$. Note that $\mathbf{w} \in [H^1(\Omega)]^2$, $\mathbf{w} = 0$ on $\Gamma_D \cup \Gamma$ and $\mathbf{R}\lambda \in [H^1_{\Gamma_D}(\Omega)]^2$. Then, by replacing (15) and (16) into (10), and using integration by parts, we deduce the Steklov–Poincaré problem: *Find* $\bar\lambda \in [H^{1/2}(\Gamma)]^2$ *such that*

$$\mathbf{S}\,\bar\lambda = \chi\,, \tag{19}$$

where $\mathbf{S} : [H^{1/2}(\Gamma)]^2 \to [H^{-1/2}(\Gamma)]^2$ is the Steklov–Poincaré operator given by

$$[\,\mathbf{S}\,\lambda, \mu\,] := \sum_{k=1}^{2} A_k\,(\mathbf{R}_k\lambda, \mathbf{R}_k\mu) \quad \forall\, \lambda,\, \mu \in [H^{1/2}(\Gamma)]^2$$

and $\chi \in [H^{-1/2}(\Gamma)]^2$ is defined as follows

$$[\,\chi, \mu\,] := \int_\Omega \mathbf{f} \cdot \mathbf{R}\mu\, dx \,-\, A(\mathbf{w} + \mathbf{g}, \mathbf{R}\mu) \quad \forall\, \mu \in [H^{1/2}(\Gamma)]^2\,.$$

The following result is necessary to prove existence and uniqueness of solution for the Steklov–Poincaré problem (19).

Lemma 3. There exist constants C_i, $\tilde C_i > 0$, $i = 1, 2$, depending only on Ω_i, such that

$$C_i\,\|\lambda\|_{[H^{1/2}(\Gamma)]^2} \le \|\mathbf{R}_i\lambda\|_{[H^1(\Omega_i)]^2} \le \tilde C_i\,\|\lambda\|_{[H^{1/2}(\Gamma)]^2} \quad \forall\, \lambda \in [H^{1/2}(\Gamma)]^2\,.$$

Proof. It follows from usual arguments for elliptic problems and the properties of the bilinear form B_N.

Lemma 4. There exists an unique $\bar{\lambda} \in [H^{1/2}(\Gamma)]^2$ such that $\mathbf{S}\,\bar{\lambda} = \chi$.

Proof. Since A_1 and A_2 are continuous bilinear forms, by using Lemma 3, it follows that $\mathbf{S}$ induces a continuous bilinear form on $[H^{1/2}(\Gamma)]^2 \times [H^{1/2}(\Gamma)]^2$. In what follows of the proof, $C > 0$ denotes a generic constant.

Now, in virtue of the Lemma 1, Korn's inequality and Lemma 3, we deduce that

$$\begin{aligned}
[\mathbf{S}\,\lambda, \lambda] &\geq C \sum_{k=1}^{2} \int_{\Omega_k} e(\mathbf{R}_k\lambda) : e(\mathbf{R}_k\lambda)\, dx \\
&= C \int_{\Omega} e(\mathbf{R}\lambda) : e(\mathbf{R}\lambda)\, dx \\
&\geq C\, \|\mathbf{R}\lambda\|^2_{[H^1(\Omega)]^2} \\
&= C \sum_{k=1}^{2} \|\mathbf{R}_k\lambda\|^2_{[H^1(\Omega_k)]^2} \\
&\geq C\, \|\lambda\|^2_{[H^{1/2}(\Gamma)]^2}
\end{aligned}$$

for all $\lambda \in [H^{1/2}(\Gamma)]^2$. Then, also $\mathbf{S}$ induces a bilinear form which is coercive on $[H^{1/2}(\Gamma)]^2$. Therefore, a direct application of the Lax–Milgram Lemma completes the proof.

The following result establishes the equivalence between the variational formulation (6) and the Steklov–Poincaré problem (19).

Theorem 2. Let $\mathbf{u}_N \in [H^1(\Omega)]^2$ and $\bar{\lambda} \in [H^{1/2}(\Gamma)]^2$ be the unique solutions of (6) and (19), respectively. Then,

$$\bar{\lambda} = \mathbf{u}_N|_\Gamma$$

and

$$\mathbf{u}_N = \mathbf{g} + \mathbf{w} + \mathbf{R}\bar{\lambda} = \begin{cases} \mathbf{u}_{N,1}(\bar{\lambda}), & \text{in } \Omega_1 \\ \mathbf{u}_{N,2}(\bar{\lambda}), & \text{in } \Omega_2\,. \end{cases}$$

Proof. It is analogous to the proof of Theorem 3 in Gatica et. al, (1998).

