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NEW CLASS OF MULTISTEP METHODS WITH ADAPTIVE COEFFICIENTS

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Abstract. A new class of multistep methods for stiff ordinary differential equations is presented. The method is based in the application of estimation functions not only for the derivatives but also for the state variables, which permits the transformation of original system in a purely algebraic system using the solutions of previous steps. From this point of view these methods adopt a semi-implicit scheme. The novelty introduced is an adaptive formula for the estimation function coefficients, which is deduced from a combined analysis of stability and convergence order. That is, the estimation function coefficients are recalculated in each time step. The convergence order of the resulting scheme is better than the equivalent linear multistep methods, while preserving A-stability. Numerical experiments are presented comparing the new method with BDF.

1 INTRODUCTION

In recent years, considerable efforts have been focused on the development of more advanced and efficient methods for stiff problems. A potentially good numerical method for the solution of stiff systems of ordinary differential equations (ODE) must have good accuracy and some reasonably wide region of absolute stability (Dahlquist 1963). The latter imposes a strong limitation on the choice of suitable methods for stiff problems. In general, the search for higher order A-stable methods has been carried out in the two main directions:

- Use of higher derivatives of the solutions
- Introduction of additional stages, off-step points, and super-future points (this leads into the large field of general linear methods (Hairer 1996)).

In the present paper a new class of multistep methods is derived, having good stability properties and improvements in the convergence order compared with equivalent linear schemes. The method is based in the application of estimation functions not only of the derivatives but also of the state variables, which leads to the transformation of original system in a purely algebraic system using the solutions of previous steps. The novelty introduced is an adaptive formula for the estimation function coefficients, which is deduced from a combined analysis of stability and convergence order. In the last section, numerical experiments are presented comparing the new method with BDF.

2 LINEAR MULTISTEP METHODS

Let us consider the following initial value problem:

$$y^{(1)}(t) = f(y),\ y(t_0) = y_0\,, \tag{2.1}$$

where $t \in [t_0, t_0 + Nh]$ (N being a natural number and h a constant time step), $y : [t_0, t_0 + Nh] \to R^m$, $y^{(1)}$ stands for the first temporal derivative, and $f : [t_0, t_0 + Nh] \times R^m \to R^m$ is continuous and differentiable.

The general multistep method can be written in the form (Ascher 1998):

$$\sum_{j=0}^{k} \alpha_j y_{n-j} = h \sum_{j=0}^{k} \beta_j f_{n-j}\,, \tag{2.2}$$

where α_j, β_j are parameters to be determined, $f_n = f(y_n)$, and $y_n = y(t_0 + nh)$, being h a constant time step.

It can be shown that a multistep method is of order p if and only if (Butcher 2003):

$$\sum_{j=0}^{k} \alpha_j j^q = q \sum_{j=0}^{k} \beta_j j^{q-1} + \mathrm{O}(h^p), \tag{2.3}$$

with $0 \le q \le p$.

A popular multistep scheme, which will be used later in this article for comparison, is the Backward Differentiation Formula (BDF) (Ascher 1998), which is given by:

$$\sum_{j=0}^{k}\alpha_j y_{n-j} = h\beta_0 f_n , \tag{2.4}$$

This scheme is a class of k-step formulas of order k. Their distinguishing feature is that f is evaluated only at the right end of the current step, (t_n, y_n).

3 HYBRID MULTISTEP METHOD (HMM)

The general multistep formula (*i.e.* Eq. 2.2) is basically a transformation of the differential Eq. 2.1 into a purely algebraic equation by means of the estimators:

$$\begin{aligned} y^{(1)} &\to \frac{1}{h}\sum_{j=0}^{k}\alpha_j y_{n-j} , \\ f &\to \sum_{j=0}^{k}\beta_j f_{n-j} . \end{aligned} \tag{3.1}$$

Alternatively, let us propose the following set of transformations:

$$\begin{aligned} y &\to \sum_{i=0}^{l} a_i y_{n-i} , \\ y^{(1)} &\to \frac{1}{h}\sum_{i=0}^{k} b_i y_{n-i} . \end{aligned} \tag{3.2}$$

which leads to the following alternative multistep algebraic equation:

$$\frac{1}{h}\sum_{i=0}^{k} b_i y_{n-i} = f\left(\sum_{i=0}^{l} a_i y_{n-i}\right). \tag{3.3}$$

4 SCHEME $k = l = 2$, $m = 1$

Let us consider the case k = l = 2 and m = 1 in order to develop a general method to determine the coefficients a_i and b_i. Eqs. 3.2 and 3.3 becomes:

$$
\begin{aligned}
&y \to \sum_{i=0}^{2} a_i y_{n-i}, \\
&y^{(1)} \to \frac{1}{h}\sum_{i=0}^{2} b_i y_{n-i}, \\
&\frac{1}{h}\sum_{i=0}^{2} b_i y_{n-i} = f\left(\sum_{i=0}^{2} a_i y_{n-i}\right).
\end{aligned} \tag{4.1}
$$

Expanding y_{n-2} and y_n in Taylor series about (*t*-*h*) leads to:

$$
\begin{aligned}
y_n &= y_{n-1}^{(0)} + h\, y_{n-1}^{(1)} + \frac{h^2}{2} y_{n-1}^{(2)} + \frac{h^3}{6} y_{n-1}^{(3)} + \mathrm{O}\left(h^4\right), \\
y_{n-2} &= y_{n-1}^{(0)} - h\, y_{n-1}^{(1)} + \frac{h^2}{2} y_{n-1}^{(2)} - \frac{h^3}{6} y_{n-1}^{(3)} + \mathrm{O}\left(h^4\right),
\end{aligned} \tag{4.2}
$$

where $y_n^{(k)}$ stands for the k derivative of y respect to time evaluated at (t-h). Combining Eqs. 4.1 and 4.2 yields:

$$
\begin{aligned}
y_n^{(1)} &= \left(b_0 - b_2\right) y_{n-1}^{(1)} + \left(b_0 + b_2\right) y_{n-1}^{(2)} \frac{h}{2} + \left(b_0 - b_2\right) y_{n-1}^{(3)} \frac{h^2}{6} + \mathrm{O}\left(h^3\right), \\
y_n &= y_{n-1}^{(0)} + \left(a_0 - a_2\right) y_{n-1}^{(1)}\, h + \left(a_0 + a_2\right) y_{n-1}^{(2)} \frac{h^2}{2} + \mathrm{O}\left(h^3\right).
\end{aligned} \tag{4.3}
$$

