Asociación Argentina de Mecánica Computacional

Mecánica Computacional Vol XXVI, pp.654-666
Sergio A. Elaskar, Elvio A. Pilotta, Germán A. Torres (Eds.)
Córdoba, Argentina, Octubre 2007

A DAUBECHIES WAVELET BEAM ELEMENT

Victoria Vampa[a,b], María T. Martín[a,c] and Lilliam Alvarez Díaz[d]

[a]*Dpto. de Matemática, Fac. Ciencias Exactas, Universidad Nacional de La Plata, victoria@mate.unlp.edu.ar*
[b]*Fac. de Ingeniería, Universidad Nacional de La Plata*
[c]*Instituto de Física La Plata, CONICET, mtmartin@fisica.unlp.edu.ar*
[d]*Ministerio de Ciencia, Tecnología y Medio Ambiente, Cuba, lilliam@citma.cu*

Keywords: Wavelet-finite element, Scaling functions, Daubechies wavelet, Connection coefficients, beam element

Abstract. In the last years, applying wavelets analysis has called the attention in a wide variety of practical problems, in particular for the numerical solutions of partial differential equations using different methods, as finite differences, semi-discrete techniques or finite element method.

Due to function wavelets have the properties of generating a direct sum of $L^2(\mathbb{R})$ and that their correspondent scaling function generates a multiresolution analysis, the wavelet bases in multiple scales combined with the finite element method provide a suitable strategy for mesh refinement.

In particular, in some mathematical models in mechanics of continuous media, the solutions may have discontinuities, singularities or high gradients, and it is necessary to approximate with interpolatory functions having good properties or capacities to efficiently localize those non-regular zones.

In some cases it is useful and convenient to use the Daubechies wavelets, due to their excellent properties of orthogonality and minimum compact support and for having vanishing moments, providing guaranty of convergence and accuracy of the approximation in a wide variety of situations.

The present work shows the feasibility of a hybrid scheme using Daubechies wavelet functions and finite element method to obtain competitive numerical solutions of some classical tests in structural mechanics.

1 INTRODUCTION

Let us first recall that Finite Element Method (FEM) is the classical and standard numerical technique to solve many engineering and physical problems in mechanics of continuous media, computing structures in civil or mechanic engineering, etc. Commonly, the method uses polynomial interpolation or any approximation functions in some steps of the calculations.

In many numerical simulation of mathematical models in physics the appearance of small scale structures that exist in only small parts of the domain is common. Wavelets provide a natural mechanism for decomposing the solution into a set of coefficients which depend on scale and location.

In this work, wavelet-based FEM in structural mechanics is proposed by using Daubechies wavelets, following the ideas presented in (Ma et al., 2003), (Chen et al., 2004). The wavelet-finite element scheme is constructed in a similar way to the conventional displacement-based FEM: the wavelet functions are used as the displacement interpolation functions and the shape functions are expressed by wavelets. Then, for the Euler Bernoulli beam model, wavelet-finite element formulations are derived.

The accuracy of this approach is investigated in some numerical test cases. The proposed wavelet finite element method shows high accuracy and good convergence properties to solve problems in structural mechanics.

2 WAVELET ANALYSIS: BASIC CONCEPTS

Wavelets are functions generated from one single function called the mother wavelet by the simple operations of dilation and translation. A mother wavelet gives rise to a decomposition of the Hilbert space $\mathrm{L}^2(\mathbb{R})$, into a direct sum of closed subspaces W_j, $j \in \mathbb{Z}$.

Let $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$ and

$$W_j = \mathrm{clos}_{\mathrm{L}^2(\mathbb{R})}[\psi_{j,k} : k \in \mathbb{Z}] \tag{1}$$

Then every $f \in \mathrm{L}^2(\mathbb{R})$ has a unique decomposition

$$f(x) = \cdots + g_{-1}(x) + g_0(x) + g_1(x) + \cdots \tag{2}$$

where $g_j \in W_j$ for all $j \in \mathbb{Z}$, it is

$$\mathrm{L}^2(\mathbb{R}) = \dot{\sum_{j\in\mathbb{Z}}} W_j = \cdots \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus \cdots \tag{3}$$