4 DISCRETE APPROXIMATIONS

We start by defining a triangulation of $\bar{\Omega}$ made up of straight and curves triangles in order to describe exactly the circle Γ_N. In fact, given $N \in \mathbf{N}$, let $0 = t_0 < t_1 \cdots < t_N = 2\pi$ be an uniform partition of $[0, 2\pi]$ with $t_{j+1} - t_j = \frac{2\pi}{N}$ for all $j = 0, N-1$. Also, we consider $z : [0, 2\pi] \to \Gamma_N$ as the parametrization of Γ_N defined by $z(t) := R\,(\cos t, \sin t)^T$ for all $t \in [0, 2\pi]$ and denote by $\Omega_{\tilde{h}}$ the annular region bounded by Γ_D and the polygonal curve $\Gamma_{N,\tilde{h}}$, with vertices $z(t_1), z(t_2), \ldots, z(t_n)$. Moreover, let $\mathcal{T}_{\tilde{h}}$ be a regular triangulation of $\Omega_{\tilde{h}}$, made up of triangles $\tilde{\tau}$ of diameter $h_{\tilde{\tau}}$, and define

$$\tilde{h} := \sup_{\tilde{\tau} \in \mathcal{T}_{\tilde{h}}} h_{\tilde{\tau}}\,.$$

Then, by replacing each triangle $\tilde{\tau} \in \mathcal{T}_{\tilde{h}}$ with a side on $\Gamma_{N,\tilde{h}}$, by the corresponding curve triangle on Γ_N, we get starting from $\mathcal{T}_{\tilde{h}}$, a triangulation $\mathcal{T}_h$ of $\bar{\Omega}$ made up of straight and curves triangles.

Let $\hat{\tau}$ be the reference triangle with vertices $\hat{P}_1 := (0,0)^T$, $\hat{P}_2 := (1,0)^T$ and $\hat{P}_3 := (0,1)^T$ and consider the family of surjective mappings $\{F_\tau\}_{\tau \in \mathcal{T}_h}$ such that $F_\tau(\hat{\tau}) = \tau$. This is, if τ is a straight triangle of $\mathcal{T}_h$, the mapping F_τ is given by

$$F_\tau(\hat{x}) := B_\tau \hat{x} + b_\tau \,, \tag{20}$$

where B_τ is a square matrix of order 2 and $b_\tau \in \mathbf{R}^2$. Now, if τ is a curve triangle with vertices P_1, P_2, P_3, such that $P_2 = z(t_{j-1}) \in \Gamma_N$ and $P_3 = z(t_j) \in \Gamma_N$, the mapping F_τ is as follows

$$F_\tau(\hat{x}) := B_\tau \hat{x} + b_\tau + G_\tau(\hat{x}) \quad \forall\, \hat{x} := (\hat{x}_1, \hat{x}_2) \in \hat{\tau}\,, \tag{21}$$

where

$$G_T(\hat{x}) :=$$

$$\frac{\hat{x}_1}{1-\hat{x}_2}\left\{z(t_{j-1} + \hat{x}_2(t_j - t_{j-1})) - [z(t_{j-1}) + \hat{x}_2(z(t_j) - z(t_{j-1}))]\right\} \,.$$

It can be shown that (21) is a C^∞*–difeomorfism* that takes the triangle $\hat{\tau}$ on the curve triangle τ in such a way that $F_\tau(\hat{P}_i) = P_i$, $i = 1, 3$ (see Zenizek, 1990). Moreover, the image of the side $\hat{P}_2\hat{P}_3$ is the curve side of τ and the other two sides of $\hat{\tau}$ are transformed linearly by (21) to the straight sides of τ.

Let $P_1(\hat{\tau})$ be the space of polynomials of degree ≤ 1 defined on $\hat{\tau}$. Then, for each triangle $\tau \in \mathcal{T}_h$, we define

$$P_1(\tau) := \{v : v = (JF_\tau)^{-1}(DF_\tau)\, \hat{v} \circ F_\tau^{-1} \quad \forall\, \hat{v} \in P_1(\hat{\tau})\}\,,$$

where JF_τ and DF_τ denote, respectively, the Jacobian and the Frêchet derivative of the mapping F_τ. With this, we define the finite element subspaces

$$\tilde{H}_{i,h} := \{v_h \in C(\bar{\Omega}_i) : v_h|_\tau \in P_1(\tau) \quad \forall\, \tau \subseteq \bar{\Omega}_i\}$$

and

$$\tilde{\Lambda}_h := \{v_h|_\Gamma : v_h \in \tilde{H}_{i,h}\}\,.$$

Moreover, we define

$$\tilde{H}^0_{1,h} := \{v_h \in \tilde{H}_{1,h} : v_h = 0 \text{ on } \Gamma_D \cup \Gamma\}\,,$$
$$\tilde{H}^0_{2,h} := \{v_h \in \tilde{H}_{2,h} : v_h = 0 \text{ on } \Gamma\}$$

and

$$H_{i,h} := \tilde{H}_{i,h} \times \tilde{H}_{i,h}\,,$$
$$\Lambda_h := \tilde{\Lambda}_h \times \tilde{\Lambda}_h\,,$$
$$H^0_{i,h} := \tilde{H}^0_{i,h} \times \tilde{H}^0_{i,h}\,.$$

