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A NEW SHELL ELEMENT FOR ELASTO-PLASTIC FINITE STRAIN ANALYZES

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Abstract. The MITC4 shell element (Dvorkin E.N. and Bathe K.J., *Engng. Computations*, Vol. 1, pp. 77-88, 1984) is very successful and it provides excellent solutions for infinitesimal strain analyzes using either elastic or elasto-plastic material models in linear or nonlinear geometrical formulations.

In (Dvorkin et al., *Comput. Meth. Appl. Mechs. Engng., vol. 125, pp.17-40*, 1995) the element formulation was extended for finite strain elasto-plastic analyzes and even tough the new element provides very good solutions it presents some room for improvements.

In previous publications we presented a new shell element formulation, the MITC4-3D that we developed for finite strain analysis using the MITC4 strains interpolation and 3D constitutive relations (hyperelastic and elasto-plastic materials).

Some of the basic features of our new element are:

- The shell geometry is interpolated using mid-surface nodes and director vectors.
- The node displacements and transverse shear strains are interpolated using the original MITC4 formulation.
- For interpolating the director vectors special care is taken to avoid spurious director vector stretches.
- Additional degrees of freedom are considered to include a linear thickness stretching. These thickness-stretching degrees of freedom are condensed at the element level.
- The elasto – plastic formulation is developed following the work of Simo and co-workers: multiplicative decomposition of the deformation gradient tensor and maximum plastic dissipation (associate plasticity).
- Special consideration is given to the formulation efficiency.

In this paper we are going to discuss the basic features of the MITC4-3D element and present further verification / validation examples.

1 INTRODUCTION

The shell element formulation presented in (Ahmad, Irons and Zienkiewicz, 1970), after many years, still constitutes the basis for modern finite element analysis of shell structures. Even tough the A-I-Z shell element was a breakthrough in the field of finite element analysis of shell structures, under the constraint of the infinitesimal strains, it suffers from the locking phenomenon and much research effort has been devoted to the development of A-I-Z type elements that do not present this problem (Bathe, 1996, Chapelle and Bathe, 2003, Zienkiewicz and Taylor, 2000).

The MITC4 shell element (Dvorkin and Bathe, 1984), which was developed to overcome the locking problem of the A-I-Z shell elements, has become, since its development in the early eighties, the standard shell element for many finite element codes. However, the limitation of infinitesimal strains is still present in the MITC4 formulation.

In 1995 Dvorkin, Pantuso and Repetto (Dvorkin et al, 1995) developed the MITC4-TLH element, that based on the original MITC4 formulation can model finite strain elasto-plastic deformations. This element imposes the condition of zero transversal stresses and its computational cost is rather high.

In previous publications we presented a new shell element formulation, the MITC4-3D that we developed for finite strain analysis (Toscano and Dvorkin, 2007) using the MITC4 strains interpolation and 3D constitutive relations (hyperelastic and elasto-plastic materials).

The most relevant differences with the original MITC4 formulation are:

- For each quadrilateral element we have 22 d.o.f.: 5 generalized displacements per node plus 2 extra d.o.f. to incorporate the through-the-thickness stretching.
- We use a general 3D constitutive relation instead of the original laminae plane stress constitutive relation.

2 THE MITC4-3D FORMULATION

Some of the basic features of our MITC4-3D element are:

- The shell geometry is interpolated using mid-surface nodes and director vectors.
- The nodal displacements and transverse shear strains are interpolated using the original MITC4 formulation (Dvorkin and Bathe, 1984).
- For interpolating the director vectors special care is taken to avoid spurious director vector stretches (Gebhardt and Schweizerhof, 1993, Simo et al., 1989).
- Two additional degrees of freedom are considered to include a linear thickness stretching. These thickness-stretching degrees of freedom are condensed at the element level (Dvorkin et al, 1995; Bischoff and Ramm, 1997).
- The elasto - plastic formulation is developed following (Simo and Hughes, 1998): multiplicative decomposition of the deformation gradient tensor and maximum plastic dissipation (associate plasticity).
- Special consideration is given to the formulation efficiency.