Using this decomposition of $\mathrm{L}^2(\mathbb{R})$, a nested sequence of closed subspaces V_j, $j \in \mathbb{Z}$, of $\mathrm{L}^2(\mathbb{R})$ can be obtained, defined by

$$V_j = \cdots \oplus W_{j-2} \oplus W_{j-1} \,. \tag{4}$$

These closed subspaces $\{V_j,\ j \in \mathbb{Z}\}$ of $\mathrm{L}^2(\mathbb{R})$, form a "multiresolution analysis" (Chui, 1992) with the following properties:

1. $\cdots \subset V_{-1} \subset V_0 \subset V_1 \cdots$
2. $\mathrm{clos}_{\mathrm{L}^2}(\bigcup V_j) = \mathrm{L}^2(\mathbb{R})$
3. $\bigcap V_j = \{0\}$

4. $V_{j+1} = V_j \oplus W_j$

5. $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}\ , j \in \mathbb{Z}$

Let $\phi \in V_0$ the so-called "scaling function" that generates the multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$ of $\mathrm{L}^2(\mathbb{R})$. Then

$$\{\phi(\cdot - k) : k \in \mathbb{Z}\} \tag{5}$$

is a basis of V_0, and by setting

$$\phi_{j,k}(x) := 2^{j/2}\phi(2^j x - k) \tag{6}$$

it follows that, for each $j \in \mathbb{Z}$, the family

$$\{\phi_{j,k} : k \in \mathbb{Z}\} \tag{7}$$

is also a basis of V_j.

Then, since $\phi \in V_0$ is in V_1 and since $\{\phi_{1,k} : k \in \mathbb{Z}\}$ is a basis of V_1, there exists a unique sequence $\{p_k\}$ that describes the following "two-scale relation":

$$\phi(x) = \sum_{k=-\infty}^{\infty} p_k\ \phi(2x - k) \tag{8}$$

of the scaling function ϕ.

2.1 Daubechies's wavelets

Different choices for ϕ may yield different multiresolution analyses, and the most useful scaling functions are those that have compact support. As an example of multiresolution analysis, a family of orthogonal Daubechies wavelets with compact support has been constructed by Daubechies (Daubechies, 1992).

A wavelet basis is orthonormal if any two translated or dilated wavelets satisfy the condition

$$\int_{-\infty}^{\infty} \psi_{n,k}(x)\psi_{m,l}(x)dx = \delta_{n,m}\delta_{k,l} \tag{9}$$

where δ is the Kronecker Delta function.

Each wavelet family is governed by a set of N (an even integer) coefficients $p_k : k = 0, 1, \ldots, N-1$ through the two-scale relation

$$\phi_N(x) = \sum_{k=0}^{N-1} p_k \phi_N(2x - k) \tag{10}$$

Based on the scaling function $\phi_N(x)$, the mother wavelet can be written as,

$$\psi_N(x) = \sum_{k=2-N}^{1} q_k \phi_N(2x - k) \tag{11}$$

Since the wavelets are orthonormal to the scaling basis the coefficients of the scaling function and the mother wavelet for the two-scale equation are related by:

$$q_k = (-1)^k p_{1-k} \tag{12}$$

In her work, Daubechies (Daubechies, 1988) found and exploited the link between vanishing moments of the wavelet ψ and regularity of wavelet and scaling functions, ψ and ϕ. The wavelet function ψ has K vanishing moments if

$$\int x^k \psi(x) dx = 0 \qquad \text{for} \quad 0 \leq k \leq K \tag{13}$$

and a necessary and sufficient condition for this to hold is that integer translates of the scaling function ϕ exactly interpolate polynomials of degree up to K. That is, for each k, $0 \leq k \leq K$ there exists constants c_l such that

$$x^k = \sum_l c_l^k \phi_l(x) \tag{14}$$

Daubechies introduced scaling functions satisfying this property and distinguished by having the shortest possible support. The scaling function ϕ_N (where N is an even integer) has support $[0, N-1]$, while the corresponding wavelet ψ_N has support in the interval $[1 - N/2, N/2]$ and has $(N/2 - 1)$ vanishing wavelet moments (Daubechies, 1988). Thus, according to equation (14) Daubechies scaling functions of order N can exactly represent any polynomial of order up to, but not greater than $N/2 - 1$.