Consequently, in virtue of the relations (17) and (18), we define, for all $\lambda_h \in \Lambda_h$, the discrete extensions $\mathbf{R}_{i,h}$, $i = 1, 2$, as follows

- $\mathbf{R}_{1,h} : \Lambda_h \ni \lambda_h \to \mathbf{R}_{1,h}\lambda_h \in H_{1,h}$ as the solution of

$$\begin{aligned} A_1\left(\mathbf{R}_{1,h}\lambda_h, v_h\right) &= 0 \quad \forall\, v_h \in H^0_{1,h}\,, \\ \mathbf{R}_{1,h}\lambda_h &= 0 \quad \text{on } \Gamma_D\,, \\ \mathbf{R}_{1,h}\lambda_h &= \lambda_h \quad \text{on } \Gamma\,. \end{aligned}$$

- $\mathbf{R}_{2,h} : \Lambda_h \ni \lambda_h \rightarrow \mathbf{R}_{2,h}\lambda_h \in H_{2,h}$ as the solution of

$$\begin{aligned} A_2\left(\mathbf{R}_{2,h}\lambda_h, v_h\right) &= 0 \quad \forall\, v_h \in H^0_{2,h}\,, \\ \mathbf{R}_{2,h}\lambda_h &= \lambda_h \quad \text{on } \Gamma\,. \end{aligned}$$

With this, we define the discrete Steklov–Poincaré operator by

$$\mathbf{S}_h := \mathbf{S}_{1,h} + \mathbf{S}_{2,h}\,,$$

where $\mathbf{S}_{i,h} : \Lambda_h \rightarrow \Lambda_h^*$, $i = 1, 2$, are the local discrete operators given by

$$[\mathbf{S}_{i,h}\lambda_h, \mu_h] := A_i(\mathbf{R}_{i,h}\lambda_h, \mathbf{R}_{i,h}\mu_h) \quad \forall\, \lambda_h, \mu_h \in \Lambda_h\,,$$

where Λ_h^* denotes the dual of Λ_h.

On the other hand, the numerical solving of (19) by means of iterative solvers needs a suitable preconditioner. This preconditioner needs to ensure that the convergence of the iteration–by–subdomain algorithm must be independent of h. To do this, we consider the *Dirichlet–Robin* type preconditioner given by $\mathbf{P}_h := \mathbf{S}_{2,h} + \mathbf{T}$, where

$$[\mathbf{T}\lambda_h, \mu_h] := c\,[\lambda_h, \mu_h] = c \int_\Gamma \lambda_h \cdot \mu_h \, ds \quad \forall\, \lambda_h, \mu_h \in \Lambda_h$$

and $c > 0$ is an arbitrary constant. In what follows, we show that κ, the spectral condition number of $\mathbf{P}_h^{-1}\,\mathbf{S}_h$, is bounded independently of h. To do this, we need some previous results.

Theorem 3. There exist constants C_1, $C_2 > 0$, independent of h, such that

$$A_1(\mathbf{R}_{1,h}\lambda_h, \mathbf{R}_{1,h}\lambda_h) \leq C_1\, \|\lambda_h\|^2_{[H^{1/2}(\Gamma)]^2} \quad \forall\, \lambda_h \in \Lambda_h$$

and

$$A_2(\mathbf{R}_{2,h}\lambda_h, \mathbf{R}_{2,h}\lambda_h) \leq C_2\, \|\lambda_h\|^2_{[H^{1/2}(\Gamma)]^2} \quad \forall\, \lambda_h \in \Lambda_h\,.$$

Proof. We note first that

$$A_1(\mathbf{R}_{1,h}\lambda_h, \mathbf{R}_{1,h}\lambda_h) \leq C\, \|\mathbf{R}_{1,h}\lambda_h\|^2_{[H^1(\Omega_1)]^2} \quad \forall\, \lambda_h \in \Lambda_h\,, \tag{22}$$

where $C > 0$ depends only on the Lamé constants. In what follows of the proof, $C > 0$ denotes a generic constant independent of h. Now, since Ω_1 is a polygonal domain and $\lambda_h \in [H^1(\Gamma)]^2$, we have the following regularity estimate (see Lions and Magenes, 1972)

$$\|\mathbf{R}_1\lambda_h\|_{[H^{3/2}(\Omega_1)]^2} \leq C\, \|\lambda_h\|_{[H^1(\Gamma)]^2}\,. \tag{23}$$