2.1 Shell element geometry

Following the MITC4 formulation we define, in the reference configuration, nodes on the shell mid-surface and at each node we define a director vector which represents, at that node, an approximation to the shell mid-surface normal (Dvorkin and Bathe, 1986, Simo et al., 1989, 1990, 1992).
Therefore, defining inside the element the natural coordinate system (r, s, t) (Bathe, 1996), for an element with constant thickness, we can write,

$$ {}^{\tau}\underline{x}(r,s,t) = h_k(r,s) \cdot {}^{\tau}\underline{x}_k + \frac{t}{2} \cdot \left({}^{\tau}\lambda_0 + {}^{\tau}\lambda_1 \cdot t\right) \cdot \frac{h_k(r,s) \cdot {}^{\tau}\underline{V}_n^k}{\left\| h_k(r,s) \cdot {}^{\tau}\underline{V}_n^k \right\|} \cdot a \tag{1} $$

Where,

$h_k(r,s)$: isoparametric 2D interpolation functions corresponding to the k-th mid-surface node (Bathe, 1996),

${}^{\tau}\underline{x}_k$: position vector of the k-th mid-surface node at time τ,

a : constant element thickness,

${}^{\tau}\underline{V}_n^k$: director vector corresponding to the k-th mid-surface node at time τ, with $\left\| {}^{\tau}\underline{V}_n^k \right\| = 1$.

In Eqn. (1) ${}^{\tau}\lambda_0$ is a constant thickness stretching and ${}^{\tau}\lambda_1$ is the through-the-thickness stretching gradient. In our formulation the element ${}^{\tau}\lambda_0$ and ${}^{\tau}\lambda_1$ are discontinuous across element boundaries and they will be condensed at the element level.

2.2 Incremental displacement

The incremental displacements to evolve from the τ-configuration to the τ+Δτ-configuration are,

$$ \underline{u} = {}^{\tau+\Delta\tau}\underline{x} - {}^{\tau}\underline{x} $$

$$ \begin{aligned} \underline{u}(r,s,t) = {} & h_k(r,s) \cdot \underline{u}_k + \frac{t}{2} \cdot \left({}^{\tau}\lambda_0 + \Delta\lambda_0 + {}^{\tau}\lambda_1 \cdot t + \Delta\lambda_1 \cdot t\right) \cdot \frac{h_k(r,s) \cdot {}^{\tau+\Delta\tau}\underline{V}_n^k}{\left\| h_k(r,s) \cdot {}^{\tau+\Delta\tau}\underline{V}_n^k \right\|} \cdot a - \\ & \frac{t}{2} \cdot \left({}^{\tau}\lambda_0 + {}^{\tau}\lambda_1 \cdot t\right) \cdot \frac{h_k(r,s) \cdot {}^{\tau}\underline{V}_n^k}{\left\| h_k(r,s) \cdot {}^{\tau}\underline{V}_n^k \right\|} \cdot a \end{aligned} \tag{2} $$

For the director vector rotations we can write,

$$ {}^{\tau+\Delta\tau}\underline{V}_n^k = {}^{\tau+\Delta\tau}_{\tau}\underline{\underline{R}} \cdot {}^{\tau}\underline{V}_n^k $$

where ${}^{\tau+\Delta\tau}_{\tau}\underline{\underline{R}}$ is a rotation tensor (Dvorkin *et al.*, 1988).

To simplify the linearization, we made the approximation $\left\| h_k(r,s) \cdot {}^{\tau+\Delta\tau}\underline{V}_n^k \right\| = \left\| h_k(r,s) \cdot {}^{\tau}\underline{V}_n^k \right\|$.

2.3 Constitutive relation

The shell element formulation developed in this paper is a fully 3D formulation since the in-layer plane stress hypothesis used in the original MITC4 formulation was not invoked in this case. This full 3D constitutive relation is based on:

- Lee´s multiplicative decomposition of the deformation gradient tensor ${}^{\tau}_{0}\underline{\underline{X}}$ (Figure 1)
- Maximum plastic dissipation
- For the elastic part an hyperelastic relation using Hooke with Hencky strains: ${}^{\tau}_{0}\underline{\underline{\Gamma}} = \underline{\underline{\underline{\underline{C}}}} : {}^{\tau}_{0}\underline{\underline{H}}^{E}$.

${}^{\tau}_{0}\Gamma^{IJ}$ components are calculated as the rotational pull-back of the contravariant components of the Kirchhoff stress tensor. For an ***isotropic*** material, the second order tensor ${}^{\tau}_{0}\underline{\underline{\Gamma}}$ is the stress measure energy-conjugate to the Hencky strain tensor (Dvorkin and Goldschmit, 2005).