The coefficients p_k in equation (10) are called scaling function filter coefficients and satisfy the following conditions based on the orthonormality and moment conditions.

$$\begin{aligned} \sum_{k=0}^{N-1} p_k &= 2 \\ \sum_{k=0}^{N-1} p_k \, p_{k+2m} &= 2\delta_{0m} && \text{for} \;\; m = 0, 1, \ldots, N/2 - 1 \\ \sum_{k=0}^{N-1} (-1)^k k^m p_k &= 0 && \text{for} \;\; m = 0, 1, \ldots, N/2 - 1 \end{aligned} \tag{15}$$

2.2 Computation of derivatives

In constructing a C^1 element, the derivatives of Daubechies scaling function have to be calculated. Because there is no explicit expression for the Daubechies scaling function, the derivatives can be obtained only on some special points. To evaluate the function or its derivatives, $\phi_N^{(m)}(x) = d^m \phi_N(x)/dx^m$, the two-scale relation is differentiated m times:

$$\phi_N^{(m)}(x) = 2^m \sum_{k=0}^{N-1} p_k \phi_N^{(m)}(2x - k) \tag{16}$$

Since supp. $\phi_N(x) \subseteq [0, N-1]$, it is certain that

$$supp.\ \phi_N^{(m)}(x) \subseteq [0, N-1] \tag{17}$$

At all integer values of the interval $[0, N-1]$, Eq. (16) gives the following N linear equations:

$$\begin{aligned} \phi_N^{(m)}(0) &= 2^m p_0 \phi_N^{(m)}(0) \\ \phi_N^{(m)}(1) &= 2^m [p_0 \phi^{(m)}(2) + p_1 \phi_N^{(m)}(1) + p_2 \phi_N^{(m)}(0)] \\ \cdots \quad & \quad \cdots \\ \phi_N^{(m)}(N-2) &= 2^m [p_{N-3} \phi_N^{(m)}(N-1) + p_{N-2} \phi_N^{(m)}(N-2) + p_{N-1} \phi_N^{(m)}(N-3)] \\ \phi_N^{(m)}(N-1) &= 2^m p_{N-1} \phi_N^{(m)}(N-1) \end{aligned} \tag{18}$$

The coefficients-matrix of the homogeneous system (18) is singular. Thus, a normalizing condition is required in order to determine a unique solution. The following important additional property of Daubechies scaling function ϕ_N can be used, (Beylkin, 1992):

$$\sum_k k^m \phi_N(x-k) = x^m + \sum_{k=1}^{m} (-1)^k \frac{m!}{(m-k)!k!} x^{m-k} \int_{-\infty}^{\infty} \phi_N(z) z^k dz \tag{19}$$

where m is a positive integer number. Differentiating m times the above equation yields:

$$\sum_k k^m \phi_N^{(m)}(x-k) = m! \tag{20}$$

Adding this normalizing condition to (18), this system of inhomogeneous equations can be solved and derivatives can be evaluated at integer values of x and used to get the values at the dyadic points. Using the two scaling relation once again the values of $\phi_N^{(m)}(x)$ at $x = \frac{i}{2^n}$ for $i = 1, 3, 5, \ldots, 2^n(N-1) - 1$ can be determined. Therefore, the functions are first evaluated at the integer points $\{0, 1, \ldots, N-1\}$ and then subsequently at half integers and so on by increasing the value of n from 0 to the desired resolution.

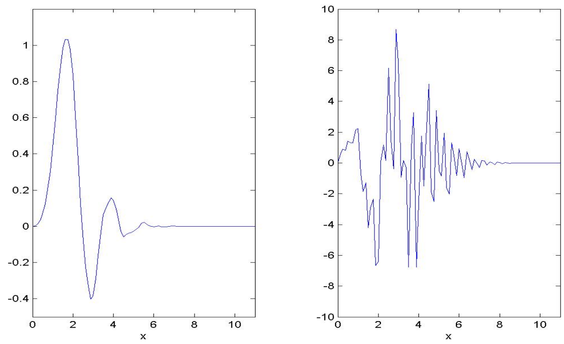

Figure 1: Daubechies scaling functions (N=12). Left: $\phi(x)$. Right: $\phi'(x)$.

2.3 Computation of Connection Coefficients

When the wavelet-finite element method is applied to solve one dimensional differential equations, different types of connection coefficients are required (Latto et al., 1995), such as the following:

$$\Gamma_{i,j}^{d_1 d_2} = \int_0^1 \phi^{(d_1)}(\xi - i)\, \phi^{(d_2)}(\xi - j)\, d\xi \tag{21}$$

where $i, j \in \mathbb{Z}$, $\phi(x)$ denotes the basis function and the superscripts d_1 and d_2 refer to differentiation. These typical coefficients give rise to the local stiffness matrices of the method. The details are discussed in the following sections.