Also, by using usual interpolation estimates (see Ciarlet, 1978), it follows that

$$\|\mathbf{R}_1\lambda_h - \mathbf{R}_{1,h}\lambda_h\|_{[H^1(\Omega_1)]^2} \leq C\, h^{1/2}\, \|\mathbf{R}_1\lambda_h\|_{[H^{3/2}(\Omega_1)]^2}\,. \tag{24}$$

Moreover, we have the inverse inequality (see Ciarlet, 1978),

$$h^{1/2}\, \|\lambda_h\|_{[H^1(\Gamma)]^2} \leq C\, \|\lambda_h\|_{[H^{1/2}(\Gamma)]^2} \quad \forall\, \lambda_h \in \Lambda_h\,. \tag{25}$$

Then, by using triangle inequality, Lemma 3, (23) and (24),

$$\begin{aligned} \|\mathbf{R}_{1,h}\lambda_h\|_{[H^1(\Omega_1)]^2} &\leq \|\mathbf{R}_1\lambda_h - \mathbf{R}_{1,h}\lambda_h\|_{[H^1(\Omega_1)]^2} + \|\mathbf{R}_1\lambda_h\|_{[H^1(\Omega_1)]^2} \\ &\leq C\,h^{1/2}\,\|\mathbf{R}_1\lambda_h\|_{[H^{3/2}(\Omega_1)]^2} + \|\mathbf{R}_1\lambda_h\|_{[H^1(\Omega_1)]^2} \\ &\leq C\,h^{1/2}\,\|\lambda_h\|_{[H^1(\Gamma)]^2} + C\,\|\lambda_h\|_{[H^{1/2}(\Gamma)]^2}\,. \end{aligned}$$

Therefore, in virtue of (25), we deduce that

$$\begin{aligned} \|\mathbf{R}_{1,h}\lambda_h\|_{[H^1(\Omega_1)]^2} &\leq C\,h^{1/2}\,h^{-1/2}\,\|\lambda_h\|_{[H^{1/2}(\Gamma)]^2} + C\,\|\lambda_h\|_{[H^{1/2}(\Gamma)]^2} \\ &\leq C\,\|\lambda_h\|_{[H^{1/2}(\Gamma)]^2} \end{aligned}$$

which, together with (22), completes the first part of the proof.

On the other hand, by using Lemma 1, we have that

$$A_2(\mathbf{R}_{2,h}\lambda_h, \mathbf{R}_{2,h}\lambda_h) \leq C\,\|\mathbf{R}_{2,h}\lambda_h\|^2_{[H^1(\Omega_2)]^2} \quad \forall\,\lambda_h \in \Lambda_h \tag{26}$$

and therefore, analogous calculations complete the proof.

Theorem 4. There exist constants C_1, $C_2 > 0$, independent of h, such that

$$C_1\,\{[\mathbf{S}_{2,h}\lambda_h, \lambda_h] + [\mathbf{T}\lambda_h, \lambda_h]\} \leq [\mathbf{S}_{1,h}\lambda_h, \lambda_h] \leq C_2\,\{[\mathbf{S}_{2,h}\lambda_h, \lambda_h] + [\mathbf{T}\lambda_h, \lambda_h]\}$$

for all $\lambda_h \in \Lambda_h$.

Proof. In virtue of the trace Theorem, the coerciveness of A_1, and denoting by $C > 0$ a generic constant independent of h, we deduce that

$$\begin{aligned} \|\lambda_h\|^2_{[H^{1/2}(\Gamma)]^2} &\leq \|\mathbf{R}_{1,h}\lambda_h\|^2_{[H^1(\Omega_1)]^2} \\ &\leq C\,A_1(\mathbf{R}_{1,h}\lambda_h, \mathbf{R}_{1,h}\lambda_h) \\ &= C\,[\mathbf{S}_{1,h}\lambda_h, \lambda_h] \end{aligned}$$

for all $\lambda_h \in \Lambda_h$. Then, by using Theorem 3, we conclude the existence of constants $\hat{C}_1$, $\hat{C}_2 > 0$, independent of h, such that

$$\hat{C}_1\,\|\lambda_h\|^2_{[H^{1/2}(\Gamma)]^2} \leq [\mathbf{S}_{1,h}\lambda_h, \lambda_h] \leq \hat{C}_2\,\|\lambda_h\|^2_{[H^{1/2}(\Gamma)]^2} \quad \forall\,\lambda_h \in \Lambda_h\,. \tag{27}$$