Likewise, expanding $f\left(\sum_{i=0}^{2} a_i y_{n-i}\right)$ around (*t*-*h*), gives:

$$
\begin{aligned}
f\left(\sum_{i=0}^{2} a_i y_{n-i}\right) &= f_{n-1}^{(0)} + \left(a_0 - a_2\right) y_{n-1}^{(1)} f_{n-1}^{(1)}\, h + \\
&\left[\left(\left(a_0 - a_2\right) y_{n-1}^{(1)}\right)^2 f_{n-1}^{(2)} + \left(a_0 + a_2\right) y_{n-1}^{(2)} f_{n-1}^{(1)}\right] \frac{h^2}{2} + \mathrm{O}\left(h^3\right),
\end{aligned} \tag{4.4}
$$

where $f_n^{(k)}$ stands for the k derivative of $f(y)$ evaluated at $y = y_n$. The relation between $f_n^{(k)}$ and $y_n^{(k)}$ can be found by successive differentiation of Eq. 2.1, that is:

$$
\begin{aligned}
y_n^{(1)} &= f_n^{(0)}, \\
y_n^{(2)} &= f_n^{(1)} f_n^{(0)}, \\
y_n^{(3)} &= f_n^{(2)} f_n^{(0)2} + f_n^{(1)2} f_n^{(0)}.
\end{aligned} \tag{4.5}
$$

Combining Eqs. 4.3 to 4.5, the requirements to satisfy Eq. 4.1 to $O(h^3)$ are given by:

$$\sum_{i=0}^{2} a_i = 1, \tag{4.6}$$

$$\sum_{i=0}^{2} b_i = 0, \tag{4.7}$$

$$(b_0 - b_2 - 1) f_{n-1}^{(0)} = 0, \tag{4.8}$$

$$(b_0 + b_2 - 2a_0 + 2a_2) f_{n-1}^{(1)} f_{n-1}^{(0)} = 0, \tag{4.9}$$

$$\left[b_0 - b_2 - 3(a_0 - a_2)^2 \right] f_{n-1}^{(2)} f_{n-1}^{(0)2} + f_{n-1}^{(1)2} f_{n-1}^{(0)} (b_0 - b_2 - 3a_0 - 3a_2) = 0. \tag{4.10}$$

Eqs. 4.6 to 4.10 is a set of 5 algebraic equations with 6 unknowns. Therefore, there is a family of coefficients a_i and b_i that ensures $O(h^3)$ convergence, that is:

$$\begin{aligned} & a_0 = \frac{1}{6} - \frac{b_1}{4} + c, \quad a_1 = \frac{2}{3} - 2c, \quad a_2 = \frac{1}{6} + \frac{b_1}{4} + c, \\ & b_0 = \frac{1}{2} - \frac{b_1}{2}, \quad b_2 = -\frac{1}{2} - \frac{b_1}{2}, \end{aligned} \tag{4.11}$$

where

$$c = \frac{f_{n-1}^{(0)} f_{n-1}^{(2)} (4 - 3b_1)}{24 \left(f_{n-1}^{(1)} \right)^2}. \tag{4.12}$$

If f is linear (*i.e.,* $f^{(2)} = 0$) then c = 0. Hence, term c can be seen as a non-linear correction, which can be applied to the coefficients in each step in order to increase the convergence order of the scheme. However, this correction is only valid when $f^{(1)} \neq 0$. In that case, Eq. 4.9 is automatically satisfied, leading to two 2-parameters family of solutions, that is:

$$\begin{cases} a_0 = -\dfrac{1}{2} a_1 + \dfrac{1}{2} \pm \dfrac{\sqrt{3}}{6}, \\ a_2 = -\dfrac{1}{2} a_1 + \dfrac{1}{2} \mp \dfrac{\sqrt{3}}{6}, \\ b_0 = \dfrac{1}{2} - \dfrac{b_1}{2}, \\ b_2 = -\dfrac{1}{2} - \dfrac{b_1}{2}. \end{cases} \tag{4.13}$$

In order to chose one particular set of coefficients from the $O(h^3)$ convergence families, A-stability is required from the scheme. A method is A-stable if applied to a stable linear set of differential equations the resulting iterative scheme is also stable independently of *h*. In that way, ensuring A-stability, *h* is determined just for precision purposes, without restrictions on linear stability. Such methods are considered good candidates to solve stiff problems (Ascher 1998).

Applying Eqs. 3.3 to the linear test equation:

$$y^{(1)} = \lambda y, \tag{4.14}$$

which leads to:

$$\sum_{i=0}^{2} (b_i - \lambda a_i) y_{n-i} = 0. \tag{4.15}$$

Eq. 4.15 is a second order linear difference equation, whose stability is ensured if the real part of the roots of the characteristic polynomial:

$$\sum_{i=0}^{2} (b_i - \lambda a_i) q^{2-i} = 0\,. \tag{4.16}$$

are negative. A-stability is then given by (Ascher 1998):

$$\mathrm{Re}(z) \geq 0\,, \tag{4.17}$$

where:

$$z = \frac{\sum_{i=0}^{2} b_i q^{2-i}}{\sum_{i=0}^{2} a_i q^{2-i}}, \tag{4.18}$$

for all (unitary) complex numbers $q = \cos\theta + i\sin\theta$, $(\theta \in [0,2\pi])$.