The typical problem that arises when using Daubechies wavelets is how to calculate these connection coefficients when $\phi(x)$ is a Daubechies-wavelet scaling function. The highly oscillatory nature of the Daubechies basis functions makes standard numerical quadrature impractical for computing connection coefficients (see Fig. 1). The numerical calculations are in general unstable and it is necessary to provide an alternative method. Latto et al. developed an exact method for evaluating connection coefficients on a un-bounded domain of integration, like the following:

$$\hat{\Gamma}^{d_1 d_2}_{i,j} = \int_{-\infty}^{\infty} \phi^{(d_1)}(\xi - i)\, \phi^{(d_2)}(\xi - j)\, d\xi \tag{22}$$

However, on a bounded domain for the evaluation of connection coefficients, $\Gamma^{d_1 d_2}_{i,j}$, some addittional considerations are required (Beylkin, 1992).

To calculate the integral in the equation (21), Beylkin proposed to do the following: to substitute the two-scaling relation, given by equation (10), into equation (21), which yields

$$\Gamma^{d_1 d_2}_{i,j} = 2^{d_1+d_2} \sum_{k,l} p_k p_l \int_0^1 \phi^{(d_1)}(2\xi - 2i - k)\, \phi^{(d_2)}(2\xi - 2j - l)\, d\xi \tag{23}$$

Recourse to adequate transformations (see (Latto et al., 1995),(Beylkin, 1992)), lead to the following expression

$$\begin{aligned}\Gamma^{d_1 d_2}_{i,j} =& 2^{d-1} \sum_{k,l} p_k p_l \Big(\int_0^1 \phi^{(d_1)}(\xi - 2i - k)\, \phi^{(d_2)}(\xi - 2j - l)\, d\xi \\ &+ \int_0^1 \phi^{(d_1)}(\xi - 2i - k)\, \phi^{(d_2)}(\xi - 2j - l)\, d\xi \Big)\end{aligned} \tag{24}$$

where $d = d_1 + d_2$. According to equation (21), equation (24) can be expressed in terms of the original connection coefficients (see (Beylkin, 1992)) as

$$\begin{aligned}\Gamma^{d_1 d_2}_{i,j} &= 2^{d-1} \sum_{k,l} p_k p_l [\Gamma^{d_1 d_2}_{2i+k,2j+l} + \Gamma^{d_1 d_2}_{2i+k-1,2j+l-1}] \\ &= 2^{d-1} \sum_{r,s} [p_{r-2i} p_{s-2j} + p_{r-2i+1} p_{s-2j+1}] \Gamma^{d_1 d_2}_{r,s}\end{aligned} \tag{25}$$

or in matrix form

$$\Gamma = 2^{d-1} P\, \Gamma \tag{26}$$

where Γ is now a column vector and P is a matrix composed of wavelet coefficients combinations.

Equation (26) can also be written as,

$$(2^{d-1} P - I)\Gamma = 0 \tag{27}$$

where I is the identity matrix.

In order to uniquely determine the connection coefficients $\Gamma^{d_1 d_2}_{i,j}$, properties of scaling functions should be employed to generate sufficient number of inhomogeneous equations.

Just as shown in equation (14), Daubechies scaling functions of order N can exactly represent any polynomials of order m, with $0 \leq m \leq N/2 - 1$.

$$x^m = \sum_k c^m_k \phi(x - k) \tag{28}$$

Thus, differentiating the expansion (28) d_1 times, the following expression is obtained

$$m(m-1)\dots(m-(d_1-1))\, x^{m-d_1} = \sum_k c^m_k \phi^{(d_1)}(x - k) \tag{29}$$

In addition, equation (29) is used with n, $0 \leq n \leq N/2-1$), and d_2 instead of m and d_1, respectively. Then, multiplying both equations and integrating the product, results

$$\int_0^1 m\ldots(m-d_1+1)n\ldots(n-d_2+1)\xi^{m+n-d}d\xi = \sum_{k,l} c_k^m c_l^n \int_0^1 \phi_k^{(d_1)}\,\phi_l^{(d_2)}\,d\xi \tag{30}$$

where $d = d_1 + d_2$. Or equivalently,

$$\frac{mn\ldots(m-(d_1-1))(n-(d_2-1))}{m+n-d+1} = \sum_{k,l} c_k^m c_l^n \Gamma_{k,l}^{d_1 d_2} \tag{31}$$

Sufficient number of inhomogeneous equations can be obtained by using different values of m and n. Adding them to equation (27) connection coefficients can finally be determined uniquely.