On the other hand, by using Lemma 1,

$$\begin{aligned} [\mathbf{S}_{2,h}\lambda_h, \lambda_h] + [\mathbf{T}\lambda_h, \lambda_h] &= \int_{\Omega_2} \sigma(\mathbf{R}_{2,h}\lambda_h) : e(\mathbf{R}_{2,h}\lambda_h)\,dx \\ &\quad + B_N(\mathbf{R}_{2,h}\lambda_h, \mathbf{R}_{2,h}\lambda_h) + [\mathbf{T}(\mathbf{R}_{2,h}\lambda_h), \mathbf{R}_{2,h}\lambda_h] \\ &\geq C\,\left\{|e(\mathbf{R}_{2,h}\lambda_h)|^2_{0,\Omega_2} + \|\mathbf{R}_{2,h}\lambda_h\|^2_{[L^2(\Gamma)]^2}\right\}\,, \end{aligned} \tag{28}$$

where

$$|e(\mathbf{R}_{2,h}\lambda_h)|^2_{0,\Omega_2} := \sum_{i,j=1}^{2} \|e_{ij}(\mathbf{R}_{2,h}\lambda_h)\|^2_{L^2(\Omega_2)}\,.$$

This induces to define the linear and continuous operator $\mathbf{C} : [H^1(\Omega_2)]^2 \to [[H^1(\Omega_2)]^2]^*$ given by

$$[[\mathbf{C}\,\mathbf{u}, \mathbf{v}]] := \int_{\Omega_2} e(\mathbf{u}) : e(\mathbf{v})\,dx + \int_\Gamma \mathbf{u}\cdot\mathbf{v}\,ds \quad \forall\,\mathbf{u}, \mathbf{v} \in [H^1(\Omega_2)]^2\,,$$

where $[[\cdot,\cdot]]$ denotes the duality pairing between $[[H^1(\Omega_2)]^2]^*$ and $[H^1(\Omega_2)]^2$. In what follows, we prove that the operator $\mathbf{C}$ is coercive on $[H^1(\Omega_2)]^2$.

In fact, we note first that $[[\mathbf{C}\,\mathbf{v},\mathbf{v}]] \geq 0$ for all $\mathbf{v} \in [H^1(\Omega_2)]^2$. Also, $[[\mathbf{C}\,\mathbf{v},\mathbf{v}]] = 0$ implies $\mathbf{v} = 0$. In fact,

$$[[\mathbf{C}\,\mathbf{v},\mathbf{v}]] = |e(\mathbf{v})|^2_{0,\Omega_2} + \|\mathbf{v}\|^2_{[L^2(\Gamma)]^2} = 0$$

shows that $\mathbf{v} \in [H^1(\Omega_2)]^2$ is a rigid body motion with $\mathbf{v} = 0$ on Γ. Moreover, by using Korn's inequality (see Ciarlet, 1988),

$$\begin{aligned}[][\mathbf{C}\,\mathbf{v},\mathbf{v}]] &\geq |e(\mathbf{v})|^2_{0,\Omega_2} \\ &\geq C\,\|\mathbf{v}\|^2_{[H^1(\Omega_2)]^2} - \|\mathbf{v}\|^2_{[L^2(\Omega_2)]^2}\end{aligned}$$

for all $\mathbf{v} \in [H^1(\Omega_2)]^2$, which shows that $\mathbf{C}$ satisfies the Gårding inequality.

The above analysis shows that $\mathbf{C}$ is coercive on $[H^1(\Omega_2)]^2$. Therefore, by using (28), the coerciveness of $\mathbf{C}$ and the trace Theorem, it follows that

$$\begin{aligned}[][\mathbf{S}_{2,h}\lambda_h,\lambda_h] + [\mathbf{T}\lambda_h,\lambda_h] &\geq C\left\{|e(\mathbf{R}_{2,h}\lambda_h)|^2_{0,\Omega_2} + \|\mathbf{R}_{2,h}\lambda_h\|^2_{L^2(\Gamma)}\right\} \\ &= C\,[[\mathbf{C}(\mathbf{R}_{2,h}\lambda_h),\mathbf{R}_{2,h}\lambda_h]] \\ &\geq C\,\|\mathbf{R}_{2,h}\lambda_h\|^2_{[H^1(\Omega_2)]^2} \\ &\geq C\,\|\lambda_h\|^2_{[H^{1/2}(\Gamma)]^2}\end{aligned} \tag{29}$$

for all $\lambda_h \in \Lambda_h$.