Eq. 4.17 is satisfied if:

$$\begin{gathered}\left[b_0\left(\cos^2\theta - \sin^2\theta\right) + b_1\cos\theta + b_2\right]\left[a_0\left(\cos^2\theta - \sin^2\theta\right) + a_1\cos\theta + a_2\right] + \\ + \left(2b_0\cos\theta + b_1\right)\left(2a_0\cos\theta + a_1\right)\sin^2\theta \geq 0,\end{gathered} \tag{4.19}$$

provided that $f_{n-1}^{(1)} \neq 0$. Combining Eqs. 4.11 and 4.19, yields:

$$b_1 (\cos\theta - 1)^2 \geq 0, \tag{4.20}$$

That is, the scheme $k = l = 2$ and $m = 1$ is A-stable if

$$b_1 \geq 0. \tag{4.21}$$

In turn, if $f_{n-1}^{(1)} = 0$, the A-stability condition requires:

$$\frac{1}{3}(\cos\theta - 1)\left(\sqrt{3}\cos\theta + 3b_1a_1(\cos\theta - 1) - 3b_1\cos\theta + \sqrt{3}\right) \geq 0. \tag{4.22}$$

Unfortunately, Eq. 4.22 is not satisfied by any real values of b_1 and a_1. In those singular iteration steps an *ad-hoc* solution should be chosen, either temporary compromising the stability or reducing the convergence order around $f_{n-1}^{(1)} = 0$. Numerical experiments indicated that the latter leads to better results. A good practical alternative is to switch to the BDF method —which is A-stable and $O(h^2)$ (Ascher 1998)— when $\left|f_{n-1}^{(1)}\right|^{-1}$ exceeds some critical threshold (figure 1). For $k = 2$, the BDF coefficients are:

$$\begin{aligned} &a_0 = \frac{3}{2}, \quad a_1 = -2, \quad a_2 = \frac{1}{2}, \\ &b_0 = 1, \quad b_1 = 0, \quad b_2 = 0. \end{aligned} \tag{4.23}$$

In practice, the implicit numerical calculation fails whenever c increases above certain value. In the particular cases analysed in the present study, good results were found when $|c| < 1$, and switching to BDF method when $\left|f_{n-1}^{(1)}\right| \approx 0$. This topic is still under investigation.

5 GENERAL CASE (*m* ≥ 1)

Eqs. 4.11 and 4.12 can be generalized to more than one variable (*i.e.* $m \geq 1$), which leads to a similar set of coefficients. The estimator's results:

$$
\begin{aligned}
& y_j \to \sum_{i=0}^{2} a_{i,j} y_{n-i,j} , \\
& y_n^{(1)} \to \frac{1}{h} \sum_{i=0}^{2} b_i y_{n-i,j} , \\
& \frac{1}{h} \sum_{i=0}^{2} b_i y_{n-i,j} = f_j \begin{pmatrix} \sum_{i=0}^{2} a_{i,0} y_{n-i,0} \\ \ldots \\ \sum_{i=0}^{2} a_{i,m} y_{n-i,m} \end{pmatrix} .
\end{aligned}
\tag{5.1}
$$

Note that the b-coefficients are the same for all the variables y_j, whereas the a-coefficients are different for each variable. The general expressions ensuring $O(h^3)$ convergence are:

$$
\begin{aligned}
& a_{0,j} = \frac{1}{6} - \frac{b_1}{4} + c_j , \\
& a_{1,j} = \frac{2}{3} - 2c_j , \\
& a_{2,j} = \frac{1}{6} + \frac{b_1}{4} + c_j , \\
& b_0 = \frac{1}{2} - \frac{b_1}{2} , \\
& b_2 = -\frac{1}{2} - \frac{b_1}{2} ,
\end{aligned}
\tag{5.2}
$$

where:

$$
c_j = \frac{\left(f^T H_{f_j} f\right)\left(4 - 3b_1\right)}{24\left(\nabla f_j^T J_f f\right)} .
\tag{5.3}
$$

In addition, Eq. 4.21 should hold for b_1 in order to comply with A-stability.

6 NUMERICAL EXPERIMENTS

In order to assess the performance of the new multistep method, it was applied to the integration of specific equations, comparing with the well-known BDF method.

6.1 Ricatti equation (*m* = 1)

Let us consider the following Ricatti equation (Abramowitz 1972):

$$
y^{(1)} = -2 - y + y^2 ,
\tag{6.1}
$$

with initial value $y_0 = 1.8$. The exact solution is given by:

$$y(t) = 2 - \frac{3}{1+14e^{-3t}}. \tag{6.2}$$

Figure 2 shows the temporal evolution of $f^{(0)}$, $f^{(1)}$ and $f^{(2)}$, the coefficients a_i and the non-linear correction term *c*. It can be seen that when $|c| = 0.083$ (0.083 was chosen randomly verifying condition 4.21, in order so to maintain the superiority of HMM respect of BDF) for *t* between 0.4 and 1.4, the value of the coefficients remain constant. Otherwise they vary according to Eqs. 4.7, which ensures $O(h^3)$ convergence during those periods.

Figures 3 and 4 show the absolute difference $|y(x_n) - y_n|$ between the analytic and the numerical solutions (figure 3 corresponds with $h = 10^{-2}$, and figure 4 with $h = 10^{-3}$). It can be observed that the numerical solution obtained with the new method is always better than the BDF solution.