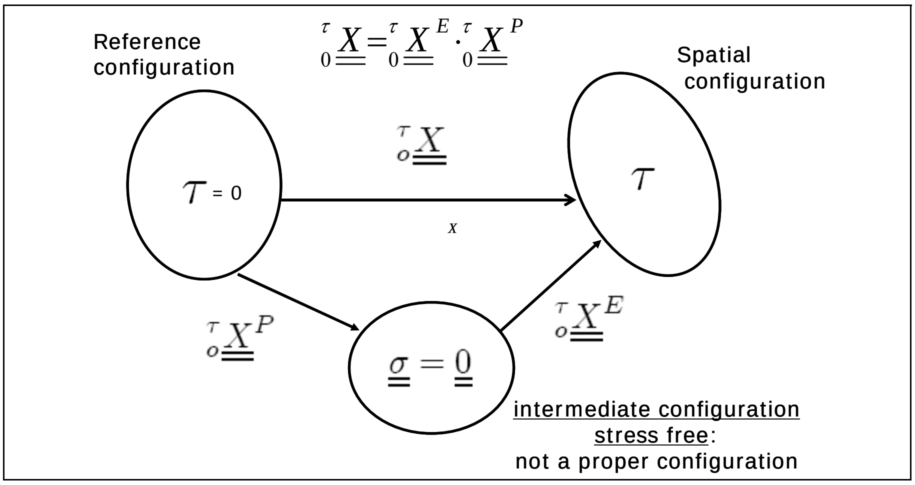

Figure 1: Lee’s multiplicative decomposition of the deformation gradient

3 THE INCREMENTAL FORMULATION

Using a total Lagrangian formulation we can write the Principle of Virtual Work for the equilibrium configuration at τ+Δτ (Bathe, 1996),

$$\int_{{}^{0}V} {}^{t+\Delta\tau}\underline{\underline{\Gamma}} : \delta \underline{\underline{H}}^{E} \cdot {}^{0}dV = {}^{t+\Delta\tau}\underline{\Re} \qquad (3)$$

where ${}^{t+\Delta\tau}\underline{\Re}$ is the virtual work of the external loads acting on the solid body in the τ+Δτ-configuration and $\underline{\underline{H}}^{E}$ is an elastic Hencky strain tensor.

But the incremental step is given considering constant the intermediate configuration, which is updated later (see Figure 2). This last t iteration is the iteration (k-1) of step $\tau + \Delta\tau$.

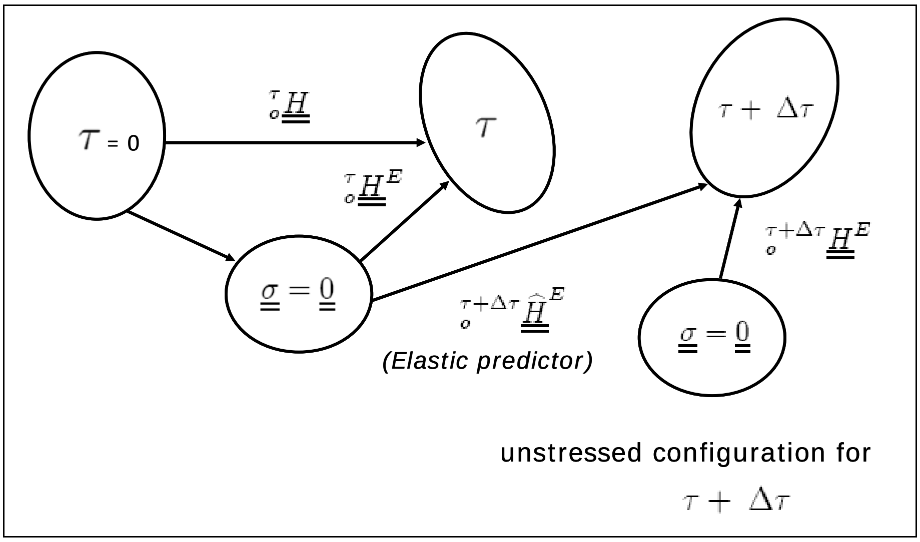

Figure 2: Incremental step

Therefore, replacing in (3),

$$\int_{{}^{0}V} {}^{t+\Delta\tau}\underline{\underline{\Gamma}} : \delta\widehat{\underline{\underline{H}}}^{E} \cdot {}^{0}dV = {}^{t+\Delta\tau}\underline{\underline{\Re}} \ . \qquad (4)$$