3 EULER-BERNOULLI BEAM EQUATIONS

The static governing equations of the Euler Bernoulli beam model can be written as follows (Bathe, 1982):

$$Q'(x) = -q(x) \tag{32}$$
$$M'(x) = Q(x) \tag{33}$$
$$X(x) = \frac{M(x)}{E(x)I(x)} \tag{34}$$

$$X(x) = \theta'(x) \tag{35}$$

$$\theta(x) = -u'(x) \tag{36}$$

where $q(x)$ is the external load, $Q(x)$ and $M(x)$ are the shear force and the bending moment respectively, $u(x)$, $\theta(x)$ and $X(x)$ are the deflection, slope and curvature functions, respectively, and the prime denotes differentiation with respect to the spatial coordinate x.

The set of differential equations (32), (33), (35) and (36) represent the equilibrium and compatibility equations, respectively, while the algebraic equation (34) is the constitutive equation relating curvature and bending moment through the spatial variable flexural stiffness $E(x)I(x)$ defined by means of the Young modulus $E(x)$ and the inertia moment $I(x)$.

Combining the compatibility and constitutive equations, given by equations (34) and (36) yields to the following second order differential equation relating the bending moment with the second derivative of deflection.

$$E(x)I(x)u''(x) = -M(x) \tag{37}$$

Having into account also the equilibrium equations (32), (35) and (33), it is obtained the Euler-Bernoulli differential governing equation, in terms of deflection only, as follows

$$[E(x)I(x)u''(x)]'' = q(x) \tag{38}$$

This is a linear fourth order differential equation of with variable coefficients, defined on a domain as the interval $0 < x < L$, where L is the length of the beam.

When this equation is integrated, there appear four arbitrary constants, so that four additional conditions must be imposed to determine the solution uniquely. These additional conditions come from the boundary conditions of the problem.

Integration of equation (37) is usually performed for statically determinate beams in view of the knowledge of the bending moment $M(x)$ through the equilibrium equations, otherwise the more general fourth order differential equation (38) has to be used.

Research fields such as fracture mechanics might require the study of beams presenting varying loads and singularities along the beam span. Moreover, cases showing abrupt changes of the cross section might result in the appearance of discontinuity in the kinematics solution function such as curvature and slope functions.

In this paper, the problem of integration of Eq.(38) is analyzed in two different cases: the first example is a beam with linearly varying load and the second one a beam with singular flexural stiffness.

3.1 Non-uniform loaded beam

A beam of length $2L$ with equal cross section and constant bending rigidity $E_0 I_0$ is non-uniform loaded. Only on the right half segment a linear load is applied, which has the expression, $q(x) = Kq_0(\frac{x}{L} - 1) \quad L < x < 2L.$

An exact solution can be obtained by integrating the differential equation and closed form solutions in terms of deflection, can be written as

$$w = \frac{q_0}{E_0 I_0}[c_2 + c_1 x + \frac{1}{2}(c_4 + \frac{KL^2}{6})x^2 + \frac{1}{6}(c_3 - \frac{KL}{2})x^3] \qquad 0 \leq x \leq L \tag{39}$$

$$w = \frac{q_0}{E_0 I_0}[\frac{Kx^5}{120L} - \frac{Kx^4}{24} + c_3\frac{x^3}{6} + c_4\frac{x^2}{2} + \frac{KL^3}{24}x - \frac{KL^4}{120}] \qquad L \leq x \leq 2L \tag{40}$$

where, the integration constants c_i are obtained by means of enforcement of boundary conditions.

3.2 Flexural stiffness discontinuity

In the second case flexural stiffness singularity is modeled by the unit step function, as it was analyzed in (Biondi and Caddemi, 2005). So it can be expressed as,

$$E(x)I(x) = E_0 I_0[1 - \gamma U(x - x_0)] \tag{41}$$

where $0 \leq x_0 \leq 2L$, $U(x - x_0)$ is the well known unit step or Heaviside function and γ is a parameter that represents the discontinuity intensity. In order to satisfy the physical constraint of non-negativity for the flexural stiffness the condition $0 \leq \gamma \leq 1$ is required.

The governing equation (38) assumes the following form

$$[E_0 I_0[1 - \gamma U(x - x_0)]u''(x)]'' = q(x) \tag{42}$$

Equation (42) models a beam model with abrupt variation of the cross section or of the Young modulus, resulting in a discontinuous flexural stiffness at the abscissa x_0 (a jump) and constant elsewhere.