6.2 Van der Pol oscillator

The Van der Pol (VDP) oscillator is a second order system that can be derived from the Rayleigh equation (Thompson 1986), and describes the behavior of nonlinear electronic circuits such as those used in the primitive radios. The system is dissipative and leads to limit cycles. A simple form of VDP in terms of first order ODE is:

$$\begin{aligned} y^{(1)}{}_1 &= y_2, \\ y^{(1)}{}_2 &= -y_1 + y_2\left(\mu - y_1^2\right). \end{aligned} \tag{6.3}$$

where μ is a constant parameter which determines the size of the limit cycle. The corresponding Jacobian and Hessians of f_1 and f_2 are:

$$J = \begin{bmatrix} 0 & 1 \\ -1-2y_1y_2 & \left(\mu - y_1^2\right) \end{bmatrix},\ H_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ H_2 = \begin{bmatrix} -2y_2 & -2y_1 \\ -2y_1 & 0 \end{bmatrix}.$$

The non-linear correction terms are given by:

$$c_1 = 0,$$

$$c_2 = \frac{\begin{pmatrix} y_2 & -y_1 + y_2\left(\mu - y_1^2\right) \end{pmatrix} \begin{pmatrix} -2y_2 & -2y_1 \\ -2y_1 & 0 \end{pmatrix} \begin{pmatrix} y_2 \\ -y_1 + y_2\left(\mu - y_1^2\right) \end{pmatrix}}{\begin{pmatrix} -1-2y_1y_2 & \left(\mu - y_1^2\right) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1-2y_1y_2 & \left(\mu - y_1^2\right) \end{pmatrix} \begin{pmatrix} y_2 \\ -y_1 + y_2\left(\mu - y_1^2\right) \end{pmatrix}} =$$

$$-\frac{-4y_1^3y_2^2-4y_1^2y_2+4\mu y_1y_2^2+2y_2^3}{\mu^3y_2-3\mu^2y_1^2y_2-\mu^2y_1+3\mu y_1^4y_2+2\mu y_1^3-4\mu y_1y_2^2-2\mu y_2-y_1^6y_2-y_1^5+4y_1^3y_2^2+4y_1^2y_2+y_1}$$

Therefore the a_i estimator coefficients for y_1 are constant.

Numerical experiments were performed for the case $\mu = 3.5$. Figure 5 shows the limit cycle in the phase plane (y_1, y_2). Figure 6 shows the temporal evolution of the second non-linear correction term, c_2, and figure 7 shows the temporal evolution of the state variables, and the coefficients $a_{0,2}$, $a_{1,2}$, $a_{2,2}$. It can be seen that the coefficients vary significantly, not only at the "elbows" of the cycle but also during the apparently smooth periods of the phase-space trajectory.

6.3 Elastic Pendulum

The elastic pendulum (Fig. 8) is conservative a fourth order system whose natural variables are the string length, *r*, the inclination angle respect to the vertical, *θ*, and their respective temporal derivatives, *z* and *w*, that is:

$$\begin{aligned}
r^{(1)} &= z,\\
\theta^{(1)} &= w,\\
z^{(1)} &= rw^2 - \frac{k}{m}(r-L) + g\cos\theta,\\
w^{(1)} &= (-g\sin\theta - 2zw)\frac{1}{r},
\end{aligned} \tag{6.4}$$

where *k* and *L* are the elastic constant and the equilibrium length of the string, *m* is the mass attached to the string, and *g* is the gravity acceleration.

The corresponding Jacobian and Hessians are:

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2rw & -k/m + w^2 & -g\sin\theta \\ -2w/r & -2z/r & (g\sin\theta + 2zw)/r^2 & -g/r\,\cos\theta \end{bmatrix},$$

$$H_r = 0,\ H_\theta = 0,\ H_z = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2r & 2w & 0 \\ 0 & 2w & 0 & 0 \\ 0 & 0 & 0 & -g\cos\theta \end{bmatrix},$$

$$H_w = \begin{bmatrix} 0 & -2/r & 2w/r^2 & 0 \\ -2/r & 0 & 2z/r^2 & 0 \\ 2w/r^2 & 2z/r^2 & -2(g\sin\theta + 2zw)/r^3 & g\cos\theta/r^2 \\ 0 & 0 & g\cos\theta/r^2 & -g\cos\theta \end{bmatrix},$$

and the non-linear correction terms:

$$c_r = c_\theta = 0$$

$$c_z = \begin{pmatrix} g^3m^3\cos^3\theta - g^3m^3\cos\theta - 4(\sin\theta)g^2m^3wz\cos\theta + 4gm^3r^2w^2\cos\theta - \\ 4gm^3w^2z^2\cos\theta + 4m^3r^3w^4 + 2m^3r^3w^2 - 4km^2r^3w^2 + 4Lkm^2r^2w^2 \end{pmatrix} *$$
$$\begin{pmatrix} 2g^3m^3\cos^3\theta - 2g^3m^3\cos\theta + Lg^2km^2\cos^2\theta - Lg^2km^2 - \\ 4z(\sin\theta)g^2m^3w\cos\theta + gk^2mr^2\cos\theta - 2gkm^2r^2w^2\cos\theta - \\ 2Lz(\sin\theta)gkm^2w + gm^3r^2w^4\cos\theta + 4z(\sin\theta)gm^3rw - k^3r^3 + \\ Lk^3r^2 + 3k^2mr^3w^2 - 2Lk^2mr^2w^2 - 3km^2r^3w^4 - 2km^2r^3w^2 + \\ Lkm^2r^2w^4 + m^3r^3w^6 + 2m^3r^3w^4 + 2m^3r^3w^2 \end{pmatrix}^{-1}$$