Where,

$${}^{t+\Delta\tau}\underline{\underline{\Gamma}} = {}^{\tau}\underline{\underline{\Gamma}} + d\underline{\underline{\Gamma}} \quad . \qquad (5)$$

Since we are interpolating the total components of ${}^{\tau}_{0}\underline{\underline{H}}$, not only the elastic part, we must calculate,

$$\delta\widehat{\underline{\underline{H}}}^{E} = \frac{\partial\widehat{\underline{\underline{H}}}^{E}}{\partial\underline{\underline{H}}} : \delta\underline{\underline{H}} \qquad (6)$$

$$d\underline{\underline{\Gamma}} = \frac{\partial\underline{\underline{\Gamma}}}{\partial\widehat{\underline{\underline{H}}}^{E}} : \frac{\partial\widehat{\underline{\underline{H}}}^{E}}{\partial\underline{\underline{H}}}\Big|^{(K-1)} : d\underline{\underline{H}} \ . \qquad (7)$$

The fourth order tensor $\underline{\underline{\underline{\underline{C}}}}_{EP} = \dfrac{\partial\underline{\underline{\Gamma}}}{\partial\widehat{\underline{\underline{H}}}^{E}}$ is the tangent constitutive elasto-plastic tensor (Dvorkin et al., 1995).

The term $\dfrac{\partial\widehat{\underline{\underline{H}}}^{E}}{\partial\underline{\underline{H}}}$ is deduced using Serrin formula (Dvorkin and Goldschmit, 2005).

In order to be able to put the strain variations in terms of the displacement variations,

$$d\underline{\underline{H}} = \frac{\partial\underline{\underline{H}}}{\partial\underline{\underline{C}}}\Big|^{(K-1)} : d\underline{\underline{C}} \ . \qquad (8)$$

Where $\underline{\underline{C}}$ is the Green second order tensor.

The Green-Lagrange strain tensor is defined in the reference configuration as,

$$ {}_0^\tau\underline{\underline{\varepsilon}} = \frac{1}{2}\cdot\left({}_0^\tau\underline{\underline{C}} - \underline{\underline{I}}\right) \quad . \qquad (9)$$

Therefore,

$$ d\underline{\underline{H}} = 2\cdot\frac{\partial \underline{\underline{H}}}{\partial \underline{\underline{C}}}\Big|^{(K-1)} : d\underline{\underline{\varepsilon}} . \qquad (10)$$

Taking into account the expressions 3 to 10, we deduce the linear and non-linear matrices and the equivalent nodal forces vector and formulate the equilibrium equation for the linearized incremental step, from time τ to $\tau + \Delta\tau$, as,

$$ \left({}_0^\tau\underline{K}_L + {}_0^\tau\underline{K}_{NL}\right)\cdot\underline{U} = {}^{\tau+\Delta\tau}\underline{\Re} - {}^\tau\underline{F} . \qquad (11)$$

4 NUMERICAL RESULTS

We have validated the new MITC4-3D element comparing its results and iterative behavior with an implementation of the MITC4 element in which the thickness is updated a posteriori of each iteration.

4.1 Infinitely long cylinder under internal pressure

In this example we analyze an infinite long elasto-plastic cylinder under internal pressure (Fig. 3).

Figure 3 also shows the equilibrium paths calculated with both elements, MITC4-3D and MITC4 with a posteriori thickness update, and the thickness stretching vs. radial displacement curve. Regarding the equilibrium path, the coincidence is perfect.

The d.o.f. ($\Delta\lambda_0$, $\Delta\lambda_1$) are condensed at the element level and (20x20) element stiffness matrices are obtained and assembled into the global stiffness matrices.