As it was mentioned before, closed form solutions can be obtained integrating the differential equation and are presented in (Biondi and Caddemi, 2005).

4 WAVELET FINITE ELEMENT EQUATION FOR A BEAM

The generalized function of potential energy for Bernoulli beam is (Bathe, 1982)

$$\pi = \frac{1}{2}\int_0^L \{[E(x)I(x)w''(x)]^2 - 2w(x)q(x)\}dx \tag{43}$$

and it is assumed that the displacement w can be approximated by Daubechies scaling functions of order N as,

$$w = \sum_{k=-(N-2)}^{0} \alpha_k \phi(\xi - k) \tag{44}$$

where α_k are the coefficients of approximation to be determined.

The minimization of equation (43), (which in this case is equivalent to ask $\delta\pi = 0$), as in standard finite element methods, provides a linear system $K\alpha = R$ which has to be solved. To obtain the stiffness matrix K and load vector R connection coefficients the equation (21) has to be used, and in case that $E(x)I(x)$ is constant, take the form,

$$\widetilde{k}_{ij}^{(e)} = \int_0^L \phi''(\xi - i)\phi''(\xi - j)d\xi \tag{45}$$

$$R_i^{(e)} = \int_0^L \phi''(\xi - i)q(\xi)d\xi \tag{46}$$

It is important to point out that when a large number of wavelet based elements are used in structure analysis, for ensure calculation efficiency, connection coefficients can be calculated and stored first. Then these connection coefficients can be used directly in the calculation of each element stiffness matrix.

This stiffness matrix is in wavelet space, and the corresponding degrees of freedom are wavelet coefficients α_k. In order to satisfy boundary conditions and compatibility at the interfaces between neighboring elements, the stiffness matrix should be transformed from wavelet space into physical space, and the elemental degrees of freedom should be transformed from wavelet space into physical space. This transformation can be expressed as,

$$\widehat{w} = T\alpha$$

where $\widehat{w}$ is the vector of elemental displacement and rotations, and T is the transformation matrix, calculated evaluating scaling function and its derivatives. The algorithm has been described in section (2.2).

The elemental stiffness matrix in physical space can be calculated from the one in wavelet space , $\widetilde{k}^{(e)}$ with

$$\overline{k}^{(e)} = (T^{-1})^t \widetilde{k}^{(e)} T^{-1} \tag{47}$$

As customary in the finite element method, all the "element-stiffness" matrices are assembled to obtain a linear system of equations and, after imposing boundary conditions, displacements in all nodes are obtained. Sometimes stresses are also required, and then the procedure is the same as in an standard FEM.

4.1 Daubechies wavelet plane beam element D12

In the finite wavelet element for Euler beam equation (38) proposed in this paper, the displacement w in equation (44), is approximated by Daubechies scaling functions of order $N = 12$. Accounting that D12 has 11 degrees of freedom, 9 nodal displacements and only two nodal rotations at the ends are considered. The degrees of freedom for each element are

$$\overline{w}^T = [w_1, \theta_1, w_2, w_3, w_4, \ldots, w_8, w_9, \theta_9] \tag{48}$$