$$c_w = \left(-4 \begin{pmatrix} 2(\sin\theta)L^2gk^2 + 4L^2k^2wz + 6(\sin\theta)Lg^2km\cos\theta - 4(\sin\theta)Lgk^2r + \\ 4(\sin\theta)Lgkmrw^2 + 12Lgkmwz\cos\theta - 8Lk^2rwz + 8Lkmrw^3z - \\ 8Lkmrwz - g^3m^2r\cos^3\theta + g^3m^2r\cos\theta + 4(\sin\theta)g^3m^2\cos^2\theta - \\ 6(\sin\theta)g^2kmr\cos\theta + 6(\sin\theta)g^2m^2rw^2\cos\theta + 4(\sin\theta)g^2m^2rwz\cos\theta + \\ 8g^2m^2wz\cos^2\theta + 2(\sin\theta)gk^2r^2 - 4(\sin\theta)gkmr^2w^2 - \\ 12gkmrwz\cos\theta + 2(\sin\theta)gm^2r^2w^4 + 12gm^2rw^3z\cos\theta + \\ 4gm^2rw^2z^2\cos\theta - 8gm^2rwz\cos\theta + 4k^2r^2wz - 8kmr^2w^3z + \\ 8kmr^2wz + 4m^2r^2w^5z - 8m^2r^2w^3z + 4m^2r^2wz \end{pmatrix} \right) *$$
$$\begin{pmatrix} 3g^3m^2\sin\theta - g^3m^2\sin 3\theta + 4gk^2r^2\sin\theta + 16m^2r^2w^3z + 8m^2r^2w^5z - \\ 8g^3m^2\cos^2\theta\sin\theta + 16g^2m^2wz + 8k^2r^2wz - 16m^2r^2wz - 8Lk^2rwz - \\ 32g^2m^2wz\cos^2\theta + 8gm^2r^2w^2\sin\theta + 4gm^2r^2w^4\sin\theta + 16gm^2w^2z^2\sin\theta \\ -4Lgk^2r\sin\theta - 16kmr^2w^3z - 8gkmr^2w^2\sin\theta + 8Lkmrw^3z + \\ 16gm^2rwz\cos\theta - 4Lg^2km\cos\theta\sin\theta - 8Lgkmwz\cos\theta + 4Lgkmrw^2\sin\theta \end{pmatrix}^{-1}$$

The following set of parameters and initial conditions were chosen for numerical integration of a particular case: $k = 7$, $L = 1$, $m = 0.1$, $g = 9.8$, $r_0 = 1$, $\theta_0 = \frac{\pi}{2}$, ${r^{(1)}}_0 = 0$ and ${\theta^{(1)}}_0 = 0$.

Figure 10 shows the temporal evolution of c_z and c_w. It can be seen that c_w fluctuates more often than c_z, which is reflected in the variations of the corresponding *a* estimator coefficients (Figs. 11 and 12). A clue to the apparently erratic behavior of the coefficient can be found in

Fig. 13 where the non-linear correction terms are plotted together with $z^{(0)}$ and $w^{(0)}$. It can be observed that the peaks of c_z and c_w are in coincidence with $z^{(0)}$, $z^{(1)}$, $w^{(0)}$ or $w^{(1)}$ approaching 0 (Figure 14 shows non-linear correction terms c_z, c_w, and the first temporal derivative $z^{(1)}, w^{(1)}$). This observation is in agreement with what is expected from Eqs. 6.5:

$$
\begin{aligned}
y_n^{(1)} &= f\left(y_n\right), \\
y_n^{(2)} &= J_f\left(y_n\right) y_n^{(0)}, \\
y_{n,i}^{(3)} &= \left(y_n^{(1)}\right)^T H_{f_i}\left(y_n\right) y_n^{(1)} + J_{f_i}\left(y_n\right) y_n^{(2)},
\end{aligned}
\tag{6.5}
$$

where the relation between $J_{f_i}(y_n)$, $H_{f_i}(y_n)$ and $y_n^{(k)}$ can be observed applying successive differentiation to Eq. 2.1.

7 CONCLUSIONS

The HMM is an A-stable method, and it is a good candidate method for the solution of stiff problems. The computation time of the HMM method is approximately the same as the BDF method, however the new method benefits of better precision and a larger stability region, as it is shown in the proposed examples.

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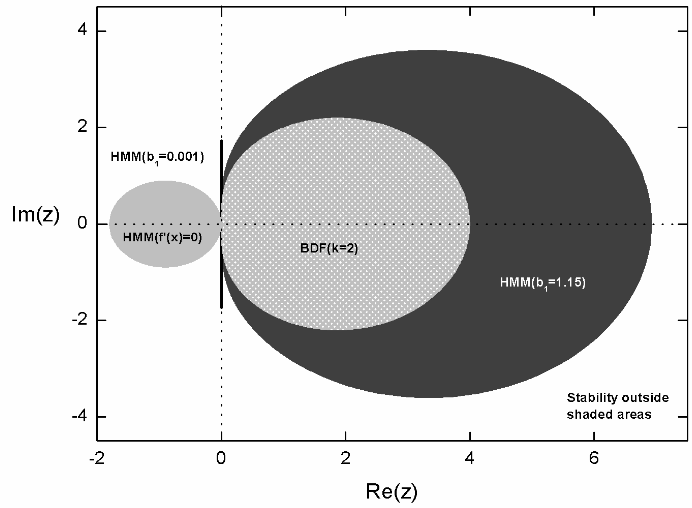

Figure 1. HMM and BDF absolute stability regions.