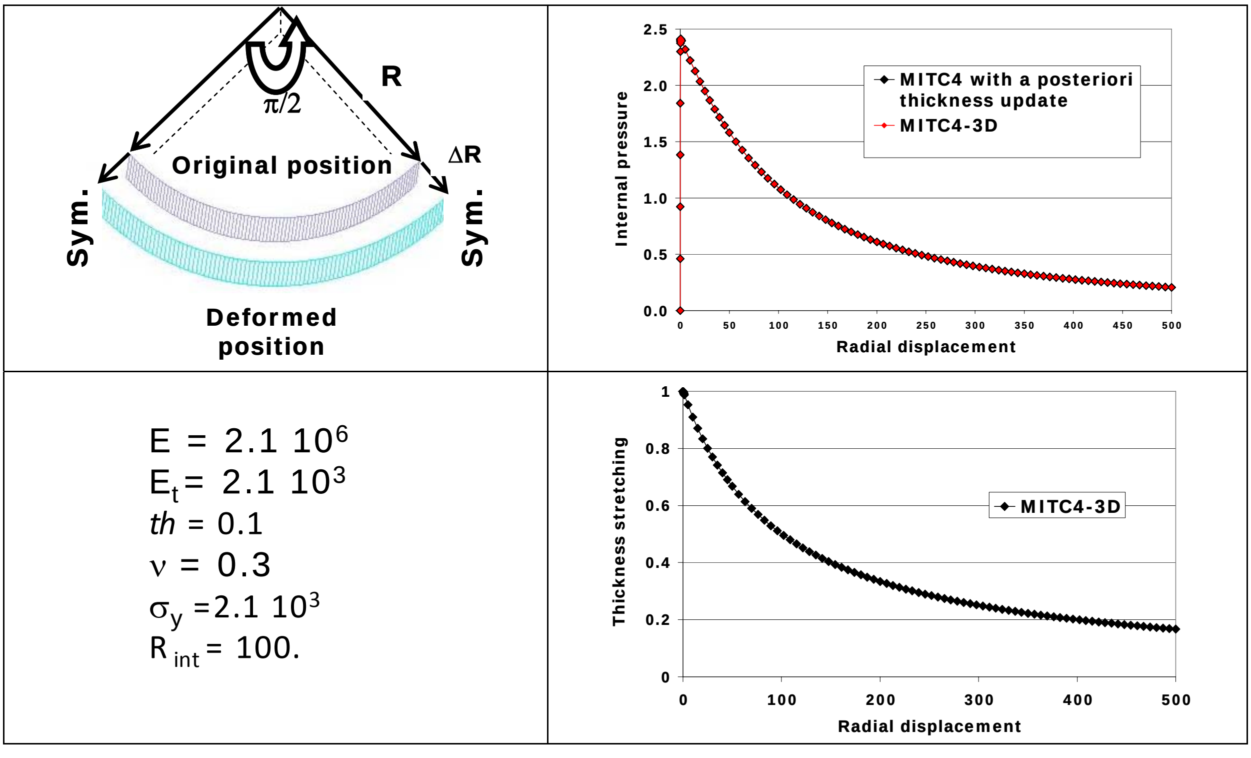

Figure 3: Inflation of an infinitely long elasto-plastic cylinder

4.2 Pipe: external collapse pressure

In this case we study the behavior of a pipe under external pressure. The material properties are: $E=21000$ kg/mm^2, $E_t=86.0$ kg/mm^2, $\nu=0.3$ and $\sigma_y=54{,}75$ kg/mm^2. Regarding the geometry of the pipe, the external diameter is 341,5mm, the thickness a=17,65mm and the ovality $ov[\%] = \dfrac{D_{\max} - D_{\min}}{D_{Average}} = 0{,}47\%$. Figure 4 shows the pipe after collapse propagation; the red area corresponds to diameter diminish while the green lateral area corresponds to diameter increase. The variable represented in the graph is the absolute value of the total displacement of each mesh point.

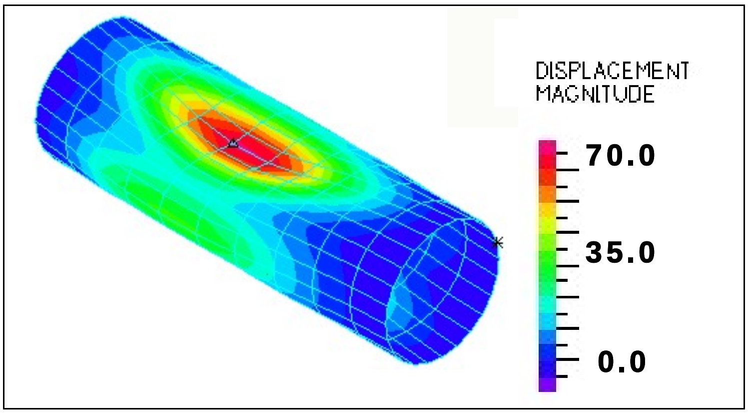

Figure 4: External collapse pressure of a pipe with ovality; total displacement of each mesh point

Figure 5 shows the External pressure vs. Diameter variation for both elements: MITC4-3D and MITC4 with a posteriori thickness update. It can be observed that the results are very similar.