Now, in virtue of Theorem 3 and the fact that $[\mathbf{T}\lambda_h,\lambda_h] = c\,\|\lambda_h\|^2_{[L^2(\Gamma)]^2} \leq c\,\|\lambda_h\|^2_{[H^{1/2}(\Gamma)]^2}$, it follows that

$$[\mathbf{S}_{2,h}\lambda_h,\lambda_h] + [\mathbf{T}\lambda_h,\lambda_h] \leq C\,\|\lambda_h\|^2_{[H^{1/2}(\Gamma)]^2} \quad \forall\,\lambda_h \in \Lambda_h\,. \tag{30}$$

Then, by using (29) and (30), we deduce the existence of constants $\tilde{C}_1, \tilde{C}_2 > 0$, independent of h, such that

$$\tilde{C}_1\,\|\lambda_h\|_{[H^{1/2}(\Gamma)]^2} \leq [\mathbf{S}_{2,h}\lambda_h,\lambda_h] + [\mathbf{T}\lambda_h,\lambda_h] \leq \tilde{C}_2\,\|\lambda_h\|_{[H^{1/2}(\Gamma)]^2} \tag{31}$$

for all $\lambda_h \in \Lambda_h$. Finally, by combining (27) and (31) the result follows.

The following Theorem shows that the spectral condition number of $\mathbf{P}_h^{-1}\mathbf{S}_h$ is independent of h.

Theorem 5. There exists $C > 0$, independent of h, such that $\kappa \leq C$.

Proof. By using Theorem 4, the proof is identical to the proof of Theorem 3 in Gatica et. al, (1988).

5 ITERATION–BY–SUBDOMAIN ALGORITHMS

In this section we deduce the corresponding iteration–by–subdomain algorithms related to the Steklov–Poincaré problem (19). To do this, we present the Richardson iteration method and the conjugate gradient method with preconditioning (PCGM). In fact, we consider the discrete problem: *Find* $\lambda_h \in \Lambda_h$ *such that*

$$\mathbf{S}_h\lambda_h = \chi_h\,, \tag{32}$$

where $\mathbf{S}_h : \Lambda_h \rightarrow \Lambda_h^*$ is the discrete Steklov–Poincaré operator, $\chi_h \in \Lambda_h^*$ and Λ_h is the discrete trace space.

Then, given a preconditioner $\mathbf{P}_h : \Lambda_h \to \Lambda_h^*$, a relaxation parameter $\alpha > 0$ and an initial guess $\lambda_h^0 \in \Lambda_h$, the Richardson method generates the following sequence to find the solution $\lambda_h \in \Lambda_h$ of (32):

$$\lambda_h^{n+1} = \lambda_h^n + \alpha\, \mathbf{P}_h^{-1}(\chi_h - \mathbf{S}_h \lambda_h^n) \quad \forall\, n \in \mathbf{N} \cup \{0\}\,. \tag{33}$$

Note that, given a small enough parameter α, the invertibility of $\mathbf{P}_h$ and $\mathbf{S}_h$ ensures the convergence of (33). However, from the computational point of view, it is necessary to show that this convergence is independent of h. Indeed, this is true if we can prove that κ, the spectral condition number of $\mathbf{P}_h^{-1}\mathbf{S}_h$, is bounded independently of h (see Quarteroni and Valli, 1999, p. 126). Therefore, in virtue of Theorem 5, we deduce that the use of (33) allows to obtain iteration–by–subdomain algorithms that are independent of h.

It is worth mentioning that the optimum parameter α, i.e., that minimizes the spectral radius of the iterative matrix $(\mathbf{I} - \alpha\, \mathbf{P}_h^{-1}\mathbf{S}_h)$, is given by $\alpha_{opt} := 2/(\sigma_1 + \sigma_m)$ and note that in this case, the spectral radius of $(\mathbf{I} - \alpha_{opt}\, \mathbf{P}_h^{-1}\mathbf{S}_h)$, well known as the error reduction rate, is $(\kappa - 1)/(\kappa + 1)$ (Quarteroni and Valli, 1999, p. 127).

On the other hand, in order to accelerate the convergence, usually PCGM is used instead Richardson. However, PCGM is only applicable to symmetric and positive definite systems. Following Concus, Golub and O'Leary (see Goluv and Van Loan, 1983, p. 540), PCGM can be written in the following way:

(**0**) Choose an initial guess $\lambda_h^0 \in \Lambda_h$.