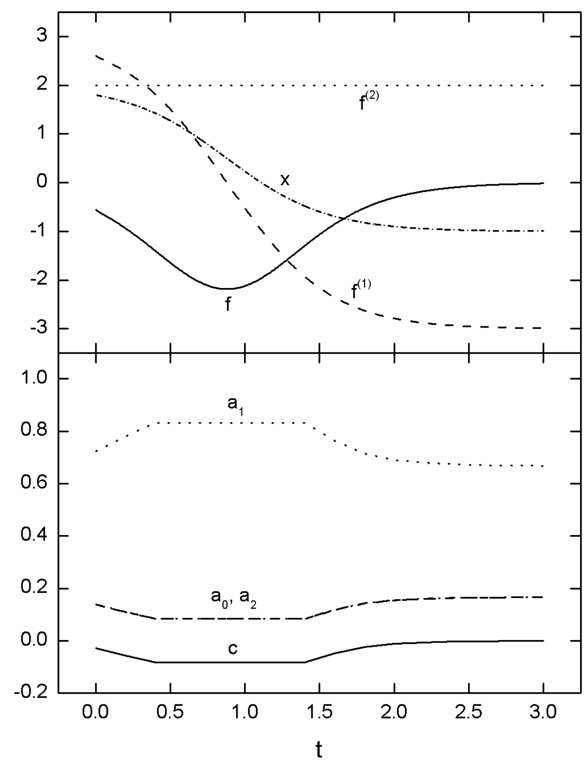

Figure 2. Temporary evolution of $f^{(0)}$, $f^{(1)}$ and $f^{(2)}$, the coefficients a_i and the non-linear correction term c.

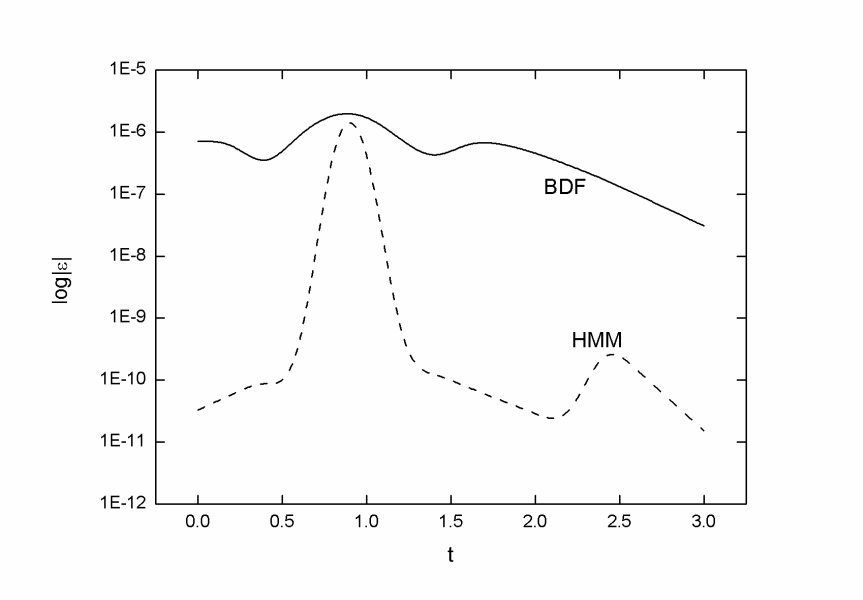

Figure 3. Calculation of the absolute difference $|y(x_n) - y_n|$ between the analytic and the numerical solutions corresponds with $h = 10^{-2}$.

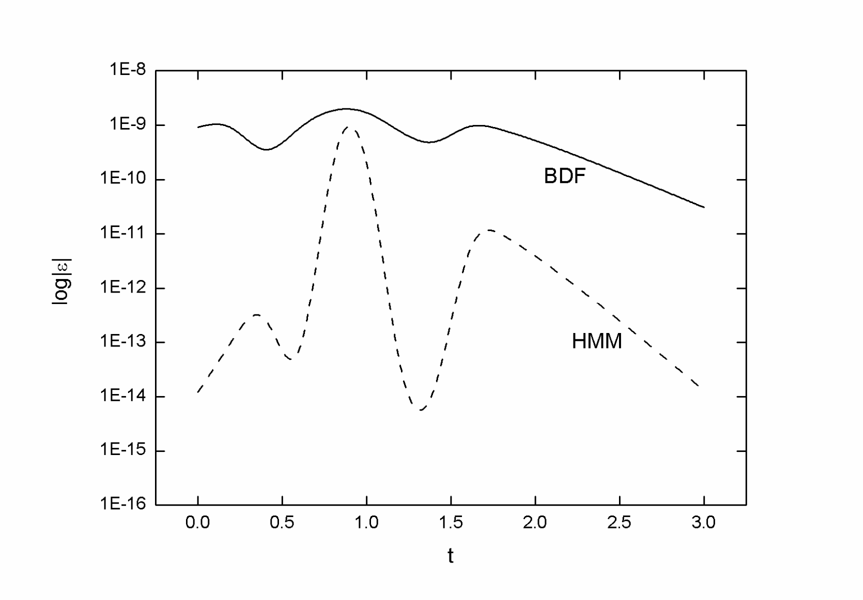

Figure 4. Calculation of the absolute difference $|y(x_n) - y_n|$ between the analytic and the numerical solutions corresponds with $h = 10^{-3}$.

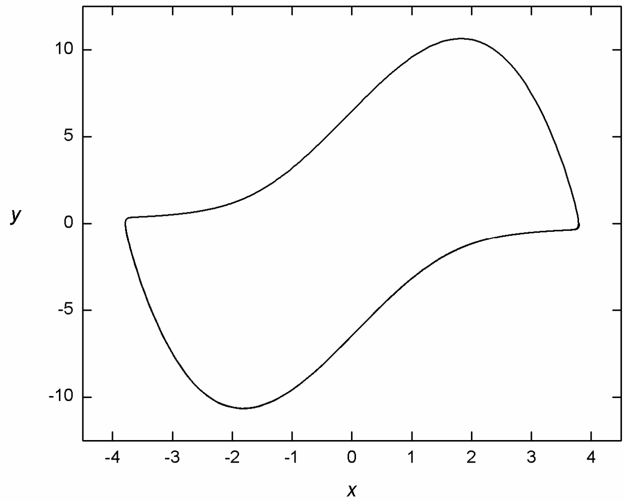

Figure 5. Phases diagram (*x* and *y*).