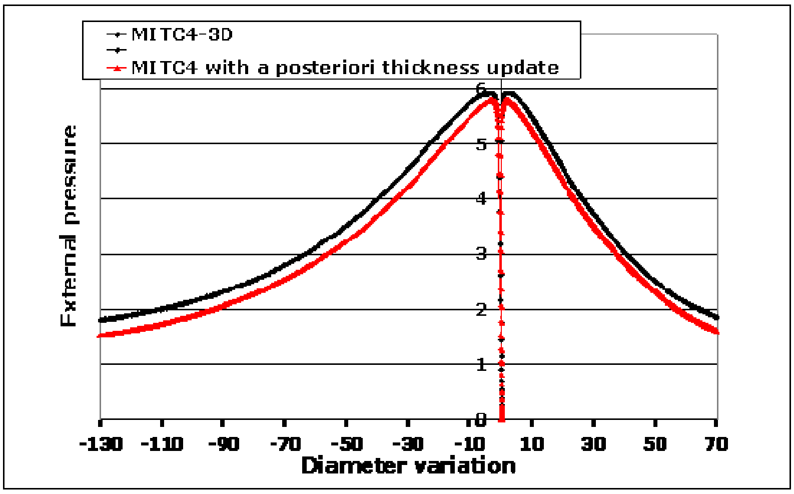

Figure 5: External pressure vs. Diameter variation

The curves Thickness stretching vs. Axial coordinates for element lines corresponding to diameters that increase and diameters that decrease are also presented (Fig. 6).

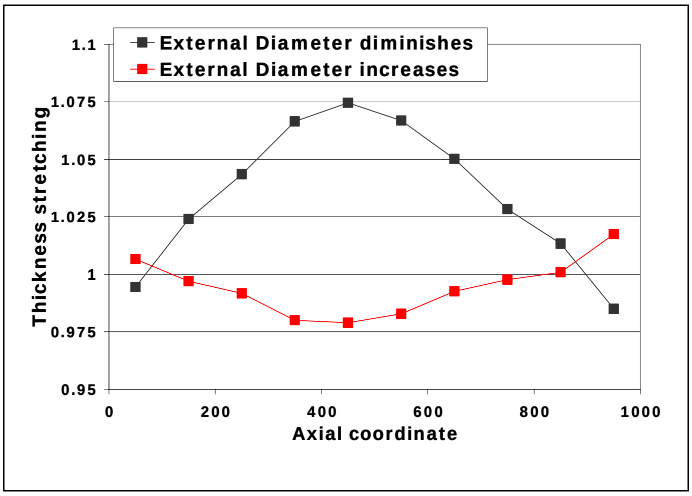

Figure 6: Thickness stretching vs. Axial coordinates

In Figure 7 we can see the comparison of the Iteration number vs. Diameter variation for MITC4-3D vs. MITC4 with a posteriori thickness update. It becomes evident that the new element shows a better efficiency.

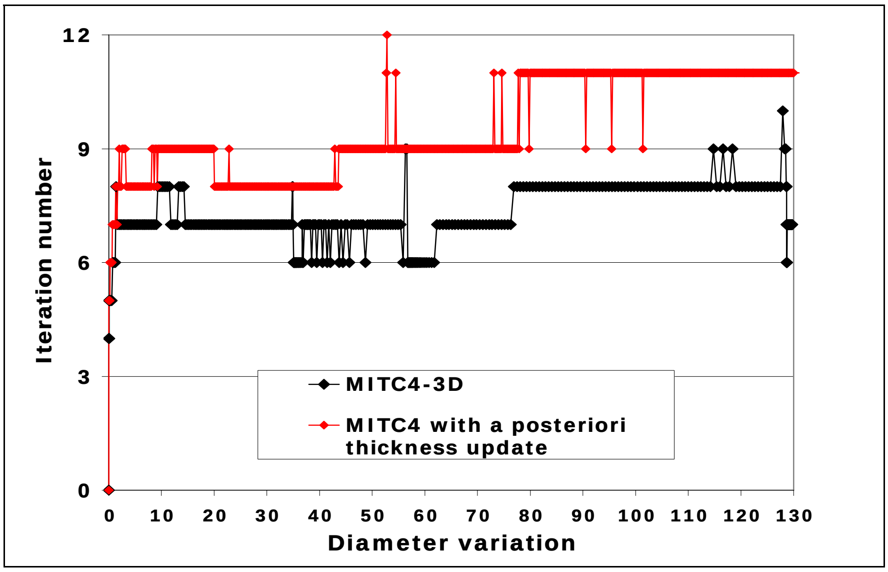

Figure 7: Iteration vs. Diameter variation (increase)