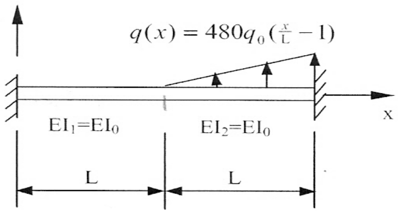

Figure 2: Clamped-clamped beam with linear varying load

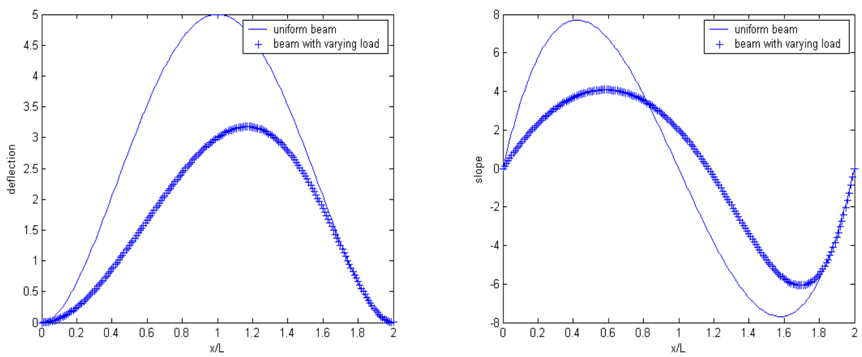

Figure 3: Left: deflection Right: slope

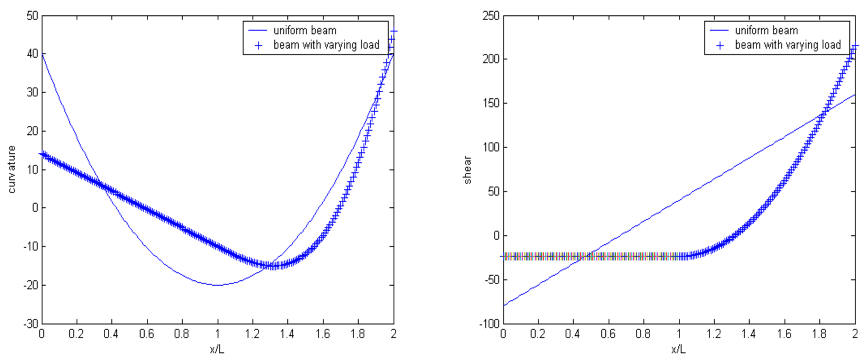

Figure 4: Left: curvature Right: shear

5 APPLICATIONS

5.1 Non-uniform loaded beam

Application of the closed form solution presented in section 3.1 is considered for a clamped-clamped beam and $K = 480$ (see Fig.2). Integration constants are then uniquely determined from boundary conditions. Deflection, slope, curvature and shear functions are plotted in Fig.3 and Fig.4. These functions are compared in each case with the ones obtained considering a uniform beam with constant load $q = \frac{K}{4L}$. A very different behaviour, produced as a consequence of the linear varying load, in particular in shear function (see Fig. 4), can be observed.

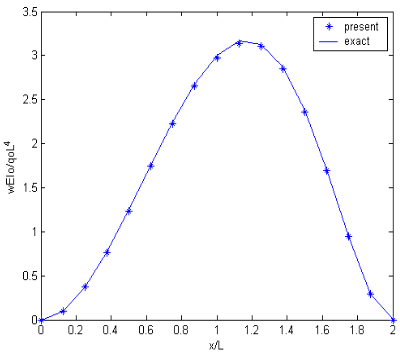

Figure 5: Numerical results: Non-uniform loaded beam

5.1.1 Numerical solutions

Using Daubechies ($N = 12$) the left and right segments are divided into one wavelet-based beam element, respectively, to analyze the problem. The deflection curve obtained in each case is compared with the exact one in Fig. 5. It can be seen that a good approximation is obtained, with a maximum relative error of 1.11.

5.2 Flexural stiffness discontinuity

A beam with a jump discontinuity as it was described in section 3.2 is considered. In this example, the beam is simply supported and shows an abrupt flexural stiffness change at $x = L$, from the value $E_0 I_0$ to the value $4E_0 I_0$, which corresponds to the value $\gamma = 3/4$ described in section 3.2. As it is shown in the Fig. 6, the beam is loaded with $q(x) = 1$. This flexural stiffness jump results in a general decrement of the deflection function with respect to the uniform beam (see Fig. 7).