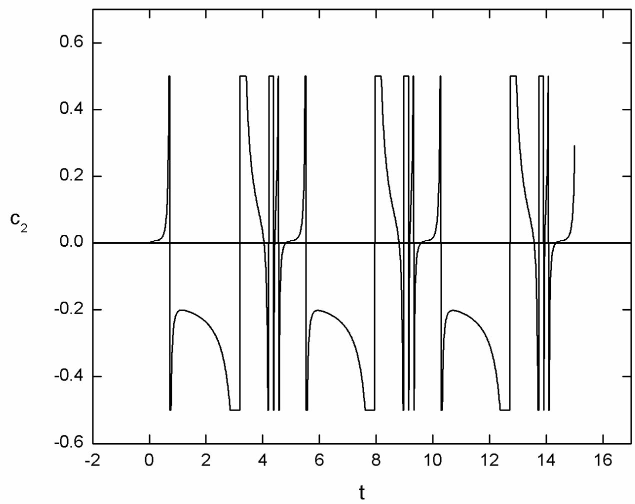

Figure 6. Temporary evolution of the second non-linear correction term, c_2.

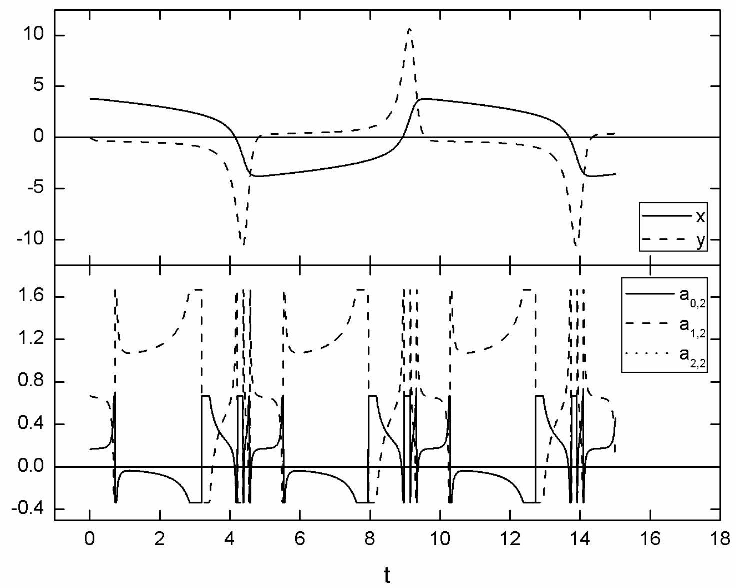

Figure 7. Temporary evolution of the state variables, and the coefficients $a_{0,2}$, $a_{1,2}$, $a_{2,2}$.

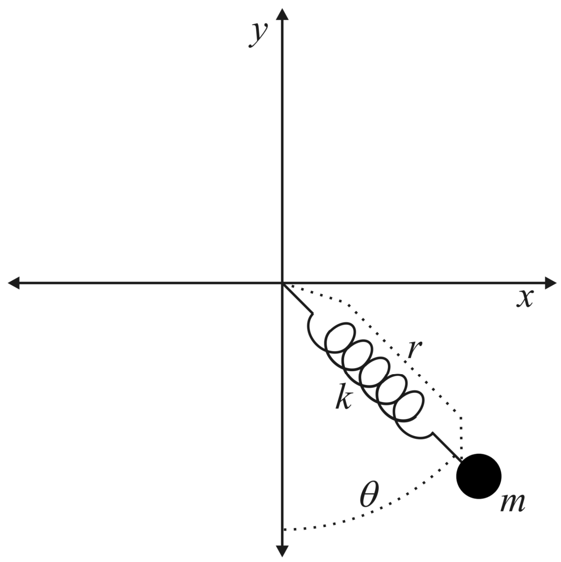

Figure 8. Elastic Pendulum.

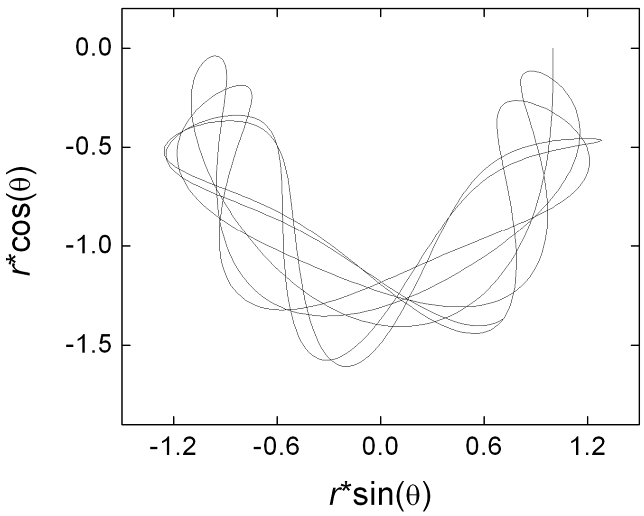

Figure 9. Trajectory of the mass in the (x, y) plane.