5 CONCLUSIONS

On the basis of the MITC4 shell element formulation, we developed the MITC4-3D shell element formulation for finite strain analyses of shell structures using general 3D constitutive models. In this paper the new element was implemented for the analyses of elasto-plastic shell structures and the results indicate that it is a very effective element.

REFERENCES

Ahmad S, Irons B.M. and Zienkiewicz O.C., "Analysis of thick and thin shell structures by curved finite elements", *Int. J. Numerical Methods in Engng., vol. 2, pp.419-451*, 1970.

Bathe K.J., *Finite Element Procedures*, Prentice Hall, Upper Saddle River, NJ., 1996.

Bischoff M. and Ramm E. , "Shear deformable shell elements for large strains and rotations", Int. J. Numerical Methods in Engng., vol. 40, pp. 4427-4449, 1997.

Chapelle D. and Bathe K.J., *The Finite Element Analysis of Shells* - Fundamentals, Springer, Berlin, 2003.

Dvorkin E.N., "Nonlinear analysis of shells using the MITC formulation", Archives Comput. Meth. Engng., vol. 2, pp.1-50, 1995.

Dvorkin E.N., "On nonlinear analysis of shells using finite elements based on mixed interpolation of tensorial components", published in Nonlinear Analysis of Shells by Finite Elements, (Ed. F.G. Rammerstorfer), CISM Courses and Lectures No. 328, Springer - Verlag, Wien - New York, 1992.

Dvorkin E.N., Pantuso D. and Repetto E.A., "A formulation of the MITC4 shell element for finite strain elasto-plastic analysis", *Comput. Meth. Appl. Mechs. Engng., vol. 125, pp.17-40*, 1995.

Dvorkin E.N., Oñate E. and Oliver J., "On a nonlinear formulation for curved Timoshenko beam elements considering large displacement/rotation increments", *Int. J. Numerical Methods in Engng.*, vol. 26, pp. 1597-1613, 1988.

Dvorkin E.N. and Bathe K.J., "A continuum mechanics based four-node shell element for general nonlinear analysis", *Engng. Computations*, Vol. 1, pp. 77-88, 1984.

Dvorkin E.N. and Goldschmit M.B. , *Nonlinear Continua*, Springer, Berlin, 2005.

Gebhardt H. and Schweizerhof K., "Interpolation of curved shell geometries by low order finite elements - Errors and modifications", *Int. J. Numerical Methods in Engng*., vol. 36, pp.287-302, 1993.

Simo J.C. , Fox D.D. and Rifai M.S., "On a stress resultant geometrically exact shell model. Part II: The linear theory; computational aspects", *Comput. Meth. Appl. Mechs. Engng.*, vol. 73, pp.53-92, 1989.

Simo J.C., Fox D.D. and Rifai M.S., "On a stress resultant geometrically exact shell model. Part III: Computational aspects of the nonlinear theory", *Comput. Meth. Appl. Mechs. Engng.*, vol. 79, pp.21-70, 1990.

Simo J.C., Fox D.D. ans Rifai M.S., "On a stress resultant geometrically exact shell model. Part IV: Variable thickness shells with through-the-thickness stretching", *Comput. Meth. Appl. Mechs. Engng.,* vol. 81, pp.91-126, 1992.

Toscano R.G. and Dvorkin E.N., "A shell element for elasto – plastic finite strain analyzes", *Proceedings 9th US National Congress on Computational Mechanics,* San Francisco, USA, July 2007.

Toscano R.G. and Dvorkin E.N., "A Shell Element for Finite Strain Analyses. Hyperelastic Material Models", *Engineering Computations*, Vol.24, N° 5, pp.514-535, 2007.

Zienkiewicz O.C. and Taylor R.L., *The Finite Element Method*, Vol. 2, 5th Edition, Butterworth-Heinemann, 2000.