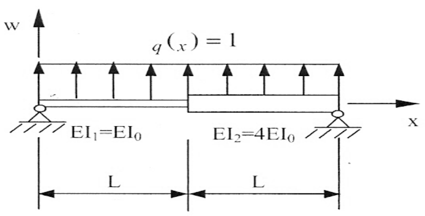

Figure 6: Simply supported beam with flexural stiffness discontinuity

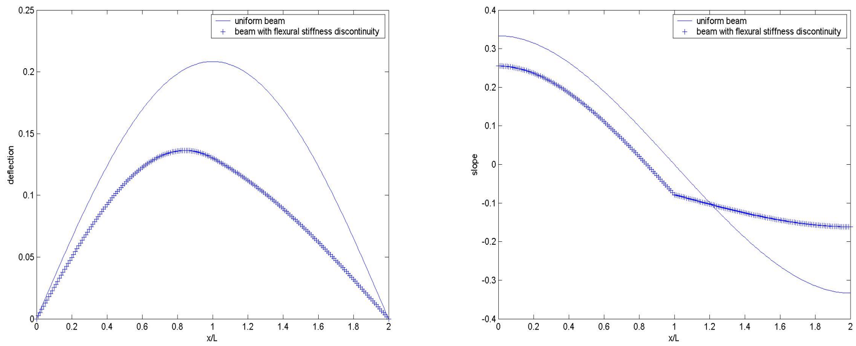

Figure 7: Left: Deflection   Right: Slope

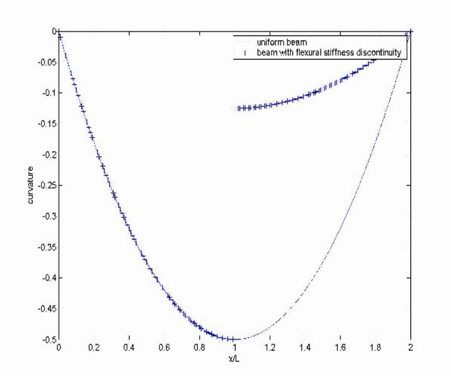

Figure 8: Curvature

The closed form solutions in terms of deflection, slope and curvature functions are plotted in Fig. 7 and 8, and compared with the solution of the uniform beam with constant flexural stiffness $E_0 I_0$.

Notice that the solution functions, in view of the singularity of the flexural stiffness, show continuous deflection and slope functions, but a curvature function showing a discontinuity at $x = L$.

5.2.1 Numerical solutions

As in the previous example, the left and right segments are divided into one wavelet beam element to analyze the problem.

Numerical results are plotted with the exact solution in Fig. 9 and good accuracy can be observed. Calculating the error displacement at nodal points, the maximum relative error was 0.38.

6 CONCLUSIONS

The numerical tests reported in this work demonstrated the feasibility and capability of using wavelet bases in the FEM. In particular, numerical examples illustrate that the wavelet-based beam element formulated in terms of Daubechies wavelet basis functions has good accuracy.

An efficient integral method to calculate stiffness and load matrices was necessary to ensure numerical stability. In particular, the use of the Daubechies wavelets, having the good

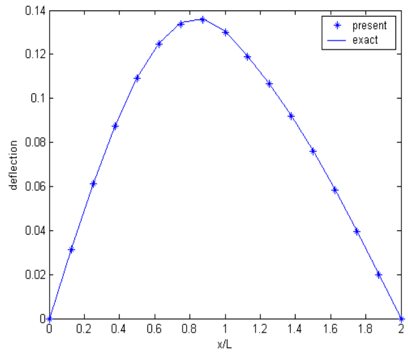

Figure 9: Numerical results: Flexural stiffness discontinuity

properties of orthogonality, compact support and nesting, guaranty the higher-order polynomial approximation.

A Matlab program was generated for the numerical experiments presented in this paper.

It can be forecasted that wavelet-based elements would play an important role to analyze more complex problems in two or three dimensions and should be studied in the future.

The authors strongly believe that combining wavelet function and finite element techniques to solve a wider class of partial differential equations is a field where still there is much to research.

ACKNOWLEDGMENTS

This work was partially supported by SECYT-CITMA (Proyecto de Cooperación Bilateral 06-07. Código CU/PA05-EIII/001).

REFERENCES

Bathe K. *Finite Element Procedures in Engineering Analysis*. Prentice-Hall, New Jersey, 1982.

Beylkin G. On the representation of operators in bases of compactly supported wavelets. *SIAM J. Numer. Anal.*, 6:1716–1740, 1992.

Biondi B. and Caddemi S. Closed form solutions of euler-bernoulli beams with singularities. *Int.J. Solid and Structures*, 42:3027–3044, 2005.

Chen X., Yang S., Ma J., and He Z. The construction of wavelet ¯finite element and its application. *Finite Elements in Analysis and Design.*, 40:541–554, 2004.

Chui C.K. *An introduction to wavelets*. Academic Press, New York, 1992.

Daubechies I. Orthonormal bases of compactly supported wavelets. *Commun. Pure Appl. Math.*, 41:909–996, 1988.

Daubechies I. *Ten Lectures on Wavelets.* MA SIAM: Philadelphia, 1992.

Latto A., Resnikoff H., and Tenenbaum E. The evaluation of connection coefficients of compactly supported wavelets. *Proceedings of the USA-French Workshop on Wavelets and Turbulence. Princeton University*, 1995.

Ma J., Xue J., Yang S., and He Z. A study of the construction and application of a daubechies wavelet-based beam element. *Finite Elements in Analysis and Design.*, 39:965–975, 2003.