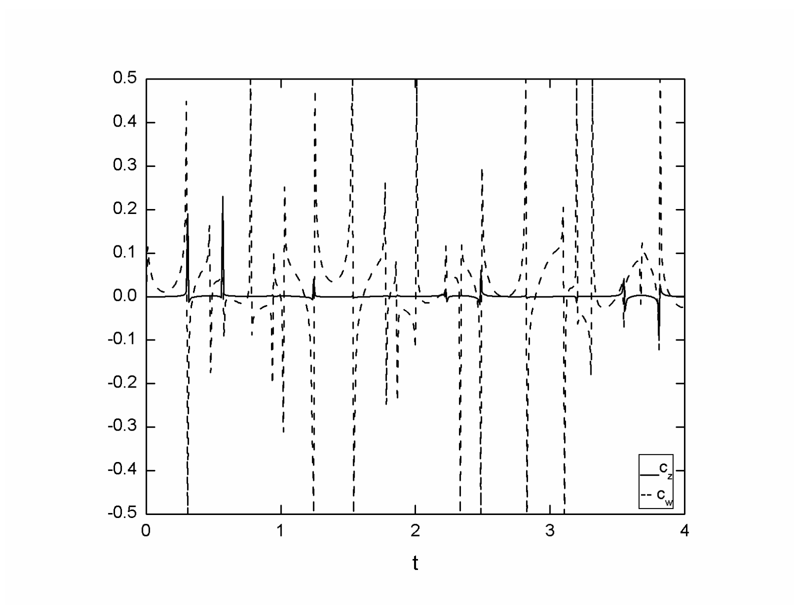

Figure 10. Temporary evolution of the non-linear correction c_z and c_w.

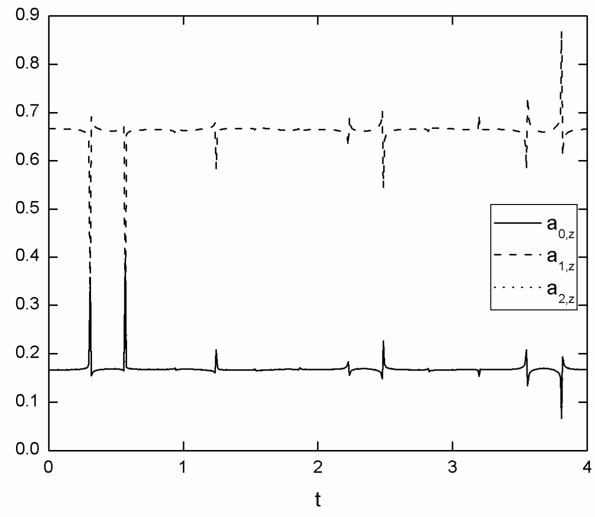

Figure 11. Temporary evolution of the coefficients $a_{0,z}$, $a_{1,z}$, $a_{2,z}$.

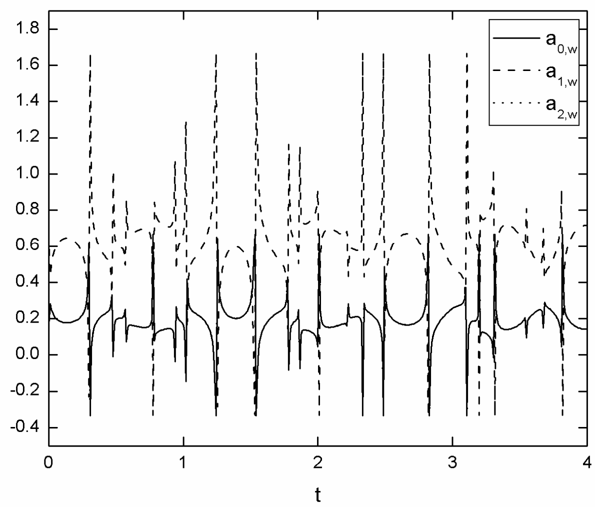

Figure 12. Temporary evolution of the coefficients $a_{0,w}$, $a_{1,w}$, $a_{2,w}$.

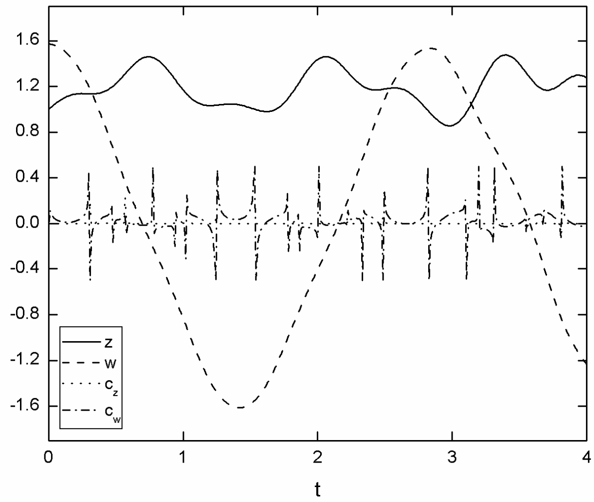

Figure 13. Temporary evolution of the state variables z and w, and the non-linear term c_z and c_w.

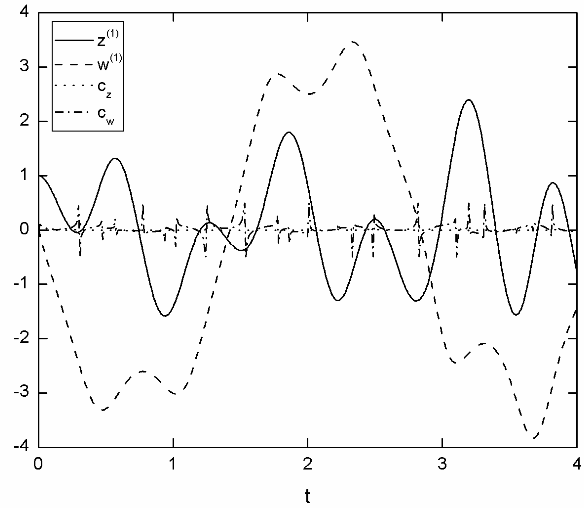

Figure 14. Temporary evolution of the first temporal derivative $z^{(1)}$ and $w^{(1)}$, and the non-linear term c_z and c_w.