(**1**) Solve

$$\begin{aligned} r^0 &= \chi_h - \mathbf{S}_h \lambda_h^0\,, \\ z^0 &= \mathbf{P}_h^{-1} r^0\,, \\ \alpha_0 &= \frac{[\mathbf{P}_h z^0, z^0]}{[\mathbf{S}_h z^0, z^0]}\,, \\ \lambda_h^1 &= \lambda_h^0 + \alpha_0\, z^0\,, \\ w_1 &= 1\,, \\ n &= 1\,. \end{aligned}$$

(**2**) Solve

$$\begin{aligned} \mathbf{P}_h z^n = r^n &:= \chi_h - \mathbf{S}_h \lambda_h^n\,, \\ \alpha_n &= \frac{[\mathbf{P}_h z^n, z^n]}{[\mathbf{S}_h z^n, z^n]}\,, \\ \frac{1}{w_{n+1}} &= 1 - \frac{\alpha_n}{\alpha_{n-1}} \frac{[\mathbf{P}_h z^n, z^n]}{[\mathbf{P}_h z^{n-1}, z^{n-1}]} \frac{1}{w_n}\,, \\ \lambda_h^{n+1} &= \lambda_h^{n-1} + w_{n+1}(\alpha_n z^n + \lambda_h^n - \lambda_h^{n-1})\,. \end{aligned}$$

(**3**) If any specified stopping criteria is not satisfied, set $n = n + 1$ and go to (**2**).

In what concerning this method, we have the following estimate in energy norm (see Quarteroni and Valli, 1999, p. 127),

$$\|\lambda_h - \lambda_h^n\|_{\mathbf{S}_h} \leq 2 \left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1} \right)^n \|\lambda_h - \lambda_h^0\|_{\mathbf{S}_h} .$$

The above estimate shows that the use of PCGM to solve the discrete Steklov–Poincaré problem leads to an algorithm with a convergence independent of h.

Strictly speaking, the iteration–by–subdomain algorithms must be written in terms of the bilinear forms associated to the discrete problems generated either by the Richardson method or PCGM. However, in order to see exactly what kind of problems must be solved, we use the continuous notation. Then, it is very important to note that the discretization of the corresponding boundary value problems must be consistent with the discrete analysis already presented in this paper.

To deduce the iteration–by–subdomain algorithms generated by the Richardson method, we first start with two subdomains and then we extend to an arbitrary number of subdomains. In fact, by replacing $\mathbf{P} = \mathbf{S}_2 + \mathbf{T}$ in the continuous version of (33), it follows that

$$\lambda^{n+1} = (1-\alpha)\,\lambda^n + \alpha\,\beta^n ,$$

where β^n satisfies

$$\mathbf{P}\,\beta^n = \chi - (\mathbf{S}_1 - \mathbf{T})\,\lambda^n .$$

Then, by using integration by parts, it follows that

$$\sigma(\mathbf{u}_{N,2}^n)\nu_2 + \mathbf{T}\,\beta^n = -\sigma(\mathbf{u}_{N,1}^n)\nu_1 + \mathbf{T}\,\lambda^n \quad \text{en } \Gamma ,$$

where $\mathbf{u}_{N,1}^n := \mathbf{u}_{N,1}(\lambda^n) = \tilde{\mathbf{w}}_1 + \mathbf{R}_1\lambda^n$ and $\mathbf{u}_{N,2}^n := \mathbf{u}_{N,2}(\beta^n) = \mathbf{w}_2 + \mathbf{R}_2\beta^n$. This, together with the fact that $\mathbf{u}_{N,1}^n\big|_\Gamma = \lambda^n$ and $\mathbf{u}_{N,2}^n\big|_\Gamma = \beta^n$, leads to the following iterative scheme to update λ^n:

(**0**) Choose an initial guess $\lambda^0 \in \Lambda_h$ and set $n = 0$.

(**1**) Solve the Dirichlet problem

$$\begin{aligned} -\operatorname{div}\sigma\,(\mathbf{u}_{N,1}^n) &= \mathbf{f} \quad \text{in } \Omega_1 , \\ \mathbf{u}_{N,1}^n &= \mathbf{g}_0 \quad \text{on } \Gamma_D , \\ \mathbf{u}_{N,1}^n &= \lambda^n \quad \text{on } \Gamma . \end{aligned}$$

(**2**) Solve the Robin–type problem

$$\begin{aligned} -\operatorname{div}\sigma\,(\mathbf{u}_{N,2}^n) &= \mathbf{f} \quad \text{in } \Omega_2 , \\ \sigma(\mathbf{u}_{N,2}^n)\nu_2 + \mathbf{T}\,\mathbf{u}_{N,2}^n &= -\sigma(\mathbf{u}_{N,1}^n)\nu_1 + \mathbf{T}\,\lambda^n \quad \text{on } \Gamma , \\ \sigma(\mathbf{u}_{N,2}^n)\nu &= \tilde{\mathbf{T}}^N(\mathbf{u}_{N,2}^n) \quad \text{on } \Gamma_N . \end{aligned}$$

(**3**) Update λ^n by means of $\lambda^{n+1} = (1-\alpha)\,\lambda^n + \alpha\,\beta^n$ where $\beta^n := \mathbf{u}_{N,2}^n\,\big|_\Gamma$. If any specified stopping criteria is not satisfied, set $n = n+1$ and go to (**1**).

In what follows, we extend this algorithm to an arbitrary number of subdomains. In fact, given an integer $p > 2$, let Γ_j, $j = 1, p-1$, be polygonal curves contained in Ω, such that the interior region to Γ_j contains $\bar{D}$ and also the interior region to Γ_{j-1}, and such that they splits Ω into p subdomains Ω_j, $j = 1, p$. In other words, Ω_j is the annular region bounded by Γ_{j-1} and Γ_j. Also, we use the notation $\Gamma_0 := \Gamma_D$, $\Gamma_p := \Gamma_N$, $\lambda_0^n := \mathbf{g}_0$, $\lambda_j^n := \lambda^n|_{\Gamma_j}$ and $\mathbf{u}_{N,j}^n := \mathbf{u}_N^n|_{\Omega_j}$. Moreover, we define $\tilde{\Omega}_1 := \bigcup_{odd\ j} \Omega_j$, $\tilde{\Omega}_2 := \bigcup_{even\ j} \Omega_j$ and $\tilde{\Gamma} := \bigcup_{j=1}^{p-1} \Gamma_j$.

Therefore, by using the algorithm in two subdomains but now considering $\tilde{\Omega}_1$, $\tilde{\Omega}_2$ and $\tilde{\Gamma}$, and by assuming, without loss of generality, that p is even, we get the following iterative scheme:

(**0**) Choose an initial guess $\lambda^0 \in \Lambda_h$ and set $n = 0$.

(**1**) Solve **in parallel**, for all odd j, $j \leq p-1$, the Dirichlet problems

$$\begin{aligned} -\operatorname{div}\sigma\,(\mathbf{u}_{N,j}^n) &= \mathbf{f} \quad \text{in } \Omega_j\,, \\ \mathbf{u}_{N,j}^n &= \lambda_{j-1}^n \quad \text{on } \Gamma_{j-1}\,, \\ \mathbf{u}_{N,j}^n &= \lambda_j^n \quad \text{on } \Gamma_j\,. \end{aligned}$$

(**2**) Solve **in parallel**, for all even j, $j \leq p-2$, the Robin–type problems

$$\begin{aligned} -\operatorname{div}\sigma\,(\mathbf{u}_{N,j}^n) &= \mathbf{f} \quad \text{in } \Omega_j\,, \\ \sigma(\mathbf{u}_{N,j}^n)\nu_j + \mathbf{T}\,\mathbf{u}_{N,j}^n &= -\sigma(\mathbf{u}_{N,j-1}^n)\nu_{j-1} + \mathbf{T}\,\lambda_{j-1}^n \quad \text{on } \Gamma_{j-1}\,, \\ \sigma(\mathbf{u}_{N,j}^n)\nu_j + \mathbf{T}\,\mathbf{u}_{N,j}^n &= -\sigma(\mathbf{u}_{N,j+1}^n)\nu_{j+1} + \mathbf{T}\,\lambda_j^n \quad \text{on } \Gamma_j \end{aligned}$$

and

$$\begin{aligned} -\operatorname{div}\sigma\,(\mathbf{u}_{N,p}^n) &= \mathbf{f} \quad \text{in } \Omega_p\,, \\ \sigma(\mathbf{u}_{N,p}^n)\nu_p + \mathbf{T}\,\mathbf{u}_{N,p}^n &= -\sigma(\mathbf{u}_{N,p-1}^n)\nu_{p-1} + \mathbf{T}\,\lambda_{p-1}^n \quad \text{on } \Gamma_{p-1}\,, \\ \sigma(\mathbf{u}_{N,p}^n)\nu &= \tilde{\mathbf{T}}^N(\mathbf{u}_{N,p}^n) \quad \text{on } \Gamma_N\,. \end{aligned}$$

(**3**) Update λ^n by means of $\lambda_i^{n+1} = (1-\alpha)\,\lambda_i^n + \alpha\,\beta_i^n$, $i = 1, p-1$, where

$$\beta^n := (\mathbf{u}_{N,2}^n|_{\Gamma_1}, \mathbf{u}_{N,2}^n|_{\Gamma_2}, \mathbf{u}_{N,4}^n|_{\Gamma_3}, \mathbf{u}_{N,4}^n|_{\Gamma_4}, \ldots, \mathbf{u}_{N,p}^n|_{\Gamma_{p-1}})\,.$$

If any specified stopping criteria is not satisfied, set $n = n+1$ and go to (**1**).

Analogously, one can deduce the iteration–by–subdomains algorithms for PCGM. For the sake of compactness, it will be presented separately in a forthcoming paper.

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